4.3 Almost sure convergence and strong laws of large numbers

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Definition 1 Suppose that ξ and $\{\xi_n, n \geq 1\}$ are random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$.

If there exists a $\Omega_0 \in \mathcal{F}$ such that $P(\Omega_0) = 0$ and for any $\omega \in \Omega \setminus \Omega_0, \xi_n(\omega) \to \xi(\omega)$, $(n \to \infty)$, then we say that ξ_n converges with probability one or almost surely to ξ, denoted by $\xi_n \to \xi$ a.s.

Theorem 1 Suppose that ξ and $\{\xi_n, n \geq 1\}$ are random variables defined on $(\Omega, \mathcal{F}, P)$.
$\xi_n(\omega) \to \xi(\omega)$ a.s. iff for any $\epsilon > 0$,

$$\lim_{n\to\infty} P(\sup_{k\geq n} |\xi_k - \xi| \geq \epsilon) = 0$$

$$i.e., \lim_{n\to\infty} P(\bigcup_{k\geq n} (|\xi_k - \xi| \geq \epsilon)) = 0.$$

Corollary 1

$$\xi_n \to \xi \ a.s. \Rightarrow \xi_n \xrightarrow{P} \xi.$$

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Corollary 2. If for any $\epsilon > 0$, $\sum_{n=1}^{\infty} P(|\xi_n - \xi| \geq \epsilon) < \infty$, then

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Corollary 2. If for any $\epsilon > 0$, $\sum_{n=1}^{\infty} P(|\xi_n - \xi| \geq \epsilon) < \infty$, then

$$\xi_n \to \xi \quad a.s.$$

Proof. Note that

$$P(|\xi_n - \xi| \geq \epsilon) \leq P\Big(\bigcup_{k \geq n} (|\xi_k - \xi| \geq \epsilon)\Big)$$

$$\leq \sum_{k=n}^{\infty} P(|\xi_k - \xi| \geq \epsilon).$$

Proof of Theorem 1. For any $\epsilon > 0$, let $A_n^\epsilon = \{|\xi_n - \xi| \ge \epsilon\}$ and $A^\epsilon = \cap_{n=1}^\infty \cup_{k\ge n} A_k^\epsilon$. Then $\xi_n(\omega) \not\to \xi(\omega)$ is equivalent to that, there is an $\epsilon_0 > 0$ such that for any N there is a $n \ge N$ for which $|\xi_n(\omega) - \xi(\omega)| \ge \epsilon_0$. This is also equivalent to that, there is an m such that for any n there is a $k \ge n$ for which $|\xi_k(\omega) - \xi(\omega)| \ge 1/m$. So

$$\{\xi_n \not\to \xi\} = \bigcup_{\epsilon>0} A^\epsilon = \bigcup_{m=1}^\infty A^{\frac{1}{m}}.$$

Proof of Theorem 1. For any $\epsilon > 0$, let $A_n^\epsilon = \{|\xi_n - \xi| \ge \epsilon\}$ and $A^\epsilon = \cap_{n=1}^\infty \cup_{k\ge n} A_k^\epsilon$. Then $\xi_n(\omega) \not\to \xi(\omega)$ is equivalent to that, there is an $\epsilon_0 > 0$ such that for any N there is a $n \ge N$ for which $|\xi_n(\omega) - \xi(\omega)| \ge \epsilon_0$. This is also equivalent to that, there is an m such that for any n there is a $k \ge n$ for which $|\xi_k(\omega) - \xi(\omega)| \ge 1/m$. So

$$\{\xi_n \not\to \xi\} = \bigcup_{\epsilon>0} A^\epsilon = \bigcup_{m=1}^\infty A^{\frac{1}{m}}.$$

By the continuity theorem, we have

$$P(A^\epsilon) = P(\bigcap_{n=1}^\infty \bigcup_{k\ge n} A_k^\epsilon) = \lim_{n\to\infty} P(\bigcup_{k\ge n} A_k^\epsilon)$$

which implies that the following relations hold:

$$
\begin{aligned}
0 = P(\{\xi_n \not\to \xi\}) \quad &\Leftrightarrow \quad P(\bigcup_{m=1}^{\infty} A^{\frac{1}{m}}) = 0 \\
&\Leftrightarrow \quad P(A^{\frac{1}{m}}) = 0, \forall m \geq 1
\end{aligned}
$$

$$
\begin{aligned}
0 = P(\{\xi_n \not\to \xi\}) &\Leftrightarrow P\left(\bigcup_{m=1}^{\infty} A^{\frac{1}{m}}\right) = 0 \\
&\Leftrightarrow P(A^{\frac{1}{m}}) = 0, \forall m \geq 1 \\
&\Leftrightarrow P\left(\bigcup_{k \geq n} A_k^{\frac{1}{m}}\right) \to 0, \forall m \geq 1 \\
&\Leftrightarrow P\left(\bigcup_{k \geq n} \left(|\xi_k - \xi| \geq \frac{1}{m}\right)\right) \to 0, \forall m \geq 1 \\
&\Leftrightarrow P\left(\bigcup_{k \geq n} (|\xi_k - \xi| \geq \epsilon)\right) \to 0, \forall \epsilon \geq 0.
\end{aligned}
$$

Corollary

If $\xi_n \xrightarrow{P} \xi$, *then there exists a sub-sequence* $\{\xi_{n_k}\}$ *such that*

$$\xi_{n_k} \to \xi \ \ a.s.$$

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$$\xi_{n_k} \to \xi \ \ a.s.$$

Proof. Let $\epsilon_k = 2^{-k}$. For any k, there exists a n_k such that

$$P(|\xi_n - \xi| \ge \epsilon_k) < \epsilon_k \ \ \forall n \ge n_k.$$

Without loss of generality, we can assume $n_1 < n_2 < \cdots < n_k < n_{k+1}$. Then for any $\epsilon > 0$, there is a k_0 such that $\epsilon_k < \epsilon$ for $k \ge k_0$.

So

$$\sum_{k=k_0}^{\infty} P(|\xi_{n_k} - \xi| \geq \epsilon) \leq \sum_{k=k_0}^{\infty} P(|\xi_{n_k} - \xi| \geq \epsilon_k) \leq \sum_{k=k_0}^{\infty} \epsilon_k < \infty.$$

Hence

$$\xi_{n_k} \to \xi \ \ a.s.$$

So

$$\sum_{k=k_0}^{\infty} P(|\xi_{n_k} - \xi| \geq \epsilon) \leq \sum_{k=k_0}^{\infty} P(|\xi_{n_k} - \xi| \geq \epsilon_k) \leq \sum_{k=k_0}^{\infty} \epsilon_k < \infty.$$

Hence

$$\xi_{n_k} \to \xi \ \ a.s.$$

Corollary

$$\xi_n - \xi_m \xrightarrow{P} 0 \ \textit{as} \ n, m \to \infty \ \textit{if any only if}$$

$$\exists \xi, \quad \xi_n \xrightarrow{P} \xi.$$

Corollary

$$\xi_n - \xi_m \xrightarrow{P} 0 \ as\ n, m \to \infty \ if\ any\ only\ if$$

$$\exists \xi, \quad \xi_n \xrightarrow{P} \xi.$$

Proof. The "if" part is obvious. For the "if" part, for each $\epsilon_k = 2^{-k}$ there exists n_k such that

$$P\left(|\xi_n - \xi_m| \geq \epsilon_k\right) \leq \epsilon_k, \quad \forall n, m \geq n_k.$$

Corollary

$$\xi_n - \xi_m \xrightarrow{P} 0 \ as\ n, m \to \infty\ if\ any\ only\ if$$

$$\exists \xi, \quad \xi_n \xrightarrow{P} \xi.$$

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$$P\left(|\xi_n - \xi_m| \geq \epsilon_k\right) \leq \epsilon_k, \quad \forall n, m \geq n_k.$$

Without loss of generality, assume $n_k < n_{k+1}$. Then

$$P\left(\left|\xi_{n_{k+1}} - \xi_{n_k}\right| \geq \epsilon_k\right) \leq \epsilon_k.$$

It follows that

$$
\begin{aligned}
&P\left(\sum_{k=1}^{\infty}|\xi_{n_{k+1}}-\xi_{n_k}|=\infty\right)\\
=&P\left(\sum_{k=k_0}^{\infty}|\xi_{n_{k+1}}-\xi_{n_k}|=\infty\right)\\
\le&P\left(\sum_{k=k_0}^{\infty}|\xi_{n_{k+1}}-\xi_{n_k}|\ge\sum_{k=k_0}^{\infty}\epsilon_k\right)\\
\le&\sum_{k=k_0}^{\infty}\epsilon_k\to 0 \text{ as } k_0\to\infty.
\end{aligned}
$$

Let $\xi_0=0$. For $\omega\in A=\{\sum_{k=1}^{\infty}|\xi_{n_{k+1}}-\xi_{n_k}|<\infty\}$, define $\xi(\omega)=\sum_{k=0}^{\infty}(\xi_{n_{k+1}}(\omega)-\xi_{n_k}(\omega))$, and for $\omega\in A$, define $\xi(\omega)=0$.

Then

$$\xi_{n_k} \to \xi \ \ a.s.$$

So,

$$\xi_{n_k} \xrightarrow{P} \xi.$$

It follows that

$$\xi_n = (\xi_n - \xi_{n_k}) + \xi_{n_k} \xrightarrow{P} \xi.$$

4.3.2 Strong laws of large numbers

Definition 2 Suppose that $\{\xi_n, n \geq 1\}$ is a sequence of random variables defined on $(\Omega, \mathcal{F}, P)$. If there exist constant sequences $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ such that

$$\frac{1}{a_n} \sum_{k=1}^{n} \xi_k - b_n \to 0 \quad a.s.,$$

we say that $\{\xi_n\}$ obeys the strong law of large numbers (SLLN).

Theorem 2 (Borel) Suppose that $\{\xi_n\}$ is a sequence of i.i.d. random variables defined on $(\Omega, \mathcal{F}, P)$ with $P(\xi_n = 1) = p$, $P(\xi_n = 0) = 1 - p$, $0 < p < 1$. Let $S_n = \sum_{k=1}^n \xi_k$, then

$$\frac{S_n}{n} - p \to 0 \quad a.s.$$

Proof. We have

$$
\begin{aligned}
& P\left(\frac{|S_n - np|}{n} \geq \epsilon\right) \\
= \; & P\left(|S_n - np|^4 \geq (\epsilon n)^4\right)
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
& P\left(\frac{|S_n - np|}{n} \geq \epsilon\right) \\
= \;& P\left(|S_n - np|^4 \geq (\epsilon n)^4\right) \\
\leq \;& \frac{1}{\epsilon^4 n^4} E|S_n - np|^4 \quad \text{(by Markov inequality).}
\end{aligned}
$$

Let $\eta_i = \xi_i - p$. Then

$$\begin{aligned} E|S_n - np|^4 &= E\Big|\sum_{i=1}^n \eta_i\Big|^4 = \sum_{i,j,l,k} E\eta_i\eta_j\eta_l\eta_k \\ &= \sum_i E\eta_i^4 + \sum_{i\neq j} E\eta_i^2\eta_j^2 \\ &= n(q^4p + p^4q) + n(n-1)(pq)^2 \leq n^2pq. \end{aligned}$$

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So,

$$\sum_{n=1}^\infty P\left(\frac{|S_n - np|}{n} \ge \epsilon\right) \le \sum_{n=1}^\infty \frac{n^2pq}{\epsilon^4 n^4} < \infty.$$

Hence $S_n/n \to p$ a.s.

Corollary Suppose that $\{\xi_n\}$ is a sequence of i.i.d. random variables defined on $(\Omega, \mathcal{F}, P)$ with $E\xi_n = \mu$, $E\xi_n^4 < \infty$. Let $S_n = \sum_{k=1}^n \xi_k$, then

$$\frac{S_n}{n} \to \mu \quad a.s.$$

Proof. We have

$$\begin{aligned}
& P\left(\frac{|S_n - n\mu|}{n} \geq \epsilon\right) \\
= \ & P\left(|S_n - n\mu|^4 \geq (\epsilon n)^4\right)
\end{aligned}$$

Proof. We have

$$\begin{aligned} & P\left(\frac{|S_n - n\mu|}{n} \ge \epsilon\right) \\ = \ & P\left(|S_n - n\mu|^4 \ge (\epsilon n)^4\right) \\ \le \ & \frac{1}{\epsilon^4 n^4} E|S_n - n\mu|^4 \quad \text{(by Markov inequality).} \end{aligned}$$

Let $\eta_i = \xi_i - \mu$. Then $E\eta_i = 0$, $E\eta_1^4 = E(\xi_1 - \mu)^4 < \infty$,

$$\begin{aligned}
E|S_n - n\mu|^4 &= E\Big|\sum_{i=1}^n \eta_i\Big|^4 = \sum_{i,j,l,k} E\eta_i\eta_j\eta_l\eta_k \\
&= \sum_i E\eta_i^4 + \sum_{i\neq j} E\eta_i^2\eta_j^2 \\
&= nE(\xi_1 - \mu)^4 + n(n-1)(Var(\xi_1))^2 \\
&\leq n^2 c_0.
\end{aligned}$$

So,

$$\sum_{n=1}^{\infty} P\left(\frac{|S_n - n\mu|}{n} \geq \epsilon\right) \leq \sum_{n=1}^{\infty} \frac{n^2 c_0}{\epsilon^4 n^4} < \infty.$$

Hence $S_n/n \to \mu$ a.s.

Theorem 3 (Kolmogorov, 1930) Suppose that $\{\xi_n, n \geq 1\}$ is a sequence of i.i.d. random variables defined on $(\Omega, \mathcal{F}, P)$ with $E|\xi_1| < \infty, E\xi_1 = \mu$. Let $S_n = \sum_{k=1}^n \xi_k$. Then

$$\frac{S_n}{n} \to \mu \quad a.s. \tag{1}$$

Theorem 3 (Kolmogorov, 1930) Suppose that $\{\xi_n, n \geq 1\}$ is a sequence of i.i.d. random variables defined on $(\Omega, \mathcal{F}, P)$ with $E|\xi_1| < \infty, E\xi_1 = \mu$. Let $S_n = \sum_{k=1}^n \xi_k$. Then

$$\frac{S_n}{n} \to \mu \quad a.s. \tag{1}$$

In fact, the converse of Theorem 2 also holds: if there exists a constant μ such that (1) holds, then the expectation of ξ_1 exists and equals to μ.

Theorem 4 Suppose that $\{\xi_n, n \geq 1\}$ is a sequence of independent random variables defined on $(\Omega, \mathcal{F}, P)$ with $E\xi_k = \mu_k$, $Var\xi_k < \infty$. Let $S_n = \sum_{k=1}^n \xi_k$. If

$$\sum_{n=1}^{\infty} \frac{Var\xi_n}{n^2} < \infty,$$

then

$$\frac{S_n - ES_n}{n} \to 0 \quad a.s.$$

Proof Theorem 3 from Theorem 4: Let $\eta_k = \xi_k I\{|\xi_k| \le k\}$. Then

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{Var(\eta_n)}{n^2} \le \sum_{n=1}^{\infty} \frac{E[\xi_1^2 I\{|\xi_1| \le n\}]}{n^2} \\
=&\sum_{n=1}^{\infty} \int_n^{n+1} \frac{E[\xi_1^2 I\{|\xi_1| \le n\}]}{n^2} dx \\
\le&2^2 \sum_{n=1}^{\infty} \int_n^{n+1} \frac{E[\xi_1^2 I\{|\xi_1| \le x\}]}{x^2} dx \\
=&2^2 \int_1^{\infty} E\left[\frac{\xi_1^2 I\{|\xi_1| \le x\}}{x^2}\right] dx \\
=&2^2 E\left[\int_1^{\infty} \frac{\xi_1^2 I\{|\xi_1| \le x\}}{x^2} dx\right] \le 4E[|\xi_1|] < \infty.
\end{aligned}
$$

By Theorem 4,

$$\frac{\sum_{k=1}^{n}(\eta_k - E\eta_k)}{n} \to 0 \quad a.s.$$

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Also,

$$\frac{\sum_{k=1}^{n} E\eta_k}{n} = \frac{\sum_{k=1}^{n} E[\xi_1 I\{|\xi_1| \le k\}]}{n}$$
$$\to E\xi_1 = \mu.$$

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It follows that

$$\frac{\sum_{k=1}^{n} \eta_k}{n} \to \mu \;\; a.s.$$

Finally,

$$P\left(\eta_k \neq \xi_k \ \ i.o.\right) = P\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{\eta_k \neq \xi_k\}\right)$$
$$= \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty}\{|\xi_k| \geq k\}\right) \leq \lim_{n\to\infty}\sum_{k=n}^{\infty} P\left(|\xi_k| \geq k\right) = 0,$$

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because

$$\sum_{k=1}^{\infty} P\left(|\xi_k| \geq k\right) = \sum_{k=1}^{\infty} P\left(|\xi_1| \geq k\right) < \infty.$$

Hence,

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\xi_k}{n}=\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\eta_k}{n}=\eta \quad a.s.$$

The proof is completed.

The necessary part: Suppose

$$\lim_{n\to\infty}\frac{S_n}{n}=\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\xi_k}{n}=\mu \quad a.s.$$

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Then

$$\frac{\xi_n}{n}=\frac{S_n}{n}-\frac{S_{n-1}}{n}=\frac{S_n}{n}-\frac{S_{n-1}}{n-1}\frac{n-1}{n}\to 0 \ \ a.s.$$

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So,

$$P\Big(|\xi_n|\ge n \ \ i.o.\Big)=0,$$

which will imply

$$\sum_{n=1}^{\infty}P\Big(|\xi_1|\ge n\Big)=\sum_{n=1}^{\infty}P\Big(|\xi_n|\ge n\Big)<\infty.$$

In fact, if

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Then

$$P\left(\bigcap_{k=n}^{\infty}\{|\xi_k| < k\}\right) = \prod_{k=n}^{\infty} P\left(|\xi_k| < k\right) = \prod_{k=n}^{\infty}\Big(1 - P\left(|\xi_k| \ge k\right)\Big)$$

$$\le \exp\left\{-\sum_{k=n}^{\infty} P\left(|\xi_k| \ge k\right)\right\} = 0.$$

So,

$$P\left(\{|\xi_n| \ge n \ \ i.o.\}^C\right) = P\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}\{|\xi_k| < k\}\right) = 0.$$

Borel-Cantelli Lemma

Lemma

(1) If $\sum_{n=1}^{\infty} P(A_n) < \infty$, *then*

$$P\left(A_n \;\; i.o.\right) = 0.$$

(2) If $\sum_{n=1}^{\infty} P(A_n) = \infty$ *and* $\{A_n\}$ *are independent events, then*

$$P\left(A_n \;\; i.o.\right) = 1.$$

(1)

$$P\left(A_k \;\; i.o.\right) = P\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k\right)$$

$$= \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty} A_k\right) \le \lim_{n\to\infty}\sum_{k=n}^{\infty} P\left(A_k\right) = 0.$$

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Then

$$\begin{aligned} P\left(\bigcap_{k=n}^{\infty} A_k^C\right) &= \prod_{k=n}^{\infty} P\left(A_k^C\right) \\ &\leq \exp\left\{-\sum_{k=n}^{\infty} P\left(A_k\right)\right\} = 0. \end{aligned}$$

So,

$$P\left(\{A_n \;\; i.o.\}^C\right) = P\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k^C\right) = 0.$$

Example. (the Monte Carlo method) Let $f(x)$ be a continuous function defined on $[0,1]$ with values in $[0,1]$, and let $\xi_1, \eta_1, \xi_2, \eta_2, \cdots$ be a sequence of independent random variables with a common uniform distribution in $[0,1]$. Define

$$\rho_i = \begin{cases} 1, & \text{if } f(\xi_i) \ge \eta_i, \\ 0, & \text{if } f(\xi_i) < \eta_i. \end{cases}$$

Then $\{\rho_i, i \ge 1\}$ are also i.i.d. random variables. Furthermore,

$$E\rho_1 = P(f(\xi_1) \ge \eta_1) \quad = \quad \int\int_{y \le f(x)} dx dy = \int_0^1 f(x) dx.$$

By Theorem 3, we have

$$\frac{1}{n}\sum_{k=1}^{n} \rho_k \to \int_0^1 f(x) dx \quad a.s.$$

Example. (the Monte Carlo method) Suppose $D \subset \mathbb{R}^d$ is a bounded area, $\int_D \big|g(\boldsymbol{x})\big| d\boldsymbol{x} < \infty$. Compute $\int_D g(\boldsymbol{x}) d\boldsymbol{x}$.

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解: Suppose that $D \subset A$ where A is a rectangle, and $\boldsymbol{\xi}$ is a random vector uniformly distributed in A. Denote

$$I_D(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \boldsymbol{x} \in D, \\ 0, & \text{otherwise.} \end{cases}.$$

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$$I_D(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \boldsymbol{x} \in D, \\ 0, & \text{otherwise.} \end{cases}.$$

Then

$$E[g(\boldsymbol{\xi}) I_D(\boldsymbol{\xi})] = \int_A \frac{g(\boldsymbol{x}) I_D(\boldsymbol{x})}{m(A)} d\boldsymbol{x} = \frac{1}{m(A)} \int_D g(\boldsymbol{x}) d\boldsymbol{x}.$$

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Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots$ be i.i.d. copies of $\boldsymbol{\xi}$. Then by the strong law of large numbers,

$$\frac{1}{n}\sum_{i=1}^{n} g(\boldsymbol{\xi}_i) I_D(\boldsymbol{\xi}_i) \to E[g(\boldsymbol{\xi}) I_D(\boldsymbol{\xi})] \quad a.s.$$

$$= \frac{1}{m(A)} \int_D g(\boldsymbol{x}) d\boldsymbol{x}.$$

Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots$ be i.i.d. copies of $\boldsymbol{\xi}$. Then by the strong law of large numbers,

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^n g(\boldsymbol{\xi}_i)I_D(\boldsymbol{\xi}_i) \to & E[g(\boldsymbol{\xi})I_D(\boldsymbol{\xi})] \ \ a.s. \\ = & \frac{1}{m(A)}\int_D g(\boldsymbol{x})d\boldsymbol{x}.\end{aligned}$$

So, for large n,

$$\int_D g(\boldsymbol{x})d\boldsymbol{x} \approx \frac{m(A)}{n}\sum_{i=1}^n g(\boldsymbol{\xi}_i)I_D(\boldsymbol{\xi}_i).$$

If D is not bounded, we choose a probability density function $f(\boldsymbol{x}) > 0$, for example the d-dimensional standard normal density. $\boldsymbol{\xi}, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots$ be i.i.d. random vectors with $f(\boldsymbol{x})$ being the pdf. Then

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\frac{g(\boldsymbol{\xi}_i)I_D(\boldsymbol{\xi}_i)}{f(\boldsymbol{\xi}_i)} \to & E\left[\frac{g(\boldsymbol{\xi})I_D(\boldsymbol{\xi})}{f(\boldsymbol{\xi})}\right] \quad a.s. \\ & = \int\left[\frac{g(\boldsymbol{x})I_D(\boldsymbol{x})}{f(\boldsymbol{x})}f(\boldsymbol{x})\right]d\boldsymbol{x} \\ & = \int_D g(\boldsymbol{x})d\boldsymbol{x}.\end{aligned}$$

Convergence rate of the SLLN:

Suppose that $\{\xi_i; \imath \geq 1\}$ be i.i.d. random variables, $E[\xi_1] = \mu$. Then

$$\frac{S_n}{n} \to \mu \ \ a.s.$$

The law of the iterated logarithm:

Theorem

Suppose $Var(\xi_1)=\sigma^2<\infty$. *Then*

$$\limsup_{n\to\infty}\frac{S_n-n\mu}{\sqrt{2n\ln\ln n}}=\sigma \quad a.s. \tag{2}$$

On the other hand, if (2) holds for some μ *and* σ, *then we must have* $Var(\xi_1)=\sigma^2$ *and* $E\xi_1=\mu$.

The law of the iterated logarithm tells that

$$\frac{S_n}{n} - \mu = O\left(\sqrt{\frac{\ln\ln n}{n}}\right) \quad a.s.$$

For the MC method, the error is about $\sqrt{\frac{\ln\ln n}{n}}$.