3.3 Characteristic functions

3.3.1 Definitions

3.3 Characteristic functions

3.3.1 Definitions

Definition 1 Suppose that ξ and η are real random variables, we call $\zeta = \xi + i\eta$ a complex random variable, where $i^2 = -1$. We call $E\zeta = E\xi + iE\eta$ the expectation of ζ.

3.3 Characteristic functions

3.3.1 Definitions

Definition 1 Suppose that ξ and η are real random variables, we call $\zeta = \xi + i\eta$ a complex random variable, where $i^2 = -1$. We call $E\zeta = E\xi + iE\eta$ the expectation of ζ.

$E\zeta$ possesses properties similar to that of a real mathematical expectation.

Modulus inequality: $|E\zeta| \leq E|\zeta|$.

Modulus inequality: $|E\zeta| \leq E|\zeta|$.

证. 记$\zeta = \xi + i\eta$. 则$|E\zeta| = \sqrt{(E\xi)^2 + (E\eta)^2}$,
$E|\zeta| = E\sqrt{\xi^2 + \eta^2}$.

Modulus inequality: $|E\zeta| \le E|\zeta|$.

证. 记$\zeta = \xi + i\eta$. 则$|E\zeta| = \sqrt{(E\xi)^2 + (E\eta)^2}$, $E|\zeta| = E\sqrt{\xi^2 + \eta^2}$. 设$a$, b 为实数, 由初等不等式

$$|a\xi + b\eta| \le \sqrt{a^2 + b^2} \cdot \sqrt{\xi^2 + \eta^2}$$

得

Modulus inequality: $|E\zeta| \le E|\zeta|$.

证. 记$\zeta = \xi + i\eta$. 则$|E\zeta| = \sqrt{(E\xi)^2 + (E\eta)^2}$, $E|\zeta| = E\sqrt{\xi^2 + \eta^2}$. 设$a$, b 为实数, 由初等不等式

$$|a\xi + b\eta| \le \sqrt{a^2 + b^2} \cdot \sqrt{\xi^2 + \eta^2}$$

得

$$|aE\xi + bE\eta| \le E|a\xi + b\eta| \le \sqrt{a^2 + b^2} \cdot E\sqrt{\xi^2 + \eta^2}.$$

取$a = E\xi$, $b = E\eta$得

$$(E\xi)^2 + (E\eta)^2 \leq \sqrt{(E\xi)^2 + (E\eta)^2} \cdot E\sqrt{\xi^2 + \eta^2}.$$

所以

$$\sqrt{(E\xi)^2 + (E\eta)^2} \leq E\sqrt{\xi^2 + \eta^2}.$$

即

$$|E\zeta| \leq E|\zeta|.$$

Definition 2 Suppose ξ is a real random variable, we call

$$f(t) = Ee^{it\xi}, \quad -\infty < t < \infty$$

the characteristic function of ξ.

Definition 2 Suppose ξ is a real random variable, we call

$$f(t) = Ee^{it\xi}, \quad -\infty < t < \infty$$

the characteristic function of ξ.

Notice $|e^{it\xi}| = 1$. $Ee^{it\xi}$ exists for all t.

Definition 2 Suppose ξ is a real random variable, we call

$$f(t) = Ee^{it\xi}, \quad -\infty < t < \infty$$

the characteristic function of ξ.

Notice $|e^{it\xi}| = 1$. $Ee^{it\xi}$ exists for all t.

$$f(t) = \int_{-\infty}^{\infty} e^{itx} dF(x).$$

If ξ is a discrete random variable with $P(\xi = x_n) = p_n$, then

$$f(t) = \sum_{n=1}^{\infty} p_n e^{itx_n}, \quad -\infty < t < \infty.$$

If ξ is a continuous random variable with the density function $p(x)$, then

$$f(t) = \int_{-\infty}^{\infty} e^{itx} p(x) dx, \quad -\infty < t < \infty,$$

which is just the Fourier transformation of $p(x)$.

Example 1. The characteristic function of the degenerate distribution $P(\xi = c) = 1$ is

$$f(t) =$$

Example 1. The characteristic function of the degenerate distribution $P(\xi = c) = 1$ is

$$f(t) = e^{ict}, \qquad -\infty < t < \infty.$$

Example 1. The characteristic function of the degenerate distribution $P(\xi = c) = 1$ is

$$f(t) = e^{ict}, \qquad -\infty < t < \infty.$$

Example 2. The characteristic function of the binomial distribution $B(n, p)$ is

$$f(t) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} e^{itk}$$

Example 1. The characteristic function of the degenerate distribution $P(\xi = c) = 1$ is

$$f(t) = e^{ict}, \qquad -\infty < t < \infty.$$

Example 2. The characteristic function of the binomial distribution $B(n, p)$ is

$$f(t) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} e^{itk} = \sum_{k=0}^{n} \binom{n}{k} (pe^{it})^k q^{n-k}$$

Example 1. The characteristic function of the degenerate distribution $P(\xi = c) = 1$ is

$$f(t) = e^{ict}, \qquad -\infty < t < \infty.$$

Example 2. The characteristic function of the binomial distribution $B(n, p)$ is

$$\begin{aligned} f(t) &= \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} e^{itk} = \sum_{k=0}^{n} \binom{n}{k} (pe^{it})^k q^{n-k} \\ &= (pe^{it} + q)^n, \quad p + q = 1. \end{aligned}$$

Example 3. The characteristic function of the Poisson distribution $P(\lambda)$ is

$$f(t) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} e^{itk}$$

Example 3. The characteristic function of the Poisson distribution $P(\lambda)$ is

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} e^{itk} \\ &= \sum_{k=0}^{\infty} \frac{(\lambda e^{it})^k}{k!} e^{-\lambda} = e^{\lambda(e^{it}-1)}. \end{aligned}$$

Example 3. The characteristic function of the Poisson distribution $P(\lambda)$ is

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} e^{itk} \\ &= \sum_{k=0}^{\infty} \frac{(\lambda e^{it})^k}{k!} e^{-\lambda} = e^{\lambda(e^{it}-1)}. \end{aligned}$$

Example 4. The characteristic function of the uniform distribution $U[a,b]$ is

$$f(t) = \int_a^b \frac{1}{b-a} e^{itx} dx$$

Example 3. The characteristic function of the Poisson distribution $P(\lambda)$ is

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} e^{itk} \\ &= \sum_{k=0}^{\infty} \frac{(\lambda e^{it})^k}{k!} e^{-\lambda} = e^{\lambda(e^{it}-1)}. \end{aligned}$$

Example 4. The characteristic function of the uniform distribution $U[a,b]$ is

$$f(t) = \int_a^b \frac{1}{b-a} e^{itx} dx = \frac{e^{itb} - e^{ita}}{i(b-a)t}.$$

Example 5. The characteristic function of the normal distribution $N(a, \sigma^2)$ is

$$f(t) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{itx - \frac{(x-a)^2}{2\sigma^2}} dx$$

Example 5. The characteristic function of the normal distribution $N(a, \sigma^2)$ is

$$\begin{aligned} f(t) &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{itx-\frac{(x-a)^2}{2\sigma^2}}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ita+it\sigma y-\frac{y^2}{2}}dy \end{aligned}$$

Example 5. The characteristic function of the normal distribution $N(a, \sigma^2)$ is

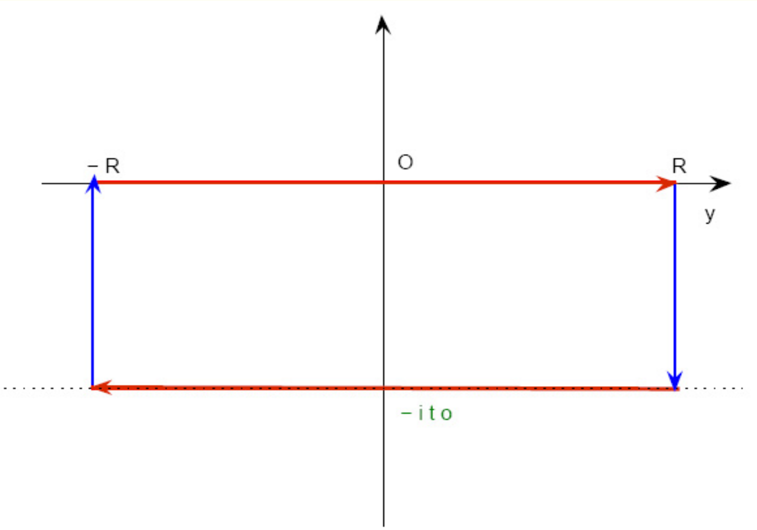
$$\begin{aligned} f(t) &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{itx-\frac{(x-a)^2}{2\sigma^2}}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ita+it\sigma y-\frac{y^2}{2}}dy \\ &= e^{iat-\frac{\sigma^2t^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(y-it\sigma)^2}{2}}dy \end{aligned}$$

Example 5. The characteristic function of the normal distribution $N(a,\sigma^2)$ is

$$
\begin{aligned}
f(t) &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{itx-\frac{(x-a)^2}{2\sigma^2}}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ita+it\sigma y-\frac{y^2}{2}}dy \\
&= e^{iat-\frac{\sigma^2t^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(y-it\sigma)^2}{2}}dy \\
&= e^{iat-\frac{\sigma^2t^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}dy \ \ (\text{围道积分})
\end{aligned}
$$

Example 5. The characteristic function of the normal distribution $N(a, \sigma^2)$ is

$$
\begin{aligned}
f(t) &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{itx-\frac{(x-a)^2}{2\sigma^2}}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ita+it\sigma y-\frac{y^2}{2}}dy \\
&= e^{iat-\frac{\sigma^2 t^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(y-it\sigma)^2}{2}}dy \\
&= e^{iat-\frac{\sigma^2 t^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}dy \quad (\text{围道积分}) \\
&= e^{iat-\frac{\sigma^2 t^2}{2}}.
\end{aligned}
$$


– R
O
R
y
– i t o

Solution (2): Let $\eta = (\xi - a)/\sigma$. Then $\eta \sim N(0,1)$ and

$$f(t) = Ee^{it(a+\sigma\eta)} = e^{ita} f_\eta(\sigma t).$$

Solution (2): Let $\eta = (\xi - a)/\sigma$. Then $\eta \sim N(0, 1)$ and

$$f(t) = Ee^{it(a+\sigma\eta)} = e^{ita} f_\eta(\sigma t).$$

So it is enough to show that $f_\eta(t) = e^{-\frac{t^2}{2}}$.

It is obvious that

$$f_\eta(t) \quad = \quad Ee^{it\eta} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{itx}e^{-\frac{x^2}{2}}dx$$

It is obvious that

$$\begin{aligned} f_\eta(t) &= Ee^{it\eta} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{itx}e^{-\frac{x^2}{2}}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos(tx)e^{-\frac{x^2}{2}}dx + i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sin(tx)e^{-\frac{x^2}{2}}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos(tx)e^{-\frac{x^2}{2}}dx. \end{aligned}$$

So

$$f_\eta'(t)$$

It is obvious that

$$
\begin{aligned}
f_\eta(t) &= Ee^{it\eta} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{itx}e^{-\frac{x^2}{2}}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \cos(tx)e^{-\frac{x^2}{2}}dx + i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sin(tx)e^{-\frac{x^2}{2}}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \cos(tx)e^{-\frac{x^2}{2}}dx.
\end{aligned}
$$

So

$$
f'_\eta(t) = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x\sin(tx)e^{-\frac{x^2}{2}}dx
$$

It is obvious that

$$\begin{aligned} f_\eta(t) &= Ee^{it\eta} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{itx}e^{-\frac{x^2}{2}}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos(tx)e^{-\frac{x^2}{2}}dx + i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sin(tx)e^{-\frac{x^2}{2}}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos(tx)e^{-\frac{x^2}{2}}dx. \end{aligned}$$

So

$$\begin{aligned} f'_\eta(t) &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x\sin(tx)e^{-\frac{x^2}{2}}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sin(tx)de^{-\frac{x^2}{2}} \end{aligned}$$

It is obvious that

$$\begin{aligned}
f_\eta(t) &= Ee^{it\eta} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{itx}e^{-\frac{x^2}{2}}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos(tx)e^{-\frac{x^2}{2}}dx + i\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sin(tx)e^{-\frac{x^2}{2}}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos(tx)e^{-\frac{x^2}{2}}dx.
\end{aligned}$$

So

$$\begin{aligned}
f'_\eta(t) &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x\sin(tx)e^{-\frac{x^2}{2}}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sin(tx)de^{-\frac{x^2}{2}} \\
&= -t\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos(tx)e^{-\frac{x^2}{2}}dx = -tf_\eta(t)
\end{aligned}$$

Next we need to solve the differential equation

$$f'_\eta(t) + t f_\eta(t) = 0.$$

Next we need to solve the differential equation

$$f_\eta'(t) + t f_\eta(t) = 0.$$

We have

$$\frac{d}{dt}\left(f_\eta(t)e^{\frac{t^2}{2}}\right) = f_\eta'(t)e^{\frac{t^2}{2}} + t f_\eta(t)e^{\frac{t^2}{2}} = 0.$$

Next we need to solve the differential equation

$$f_\eta'(t) + t f_\eta(t) = 0.$$

We have

$$\frac{d}{dt}\left(f_\eta(t)e^{\frac{t^2}{2}}\right) = f_\eta'(t)e^{\frac{t^2}{2}} + t f_\eta(t)e^{\frac{t^2}{2}} = 0.$$

Hence

$$f_\eta(t)e^{\frac{t^2}{2}} = C$$

Next we need to solve the differential equation

$$f_\eta'(t) + t f_\eta(t) = 0.$$

We have

$$\frac{d}{dt}\left(f_\eta(t)e^{\frac{t^2}{2}}\right) = f_\eta'(t)e^{\frac{t^2}{2}} + t f_\eta(t)e^{\frac{t^2}{2}} = 0.$$

Hence

$$f_\eta(t)e^{\frac{t^2}{2}} = C = f_\eta(0) = 1.$$

So

Next we need to solve the differential equation

$$f_\eta'(t) + t f_\eta(t) = 0.$$

We have

$$\frac{d}{dt}\left(f_\eta(t)e^{\frac{t^2}{2}}\right) = f_\eta'(t)e^{\frac{t^2}{2}} + t f_\eta(t)e^{\frac{t^2}{2}} = 0.$$

Hence

$$f_\eta(t)e^{\frac{t^2}{2}} = C = f_\eta(0) = 1.$$

So

$$f_\eta(t) = e^{-\frac{t^2}{2}}.$$

Solution (3): Let $\eta = (\xi - a)/\sigma$. Then $\eta \sim N(0,1)$ and

$$f(t) = Ee^{it(a+\sigma\eta)} = e^{ita} f_\eta(\sigma t).$$

Solution (3): Let $\eta = (\xi - a)/\sigma$. Then $\eta \sim N(0,1)$ and

$$f(t) = Ee^{it(a+\sigma\eta)} = e^{ita} f_\eta(\sigma t).$$

So it is enough to show that $f_\eta(t) = e^{-\frac{t^2}{2}}$.

$$f_\eta(t) = Ee^{it\eta} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{itx} e^{-\frac{x^2}{2}} dx$$

$$
\begin{aligned}
f_\eta(t) &= Ee^{it\eta} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} e^{-\frac{x^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(itx)^n}{n!} e^{-\frac{x^2}{2}} dx \\
&= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}} dx
\end{aligned}
$$

$$
\begin{aligned}
f_\eta(t) &= Ee^{it\eta} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{itx}e^{-\frac{x^2}{2}}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{(itx)^n}{n!}e^{-\frac{x^2}{2}}dx \\
&= \sum_{n=0}^{\infty}\frac{(it)^n}{n!}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}x^n e^{-\frac{x^2}{2}}dx \\
&= \sum_{n=0}^{\infty}\frac{(it)^{2n}}{(2n)!}E\eta^{2n} = \sum_{n=0}^{\infty}\frac{(it)^{2n}}{(2n)!}(2n-1)!!
\end{aligned}
$$

$$
\begin{aligned}
f_\eta(t) &= Ee^{it\eta} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{itx}e^{-\frac{x^2}{2}}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{(itx)^n}{n!}e^{-\frac{x^2}{2}}dx \\
&= \sum_{n=0}^{\infty}\frac{(it)^n}{n!}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}x^n e^{-\frac{x^2}{2}}dx \\
&= \sum_{n=0}^{\infty}\frac{(it)^{2n}}{(2n)!}E\eta^{2n} = \sum_{n=0}^{\infty}\frac{(it)^{2n}}{(2n)!}(2n-1)!! \\
&= \sum_{n=0}^{\infty}\frac{(it)^{2n}}{(2n)!}\frac{(2n)!}{2^n n!} = \sum_{n=0}^{\infty}\left(-\frac{t^2}{2}\right)^n\frac{1}{n!} = e^{-\frac{t^2}{2}}.
\end{aligned}
$$

Example 6. The characteristic function of the Cauchy distribution is

$$f(t) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{\pi(1+x^2)} dx$$

Example 6. The characteristic function of the Cauchy distribution is

$$f(t) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{\pi(1+x^2)} dx = e^{-|t|}.$$

Example 6. The characteristic function of the Cauchy distribution is

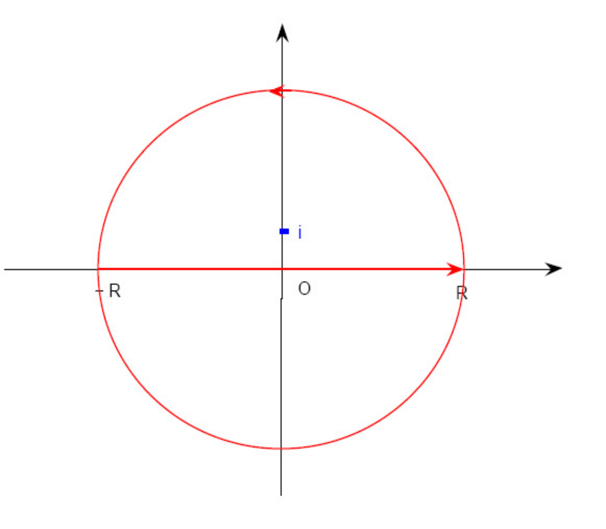
$$f(t) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{\pi(1+x^2)} dx = e^{-|t|}.$$

In fact, when $t > 0$,

$$\int_{-R}^{R} e^{itx} \frac{1}{\pi(1+x^2)} dx + \int_{semicircle} \frac{e^{itz}}{\pi(1+z^2)} dz$$

$$= 2\pi i Res \left(\frac{e^{itz}}{\pi(1+z^2)} \text{ at } i \right)$$

$$= 2\pi i (z-i) \frac{e^{itz}}{\pi(1+iz)(1-iz)} \Big|_{z=i} = e^{-t}.$$


i
-R
O
R

$$
\begin{aligned}
\left| \int_{semicircle} \frac{e^{itz}}{\pi(1+z^2)} dz \right| \leq & \int_{semicircle} \frac{1}{\pi(R^2-1)} dz \\
= & \frac{\pi R}{\pi(R^2-1)} \to 0.
\end{aligned}
$$

3.3.2 Properties

1. $|f(t)| \leq f(0) = 1;\ f(-t) = \overline{f(t)}.$

Proof.

3.3.2 Properties

1. $|f(t)| \leq f(0) = 1$; $f(-t) = \overline{f(t)}$.

Proof. Obviously,

$$|f(t)| = |\int_{-\infty}^{\infty} e^{itx} dF(x)| \leq \int_{-\infty}^{\infty} |e^{itx}| dF(x) = 1$$

and

3.3.2 Properties

1. $|f(t)| \le f(0) = 1$; $f(-t) = \overline{f(t)}$.

Proof. Obviously,

$$|f(t)| = \left|\int_{-\infty}^{\infty} e^{itx} dF(x)\right| \le \int_{-\infty}^{\infty} |e^{itx}| dF(x) = 1$$

and

$$f(0) = \int_{-\infty}^{\infty} e^{i0x} dF(x) = 1.$$

Also,

$$f(-t) = \int_{-\infty}^{\infty} e^{-itx} dF(x)$$

Also,

$$f(-t) = \int_{-\infty}^{\infty} e^{-itx} dF(x) = \int_{-\infty}^{\infty} \overline{e^{itx}} dF(x)$$

Also,

$$\begin{aligned} f(-t) &= \int_{-\infty}^{\infty} e^{-itx} dF(x) = \int_{-\infty}^{\infty} \overline{e^{itx}} dF(x) \\ &= \overline{\int_{-\infty}^{\infty} e^{itx} dF(x)} = \overline{f(t)}, \end{aligned}$$

as desired.

2. $f(t)$ is uniformly continuous on $(-\infty, \infty)$.

2. $f(t)$ is uniformly continuous on $(-\infty, \infty)$.

Proof.

2. $f(t)$ is uniformly continuous on $(-\infty, \infty)$.

Proof. For any $t \in (-\infty, \infty)$ and $\varepsilon > 0$,

2. $f(t)$ is uniformly continuous on $(-\infty, \infty)$.

Proof. For any $t \in (-\infty, \infty)$ and $\varepsilon > 0$, it follows that

$$\begin{aligned} & |f(t+h) - f(t)| \\ = \; & \left| \int_{-\infty}^{\infty} (e^{itx} e^{ihx} - e^{itx}) dF(x) \right| \end{aligned}$$

2. $f(t)$ is uniformly continuous on $(-\infty, \infty)$.

Proof. For any $t \in (-\infty, \infty)$ and $\varepsilon > 0$, it follows that

$$
\begin{aligned}
& |f(t+h) - f(t)| \\
= \ & \left| \int_{-\infty}^{\infty} (e^{itx} e^{ihx} - e^{itx}) dF(x) \right| \\
\leq \ & \int_{-\infty}^{\infty} |e^{ihx} - 1| dF(x)
\end{aligned}
$$

2. $f(t)$ is uniformly continuous on $(-\infty, \infty)$.

Proof. For any $t \in (-\infty, \infty)$ and $\varepsilon > 0$, it follows that

$$
\begin{aligned}
& |f(t+h) - f(t)| \\
= \ & \left| \int_{-\infty}^{\infty} (e^{itx} e^{ihx} - e^{itx}) dF(x) \right| \\
\le \ & \int_{-\infty}^{\infty} |e^{ihx} - 1| dF(x) \\
\le \ & \left(\int_{|x| \ge A} + \int_{|x| < A} \right) |e^{ihx} - 1| dF(x).
\end{aligned}
$$

Note that $|e^{ihx} - 1| \le 2$ and

$$\begin{aligned} |e^{ihx} - 1| &= |e^{i\frac{h}{2}x}||e^{i\frac{h}{2}x} - e^{-i\frac{h}{2}x}| = 2|\sin\frac{hx}{2}| \\ &\le |hx| \le A|h|, \quad \text{when } |x| \le A. \end{aligned}$$

Note that $|e^{ihx} - 1| \le 2$ and

$$\begin{aligned} |e^{ihx} - 1| &= |e^{i\frac{h}{2}x}||e^{i\frac{h}{2}x} - e^{-i\frac{h}{2}x}| = 2|\sin\frac{hx}{2}| \\ &\le |hx| \le A|h|, \quad \text{when } |x| \le A. \end{aligned}$$

We have

$$\begin{aligned} |f(t+h) - f(t)| &\le 2\int_{|x|\ge A} dF(x) + A|h| \int_{|x|<A} dF(x) \\ &\le 2\int_{|x|\ge A} dF(x) + |h|A. \end{aligned}$$

Note that $|e^{ihx} - 1| \le 2$ and

$$\begin{aligned} |e^{ihx} - 1| &= |e^{i\frac{h}{2}x}||e^{i\frac{h}{2}x} - e^{-i\frac{h}{2}x}| = 2|\sin\frac{hx}{2}| \\ &\le |hx| \le A|h|, \quad \text{when } |x| \le A. \end{aligned}$$

We have

$$\begin{aligned} |f(t+h) - f(t)| &\le 2\int_{|x|\ge A} dF(x) + A|h| \int_{|x|<A} dF(x) \\ &\le 2\int_{|x|\ge A} dF(x) + |h|A. \end{aligned}$$

Choose A such that $\int_{|x|\ge A} dF(x) < \epsilon/4$.

Note that $|e^{ihx} - 1| \le 2$ and

$$\begin{aligned} |e^{ihx} - 1| &= |e^{i\frac{h}{2}x}||e^{i\frac{h}{2}x} - e^{-i\frac{h}{2}x}| = 2|\sin\frac{hx}{2}| \\ &\le |hx| \le A|h|, \quad \text{when } |x| \le A. \end{aligned}$$

We have

$$\begin{aligned} |f(t+h) - f(t)| &\le 2\int_{|x|\ge A} dF(x) + A|h| \int_{|x|<A} dF(x) \\ &\le 2\int_{|x|\ge A} dF(x) + |h|A. \end{aligned}$$

Choose A such that $\int_{|x|\ge A} dF(x) < \epsilon/4$. And then take $\delta = \varepsilon/(2A)$. Consequently, $|f(t+h) - f(t)| < \varepsilon$ for all t whenever $|h| < \delta$.

3. $f(t)$ is non-negative definite, i.e., for an arbitrary integer n, any real numbers $t_1, \cdots, t_n$ and complex numbers $\lambda_1, \cdots, \lambda_n$, it follows

$$\sum_{k=1}^{n} \sum_{j=1}^{n} f(t_k - t_j)\lambda_k \overline{\lambda_j} \geq 0.$$

Proof.

Proof.

$$
\begin{aligned}
& \sum_{k=1}^{n}\sum_{j=1}^{n} f(t_k - t_j)\lambda_k\overline{\lambda_j} \\
= \; & \sum_{k=1}^{n}\sum_{j=1}^{n} \int_{-\infty}^{\infty} e^{i(t_k-t_j)x} dF(x)\lambda_k\overline{\lambda_j} \\
= \; & \int_{-\infty}^{\infty} \left(\sum_{k=1}^{n} e^{it_k x}\lambda_k\right)\left(\sum_{j=1}^{n} e^{-it_j x}\overline{\lambda_j}\right) dF(x)
\end{aligned}
$$

Proof.

$$
\begin{aligned}
&\sum_{k=1}^{n}\sum_{j=1}^{n} f(t_k - t_j)\lambda_k\overline{\lambda_j} \\
= \;& \sum_{k=1}^{n}\sum_{j=1}^{n} \int_{-\infty}^{\infty} e^{i(t_k - t_j)x} dF(x)\lambda_k\overline{\lambda_j} \\
= \;& \int_{-\infty}^{\infty} \left(\sum_{k=1}^{n} e^{it_k x}\lambda_k\right)\left(\sum_{j=1}^{n} e^{-it_j x}\overline{\lambda_j}\right) dF(x) \\
= \;& \int_{-\infty}^{\infty} (\sum_{k=1}^{n} e^{it_k x}\lambda_k)\overline{(\sum_{j=1}^{n} e^{it_j x}\lambda_j)} dF(x)
\end{aligned}
$$

Proof.

$$
\begin{aligned}
& \sum_{k=1}^{n}\sum_{j=1}^{n} f(t_k - t_j)\lambda_k\overline{\lambda_j} \\
= & \sum_{k=1}^{n}\sum_{j=1}^{n}\int_{-\infty}^{\infty} e^{i(t_k-t_j)x}dF(x)\lambda_k\overline{\lambda_j} \\
= & \int_{-\infty}^{\infty}\left(\sum_{k=1}^{n} e^{it_kx}\lambda_k\right)\left(\sum_{j=1}^{n} e^{-it_jx}\overline{\lambda_j}\right)dF(x) \\
= & \int_{-\infty}^{\infty}(\sum_{k=1}^{n} e^{it_kx}\lambda_k)\overline{(\sum_{j=1}^{n} e^{it_jx}\lambda_j)}dF(x) \\
= & \int_{-\infty}^{\infty}|\sum_{k=1}^{n} e^{it_kx}\lambda_k|^2 dF(x) \geq 0.
\end{aligned}
$$

Bochner-Khinchine Theorem.

The function $f(t)$ is a characteristic function if and only if $f(t)$ is non-negative definite, continuous and $f(0) = 1$.

4. Assume that $\xi_1, \cdots, \xi_n$ are indept., then

$$f_{\xi_1+\xi_2+\cdots+\xi_n}(t) = f_{\xi_1}(t) f_{\xi_2}(t) \cdots f_{\xi_n}(t).$$

(Proof?)

4. Assume that $\xi_1, \cdots, \xi_n$ are indept., then

$$f_{\xi_1+\xi_2+\cdots+\xi_n}(t) = f_{\xi_1}(t) f_{\xi_2}(t) \cdots f_{\xi_n}(t).$$

(Proof?)

5. If $E\xi^n$ exists, then $f(t)$ is differentiable of n orders, and when $k \le n$

$$f^{(k)}(0) = i^k E\xi^k.$$

4. Assume that $\xi_1, \cdots, \xi_n$ are indept., then

$$f_{\xi_1+\xi_2+\cdots+\xi_n}(t) = f_{\xi_1}(t) f_{\xi_2}(t) \cdots f_{\xi_n}(t).$$

(Proof?)

5. If $E\xi^n$ exists, then $f(t)$ is differentiable of n orders, and when $k \le n$

$$f^{(k)}(0) = i^k E\xi^k.$$

4. Assume that $\xi_1, \cdots, \xi_n$ are indept., then

$$f_{\xi_1+\xi_2+\cdots+\xi_n}(t) = f_{\xi_1}(t) f_{\xi_2}(t) \cdots f_{\xi_n}(t).$$

(Proof?)

5. If $E\xi^n$ exists, then $f(t)$ is differentiable of n orders, and when $k \le n$

$$f^{(k)}(0) = i^k E\xi^k.$$

In particular, when $E\xi^2$ exists, $E\xi = -if'(0)$, $E\xi^2 = -f''(0)$, $Var\xi = -f''(0) + [f'(0)]^2$.

Proof. Since

$$|\frac{d^k}{dt^k}e^{itx}| = |i^k x^k e^{itx}| = |x|^k,$$

Proof. Since

$$\left|\frac{d^k}{dt^k}e^{itx}\right| = |i^k x^k e^{itx}| = |x|^k,$$

$$\int_{-\infty}^{\infty} |x^k| dF(x) = E|\xi|^k < \infty,$$

Proof. Since

$$|\frac{d^k}{dt^k}e^{itx}| = |i^k x^k e^{itx}| = |x|^k,$$

$$\int_{-\infty}^{\infty} |x^k| dF(x) = E|\xi|^k < \infty,$$

then $\int_{-\infty}^{\infty} \frac{d^k}{dt^k} e^{itx} dF(x)$ uniformly converges in t.

Proof. Since

$$|\frac{d^k}{dt^k}e^{itx}| = |i^k x^k e^{itx}| = |x|^k,$$
$$\int_{-\infty}^{\infty} |x^k| dF(x) = E|\xi|^k < \infty,$$

then $\int_{-\infty}^{\infty} \frac{d^k}{dt^k} e^{itx} dF(x)$ uniformly converges in t. So, $f^{(k)}(t)$ exists and

Proof. Since

$$|\frac{d^k}{dt^k}e^{itx}| = |i^k x^k e^{itx}| = |x|^k,$$

$$\int_{-\infty}^{\infty} |x^k| dF(x) = E|\xi|^k < \infty,$$

then $\int_{-\infty}^{\infty} \frac{d^k}{dt^k} e^{itx} dF(x)$ uniformly converges in t. So, $f^{(k)}(t)$ exists and

$$f^{(k)}(t) = \int_{-\infty}^{\infty} \frac{d^k}{dt^k} e^{itx} dF(x) = i^k \int_{-\infty}^{\infty} x^k e^{itx} dF(x),$$

Proof. Since

$$|\frac{d^k}{dt^k}e^{itx}| = |i^k x^k e^{itx}| = |x|^k,$$

$$\int_{-\infty}^{\infty} |x^k| dF(x) = E|\xi|^k < \infty,$$

then $\int_{-\infty}^{\infty} \frac{d^k}{dt^k} e^{itx} dF(x)$ uniformly converges in t. So, $f^{(k)}(t)$ exists and

$$f^{(k)}(t) = \int_{-\infty}^{\infty} \frac{d^k}{dt^k} e^{itx} dF(x) = i^k \int_{-\infty}^{\infty} x^k e^{itx} dF(x),$$

$$f^{(k)}(0) = i^k \int_{-\infty}^{\infty} x^k dF(x) = i^k E\xi^k.$$

反过来, 若n为偶数, 且$f^{(n)}(0)$存在, 则$E\xi^n$存在.

反过来, 若n为偶数, 且$f^{(n)}(0)$存在, 则$E\xi^n$存在.

Proof. 我们用数学归纳法来证明. 当 $n=2$ 时,

$$
\begin{aligned}
f''(0) &= \lim_{h\to 0}\frac{f(h)-2f(0)+f(-h)}{h^2}\\
&= \lim_{h\to 0}\int_{-\infty}^{\infty}\frac{e^{\mathrm{i}hx}-2+e^{-\mathrm{i}hx}}{h^2}dF(x)\\
&= -\lim_{h\to 0}\int_{-\infty}^{\infty}2\frac{1-\cos hx}{h^2}dF(x).
\end{aligned}
$$

反过来, 若n为偶数, 且$f^{(n)}(0)$存在, 则$E\xi^n$存在.

Proof. 我们用数学归纳法来证明. 当 $n = 2$ 时,

$$
\begin{aligned}
f''(0) &= \lim_{h\to 0} \frac{f(h) - 2f(0) + f(-h)}{h^2} \\
&= \lim_{h\to 0} \int_{-\infty}^{\infty} \frac{e^{\mathrm{i}hx} - 2 + e^{-\mathrm{i}hx}}{h^2} dF(x) \\
&= -\lim_{h\to 0} \int_{-\infty}^{\infty} 2\frac{1-\cos hx}{h^2} dF(x).
\end{aligned}
$$

注意到, $0 \le 2(1-\cos hx)/h^2 \le x^2$, 并

且 $\lim_{h\to 0} 2(1-\cos hx)/h^2 = x^2$ 关于 x 在任一有限区间内一致成立.

因此对任意 $a > 0$ 有

$$
\begin{aligned}
-f''(0) &\geq \lim_{h\to 0} \int_{-a}^{a} 2\frac{1-\cos hx}{h^2} dF(x) \\
&= \int_{-a}^{a} \lim_{h\to 0} 2\frac{1-\cos hx}{h^2} dF(x) = \int_{-a}^{a} x^2 dF(x).
\end{aligned}
$$

令 $a \to \infty$ 得 $\int_{-\infty}^{\infty} x^2 dF(x) \leq -f''(0)$, 即 $E\xi^2$ 存在.

现设 $f^{(2k)}(0)$ 存在, 同时归纳假设 $E\xi^{2k-2}$ 也存在. 由第一部分结论, $f(t)$ 是 $2k-2$ 次可微的, 且

$$
\begin{aligned}
f^{(2k-2)}(t) =& i^{2k-2}\int_{-\infty}^{\infty} e^{\mathrm{i}tx}x^{2k-2}dF(x) \\
=&(-1)^{k-1}\int_{-\infty}^{\infty} e^{\mathrm{i}tx}x^{2k-2}dF(x).
\end{aligned}
$$

记 $G(y)=\int_{-\infty}^{y} x^{2k-2}dF(x)$, 其中 $G(\infty)=E\xi^{2k-2}$, 则 $G(y)/G(\infty)$ 为分布函数,

$G(y)/G(\infty)$的特征函数为

$$\begin{aligned} g(t) =& \frac{1}{G(\infty)}\int_{-\infty}^{\infty} e^{ity}dG(y) \\ =& \frac{1}{G(\infty)}\int_{-\infty}^{\infty} e^{ity}y^{2k-2}dF(y) = \frac{(-1)^{k-1}}{G(\infty)}f^{(2k-2)}(t). \end{aligned}$$

从而 $g''(0) = (-1)^{k-1}f^{(2k)}(0)/G(\infty)$ 存在. 由已证的 $n = 2$ 时的结论知

$$\frac{1}{G(\infty)}\int_{-\infty}^{\infty} x^{2k}dF(x) = \frac{1}{G(\infty)}\int_{-\infty}^{\infty} y^2dG(y)$$

存在. 即 $E\xi^{2k}$ 存在, 结论得证.

6. Let $\eta = a\xi + b$, where a, b are arbitrary constants. Then

$$f_\eta(t) = e^{ibt} f(at).$$

6. Let $\eta = a\xi + b$, where a, b are arbitrary constants. Then

$$f_\eta(t) = e^{ibt} f(at).$$

Proof.

$$Ee^{i(a\xi+b)t} = Ee^{iat\xi} \cdot e^{ibt} = e^{ibt} f(at).$$

Example 7. Are the following functions characteristic functions of some random variables?

(1) $f(t) = \sin t$;

(2) $f(t) = \ln(e + |t|)$;

(3) $f(t) = 0$ when $t < 0$; $f(t) = 1$ when $t \geq 0$.

Solution……

3.3.3 Inverse formula and uniqueness theorem

Theorem 1 (Inverse formula) Suppose that $f(t)$ is a c.f. corresponding to cdf $F(x)$. Let x_1, x_2 be two continuity points of $F(x)$, then

$$F(x_2) - F(x_1) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-itx_1} - e^{-itx_2}}{it} f(t)dt.$$

Proof. Suppose $\xi \sim F(x)$. Without loss of generality, we assume that $x_1 < x_2$.

$$\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-itx_1}-e^{-itx_2}}{it}f(t)dt$$

Proof. Suppose $\xi \sim F(x)$. Without loss of generality, we assume that $x_1 < x_2$.

$$
\begin{aligned}
& \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-itx_1} - e^{-itx_2}}{it} f(t)dt \\
= & \frac{1}{2\pi}\int_{-T}^{T} \int_{-\infty}^{\infty} \frac{e^{-itx_1} - e^{-itx_2}}{it} e^{itx} dF(x)dt
\end{aligned}
$$

Proof. Suppose $\xi \sim F(x)$. Without loss of generality, we assume that $x_1 < x_2$.

$$
\begin{aligned}
& \frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-itx_1}-e^{-itx_2}}{it}f(t)dt \\
= & \frac{1}{2\pi}\int_{-T}^{T}\int_{-\infty}^{\infty}\frac{e^{-itx_1}-e^{-itx_2}}{it}e^{itx}dF(x)dt \\
= & \frac{1}{2\pi}\int_{-\infty}^{\infty}\{\int_{0}^{T}[\frac{e^{it(x-x_1)}-e^{-it(x-x_1)}}{it} \\
& \qquad -\frac{e^{it(x-x_2)}-e^{-it(x-x_2)}}{it}]dt\}dF(x)
\end{aligned}
$$

Proof. Suppose $\xi \sim F(x)$. Without loss of generality, we assume that $x_1 < x_2$.

$$\begin{aligned}
& \frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-itx_1}-e^{-itx_2}}{it}f(t)dt \\
= & \frac{1}{2\pi}\int_{-T}^{T}\int_{-\infty}^{\infty}\frac{e^{-itx_1}-e^{-itx_2}}{it}e^{itx}dF(x)dt \\
= & \frac{1}{2\pi}\int_{-\infty}^{\infty}\{\int_{0}^{T}[\frac{e^{it(x-x_1)}-e^{-it(x-x_1)}}{it} \\
& \qquad -\frac{e^{it(x-x_2)}-e^{-it(x-x_2)}}{it}]dt\}dF(x) \\
= & \frac{1}{\pi}\int_{-\infty}^{\infty}\{\int_{0}^{T}[\frac{\sin t(x-x_1)}{t}-\frac{\sin t(x-x_2)}{t}]dt\}dF(x)
\end{aligned}$$

Proof. Suppose $\xi \sim F(x)$. Without loss of generality, we assume that $x_1 < x_2$.

$$
\begin{aligned}
& \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-itx_1} - e^{-itx_2}}{it} f(t)dt \\
= \ & \frac{1}{2\pi}\int_{-T}^{T}\int_{-\infty}^{\infty} \frac{e^{-itx_1} - e^{-itx_2}}{it} e^{itx} dF(x)dt \\
= \ & \frac{1}{2\pi}\int_{-\infty}^{\infty}\{\int_{0}^{T}[\frac{e^{it(x-x_1)} - e^{-it(x-x_1)}}{it} \\
& \qquad - \frac{e^{it(x-x_2)} - e^{-it(x-x_2)}}{it}]dt\}dF(x) \\
= \ & \frac{1}{\pi}\int_{-\infty}^{\infty}\{\int_{0}^{T}[\frac{\sin t(x-x_1)}{t} - \frac{\sin t(x-x_2)}{t}]dt\}dF(x) \\
\triangleq \ & \int_{-\infty}^{\infty} g(T, x, x_1, x_2)dF(x) = Eg(T, \xi, x_1, x_2).
\end{aligned}
$$

Notice

$$\int_0^T \frac{\sin at}{t} dt = \int_0^{Ta} \frac{\sin t}{t} dt$$

$$\to sgn(a) \int_0^\infty \frac{\sin t}{t} dt = \begin{cases} \frac{\pi}{2}, & \text{if } a > 0, \\ 0, & \text{if } a = 0, \\ -\frac{\pi}{2}, & \text{if } a < 0. \end{cases}$$

and

$$\int_0^x \frac{\sin t}{t} dt \text{ is a bounded function .}$$

It follows that

$$
\begin{aligned}
& \lim_{T\to\infty} g(T,x,x_1,x_2) \\
= & \lim_{T\to\infty} \frac{1}{\pi}\{\int_0^T [\frac{\sin t(x-x_1)}{t} - \frac{\sin t(x-x_2)}{t}]dt\} \\
= & \begin{cases} 0, & x<x_1 \text{ or } x>x_2, \\ \frac{1}{2}, & x=x_1 \text{ or } x=x_2, \\ 1, & x_1<x<x_2. \end{cases}
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \lim_{T\to\infty} g(T, x, x_1, x_2) \\
= & \lim_{T\to\infty} \frac{1}{\pi}\{\int_0^T [\frac{\sin t(x-x_1)}{t} - \frac{\sin t(x-x_2)}{t}]dt\} \\
= & \begin{cases} 0, & x < x_1 \text{ or } x > x_2, \\ \frac{1}{2}, & x = x_1 \text{ or } x = x_2, \\ 1, & x_1 < x < x_2. \end{cases} \quad \triangleq g(x)
\end{aligned}
$$

and

$$|g(T, x, x_1, x_2)| < M.$$

It follows that

$$
\begin{aligned}
& \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-itx_1} - e^{-itx_2}}{it} f(t)dt \\
= & \lim_{T\to\infty} Eg(T, \xi, x_1, x_2)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-itx_1}-e^{-itx_2}}{it}f(t)dt \\
= & \lim_{T\to\infty} Eg(T,\xi,x_1,x_2) = Eg(\xi)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-itx_1} - e^{-itx_2}}{it} f(t)dt \\
= & \lim_{T\to\infty} Eg(T, \xi, x_1, x_2) = Eg(\xi) \\
= & P(x_1 < \xi < x_2) + \frac{1}{2}\Big(P(\xi = x_1) + P(\xi = x_2)\Big)
\end{aligned}
$$

It follows that

$$\begin{aligned}
& \lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-itx_1}-e^{-itx_2}}{it}f(t)dt \\
= & \lim_{T\to\infty}Eg(T,\xi,x_1,x_2)=Eg(\xi) \\
= & P(x_1<\xi<x_2)+\frac{1}{2}\big(P(\xi=x_1)+P(\xi=x_2)\big) \\
= & F(x_2)-F(x_1).
\end{aligned}$$

Theorem 2. (Uniqueness) A distribution function can be uniquely determined by its characteristic function.

Theorem 2. (Uniqueness) A distribution function can be uniquely determined by its characteristic function.

Proof.

Theorem 2. (Uniqueness) A distribution function can be uniquely determined by its characteristic function.

Proof. By inverse formula, if $y < x$ are continuous points of $F(x)$, then

$$F(x) - F(y) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ity} - e^{-itx}}{it} f(t)dt.$$

Theorem 2. (Uniqueness) A distribution function can be uniquely determined by its characteristic function.

Proof. By inverse formula, if $y < x$ are continuous points of $F(x)$, then

$$F(x) - F(y) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ity} - e^{-itx}}{it} f(t)dt.$$

Letting $y \to -\infty$ along continuity points of $F(x)$, we have

$$F(x) = \lim_{y\to-\infty} \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ity} - e^{-itx}}{it} f(t)dt.$$

Thus it is easy to see that $f(t)$ determines the value of $F(x)$ at its continuity points. As for the discontinuous points, in view of right continuity of $F(x)$, it suffices to take right limits along continuity points. The theorem is proved.

Theorem 3. (Inverse Fourier transform) Suppose that $f(t)$ is a c.f. and $\int_{-\infty}^{\infty}|f(t)|dt < \infty$, then $F'(x)$ exists and is continuous. Moreover

$$F'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} f(t)dt.$$

Proof. Since $f(t)$ is absolutely integrable and

$$\left|\frac{e^{-itx}-e^{-ity}}{it}\right| \le |y-x|,$$

it follows that

$$F(y)-F(x)=\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-itx}-e^{-ity}}{it}f(t)dt$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-itx}-e^{-ity}}{it}f(t)dt \triangleq H(x,y),$$

whenever x, y are continuous points of $F(x)$.

Also, $H(x, y)$ is a continuous function of (x, y). So, $F(x)$ must be a continuous function and the above equality holds for all x, y.

Also, $H(x, y)$ is a continuous function of (x, y). So, $F(x)$ must be a continuous function and the above equality holds for all x, y.

Now, applying the dominated convergence theorem yields

$$F'(x) = \lim_{y \to x} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-itx} - e^{-ity}}{it(y - x)} f(t) dt$$

Also, $H(x,y)$ is a continuous function of (x,y). So, $F(x)$ must be a continuous function and the above equality holds for all x,y.

Now, applying the dominated convergence theorem yields

$$\begin{aligned}
F'(x) &= \lim_{y\to x}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-itx}-e^{-ity}}{it(y-x)}f(t)dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{y\to x}\frac{e^{-itx}-e^{-ity}}{it(y-x)}f(t)dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}f(t)dt.
\end{aligned}$$

For same reason,

$$\lim_{y\to x} F'(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{y\to x} e^{-ity} f(t)dt = F'(x).$$

So, $F'(x)$ is continuous.

Discrete random variables: Assume $P(\xi = k) = p_k, k = 0, 1, 2, \cdots,$ then

$$f(t) = \sum_{k=0}^{\infty} p_k e^{itk}.$$

Discrete random variables: Assume $P(\xi = k) = p_k, k = 0, 1, 2, \cdots$, then

$$f(t) = \sum_{k=0}^{\infty} p_k e^{itk}.$$

If $f(t)$ is given, then we can multiply both sides by e^{-itk} and integrate. Noting that

$$\int_0^{2\pi} e^{int} dt = \begin{cases} 2\pi, & n = 0, \\ 0, & n \neq 0, \end{cases}$$

we have

$$p_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-itk} f(t) dt.$$

Example 8. Show $f(t) = \cos t$ is a characteristic function of some random variable, and find its distribution function.

Solution……

Example 8. Show $f(t) = \cos t$ is a characteristic function of some random variable, and find its distribution function.

Solution......

In general, if $f(t)$ can be written as $\sum a_n e^{ix_n t}$, where $a_n > 0$ and $\sum a_n = 1$, then $f(t)$ is a characteristic function, whose corresponding random variable has distribution sequence $P(\xi = x_n) = a_n$, $n = 1, 2, \cdots$.

Example 9. If $f(t)$ is a characteristic function of some random variable, show so are $\overline{f(t)}$ and $|f(t)|^2$.

Solution.......

3.3.4 Additivity of distribution functions

The additivity, also called regenerativity, means that if ξ and η are independent and follow a common type of distributions, then so do their sum $\xi + \eta$ and the parameter is the sum of parameters of ξ and η.

Example 10. If ξ_j $(j = 1, 2, \cdots, k)$ follows the binomial distribution $B(n_j, p)$ respectively and are indept. of each other. Find the distribution of $\sum_{j=1}^{k} \xi_j$.

Solution.

Example 10. If ξ_j $(j = 1, 2, \cdots, k)$ follows the binomial distribution $B(n_j, p)$ respectively and are indept. of each other. Find the distribution of $\sum_{j=1}^{k} \xi_j$.

Solution. The c.f. $f_j(t)$ of ξ_j is $(pe^{it} + q)^{n_j}$. By Property 4, the c.f. of $\sum_{j=1}^{k} \xi_j$ is

Example 10. If ξ_j $(j = 1, 2, \cdots, k)$ follows the binomial distribution $B(n_j, p)$ respectively and are indept. of each other. Find the distribution of $\sum_{j=1}^{k} \xi_j$.

Solution. The c.f. $f_j(t)$ of ξ_j is $(pe^{it} + q)^{n_j}$. By Property 4, the c.f. of $\sum_{j=1}^{k} \xi_j$ is

$$\prod_{j=1}^{k} f_j(t) = (pe^{it} + q)^{\sum_{j=1}^{k} n_j}.$$

Example 10. If ξ_j $(j = 1, 2, \cdots, k)$ follows the binomial distribution $B(n_j, p)$ respectively and are indept. of each other. Find the distribution of $\sum_{j=1}^k \xi_j$.

Solution. The c.f. $f_j(t)$ of ξ_j is $(pe^{it} + q)^{n_j}$. By Property 4, the c.f. of $\sum_{j=1}^k \xi_j$ is

$$\prod_{j=1}^k f_j(t) = (pe^{it} + q)^{\sum_{j=1}^k n_j}.$$

In turn, by the uniqueness theorem, $\sum_{j=1}^k \xi_j \sim B(\sum_{j=1}^k n_j, p)$.

Example 11. Suppose that $\xi_1, \cdots, \xi_n$ are indept., and $\xi_k \sim N(a_k, \sigma_k^2)$, $k = 1, \cdots, n$. Find the distribution of $\sum_{k=1}^n \xi_k$.

Solution.

Example 11. Suppose that $\xi_1, \cdots, \xi_n$ are indept., and $\xi_k \sim N(a_k, \sigma_k^2)$, $k = 1, \cdots, n$. Find the distribution of $\sum_{k=1}^n \xi_k$.

Solution. It is known that the c.f. of ξ_k is $e^{ia_k t - \sigma_k^2 t^2/2}$, so the c.f. of $\sum_{k=1}^n \xi_k$ is

$$\prod_{k=1}^n e^{ia_k t - \frac{\sigma_k^2 t^2}{2}} = \exp\{i \sum_{k=1}^n a_k t - \frac{\sum_{k=1}^n \sigma_k^2 t^2}{2}\}.$$

Example 11. Suppose that $\xi_1, \cdots, \xi_n$ are indept., and $\xi_k \sim N(a_k, \sigma_k^2)$, $k = 1, \cdots, n$. Find the distribution of $\sum_{k=1}^n \xi_k$.

Solution. It is known that the c.f. of ξ_k is $e^{ia_kt-\sigma_k^2t^2/2}$, so the c.f. of $\sum_{k=1}^n \xi_k$ is

$$\prod_{k=1}^n e^{ia_kt-\frac{\sigma_k^2t^2}{2}} = \exp\{i\sum_{k=1}^n a_kt - \frac{\sum_{k=1}^n \sigma_k^2t^2}{2}\}.$$

Thus $\sum_{k=1}^n \xi_k \sim N\left(\sum_k a_k, \sum_k \sigma_k^2\right)$.

3.3.5 Multivariate characteristic functions

Definition 3 Suppose the random vector $\xi = (\xi_1, \cdots, \xi_n)'$ has distribution function $F(x_1, \cdots, x_n)$, then its characteristic function is defined by

$$f(t_1, \cdots, t_n) = Ee^{i(t_1\xi_1 + \cdots + t_n\xi_n)}$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{i(t_1x_1 + \cdots + t_nx_n)} dF(x_1, \cdots, x_n).$$

3.3.5 Multivariate characteristic functions

Definition 3 Suppose the random vector $\xi = (\xi_1, \cdots, \xi_n)'$ has distribution function $F(x_1, \cdots, x_n)$, then its characteristic function is defined by

$$f(t_1, \cdots, t_n) = Ee^{i(t_1\xi_1 + \cdots + t_n\xi_n)}$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{i(t_1x_1 + \cdots + t_nx_n)} dF(x_1, \cdots, x_n).$$

$$f(\boldsymbol{t}) = Ee^{i\boldsymbol{t}'\boldsymbol{\xi}} = \int_{\boldsymbol{R}^n} e^{i\boldsymbol{t}'\boldsymbol{x}} dF(\boldsymbol{x}),$$

where $\boldsymbol{t} = (t_1, \cdots, t_n)'$, $\boldsymbol{x} = (x_1, \cdots, x_n)'$.

1. The c.f. of $\eta = a_1\xi_1 + \cdots + a_n\xi_n$ is

$$\begin{aligned} f_\eta(t) &= Ee^{it\eta} = Ee^{it\sum a_k\xi_k} \\ &= Ee^{i\sum(a_kt)\xi_k} = f(a_1t, \cdots, a_nt). \end{aligned}$$

2. If the c.f. of $(\xi_1, \cdots, \xi_n)'$ is $f(t_1, \cdots, t_n)$, then k-dimensional sub-vector $(\xi_{l_1}, \cdots, \xi_{l_k})'$ has c.f.

$$f(0, \cdots, 0, t_{l_1}, 0, \cdots, 0, t_{l_k}, 0, \cdots, 0).$$

3. Assume that ξ_j has c.f. $f_j(t)$, $j = 1, \cdots, n$, then $\xi_1, \cdots, \xi_n$ are indept. iff the c.f. of $(\xi_1, \cdots, \xi_n)'$ is such that

$$f(t_1, \cdots, t_n) = f_1(t_1) \cdots f_n(t_n).$$

4. $(\xi_1, \cdots, \xi_k)'$ and $(\xi_{k+1}, \cdots, \xi_n)'$ are indept. iff the product of their c.f.s is just equal to the c.f. of $(\xi_1, \cdots, \xi_n)'$.