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(Mathematical) expectation: mean of the values of a random variable;

Variance: to express the extent to which a random variable diverts from the mean.

Characteristic function: a powerful tool to analyze random variables.

3.1.1 Expectations for discrete random variables

Example 1. In order to evaluate A's shooting level, randomly observe his ten shootings and record the number of cycles he hits each time and the frequency as below.

x_k	8	9	10
v_k	2	5	3
$f_k = v_k/N$	0.2	0.5	0.3

3.1.1 Expectations for discrete random variables

Example 1. In order to evaluate A's shooting level, randomly observe his ten shootings and record the number of cycles he hits each time and the frequency as below.

x_k	8	9	10
v_k	2	5	3
$f_k = v_k/N$	0.2	0.5	0.3

The average number of cycles is

$$\sum x_k f_k = 8 * 0.2 + 9 * 0.5 + 10 * 0.3 = 9.1.$$

一般地

$$\overline{x} = \left(\sum x_k v_k\right)/N = \sum x_k f_k.$$

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$$\overline{x} = (\sum x_k v_k)/N = \sum x_k f_k.$$

当N越来越大时，频率f_k会稳定到概率p_k，从而平均值$\overline{x}$会稳定到

$$\sum x_k p_k.$$

Definition 1 Suppose that a discrete random variable ξ has the distribution sequence

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_k & \cdots \\ p_1 & p_2 & \cdots & p_k & \cdots \end{pmatrix}.$$

If the series $\sum_k x_k p_k$ converges absolutely, that is, $\sum_k |x_k| p_k < \infty$, the sum is called mathematical expectation or mean of ξ, written as

$$E\xi = \sum_k x_k p_k.$$

Example 2. The degenerate distribution $P(\xi = a) = 1$ has mathematical expectation $E\xi = a$. In other words, the expectation of a constant is just itself.

Example 3. Calculate the mathematical expectation of the binomial distribution

$$P(\xi = k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, 2, \cdots, n.$$

Solution.

$$E\xi = \sum_{k=0}^{n} k p_k$$

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$$= \sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} p^k q^{n-k}$$

Solution.

$$
\begin{aligned}
E\xi &= \sum_{k=0}^{n} k p_k \\
&= \sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} p^k q^{n-k} \\
&= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)![(n-1)-(k-1)]!} p^{k-1} q^{n-1-(k-1)}
\end{aligned}
$$

Solution.

$$
\begin{aligned}
E\xi &= \sum_{k=0}^{n} k p_k \\
&= \sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} p^k q^{n-k} \\
&= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)![(n-1)-(k-1)]!} p^{k-1} q^{n-1-(k-1)} \\
&= np \sum_{r=0}^{n-1} \frac{(n-1)!}{r![(n-1)-r]!} p^r q^{n-1-r}
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$$
\begin{aligned}
E\xi &= \sum_{k=0}^{n} k p_k \\
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&= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)![(n-1)-(k-1)]!} p^{k-1} q^{n-1-(k-1)} \\
&= np \sum_{r=0}^{n-1} \frac{(n-1)!}{r![(n-1)-r]!} p^r q^{n-1-r} \\
&= np(p+q)^{n-1} = np.
\end{aligned}
$$

Example 4. Calculate the mathematical expectation of the Poisson distribution

$$P(\xi = k) = \frac{\lambda^k}{k!}e^{-\lambda}, k = 0, 1, 2, \cdots .$$

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$$\begin{aligned} E\xi &= \sum_{k=0}^{\infty} kP(\xi = k) = \sum_{k=0}^{\infty} k\frac{\lambda^k}{k!}e^{-\lambda} \\ &= \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}e^{-\lambda} \end{aligned}$$

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Example 5. Calculate the mathematical expectation of the geometric distribution

$$P(\xi = k) = pq^{k-1}, k = 1, 2, \cdots, 0 < p < 1.$$

Solution.

$$E\xi = \sum_{k=1}^{\infty} kpq^{k-1}$$

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$$= p(\frac{x}{1-x})'|_{x=q}$$

Solution.

$$
\begin{aligned}
E\xi &= \sum_{k=1}^{\infty} kpq^{k-1} \\
&= p\sum_{k=1}^{\infty}(x^k)'|_{x=q} = p(\sum_{k=1}^{\infty} x^k)'|_{x=q}) \\
&= p(\frac{x}{1-x})'|_{x=q} \\
&= p\frac{1}{(1-x)^2}|_{x=q} = \frac{1}{p}.
\end{aligned}
$$

Example 6. Suppose that

$$P(\xi = (-1)^k \frac{2^k}{k}) = \frac{1}{2^k}, \quad k = 1, 2, \cdots .$$

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$$\sum_{k=1}^{\infty} x_k p_k = \sum_{k=1}^{\infty} (-1)^k \frac{1}{k} = ?.$$

Note that

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We say that $E\xi$ does not exist, although $\sum_{k=1}^{\infty} x_k p_k$ is convergent.

Basic properties of expectations of discrete random variables

Property 1 (*Absolute integrability*): $E\xi$ is finite if and only if $E|\xi| < \infty$. Further

$$E\xi = E\xi^+ - E\xi^-, \quad E|\xi| = E\xi^+ + E\xi^-.$$

Property 2 (*Linearity*) (1): $E(a\xi) = aE\xi$.

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In fact, if the pmf of ξ is $P(\xi = x_i) = p_i$, then the pmf of $a\xi$ is $P(a\xi = ax_i) = p_i$. So

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$$E(a\xi) = \sum_i (ax_i)p_i = a\sum_i x_i p_i = aE(\xi).$$

Property 2(2): $E(\xi+\eta)=E\xi+E\eta$.

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$$\begin{aligned} E(\zeta) &= \sum_l z_l P(\zeta = z_l) = \sum_l z_l P(\xi + \eta = z_l) \\ &= \sum_l z_l \sum_{i,j: x_i + y_j = z_l} P(\xi = x_i, \eta = y_j) \end{aligned}$$

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\begin{aligned}
E(\zeta) &= \sum_l z_l P(\zeta=z_l)=\sum_l z_l P(\xi+\eta=z_l)\\
&= \sum_l z_l \sum_{i,j:x_i+y_j=z_l} P(\xi=x_i,\eta=y_j)\\
&= \sum_l \sum_{i,j:x_i+y_j=z_l} (x_i+y_j)P(\xi=x_i,\eta=y_j)
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&= \sum_l \sum_{i,j:x_i+y_j=z_l} (x_i+y_j)P(\xi=x_i,\eta=y_j)\\
&= \sum_{i,j}(x_i+y_j)P(\xi=x_i,\eta=y_j)\\
&= \sum_{i,j}x_iP(\xi=x_i,\eta=y_j)+\sum_{i,j}y_jP(\xi=x_i,\eta=y_j)
\end{aligned}$$

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&= \sum_l z_l \sum_{i,j:x_i+y_j=z_l} P(\xi=x_i,\eta=y_j)\\
&= \sum_l \sum_{i,j:x_i+y_j=z_l} (x_i+y_j)P(\xi=x_i,\eta=y_j)\\
&= \sum_{i,j}(x_i+y_j)P(\xi=x_i,\eta=y_j)\\
&= \sum_{i,j} x_i P(\xi=x_i,\eta=y_j)+\sum_{i,j} y_j P(\xi=x_i,\eta=y_j)\\
&= \sum_i x_i P(\xi=x_i)+\sum_j y_j P(\eta=y_j)=E\xi+E\eta.
\end{aligned}$$

Property 3 (*Monotonicity*): If $\xi \leq \eta$ and the expectations of ξ and η exist, then $E\xi \leq E\eta$.

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Proof. $\eta - \xi$ is also a discrete random variable. By Property 2. $E[\eta - \xi] = E\eta - E\xi$ exists.

Property 3 (*Monotonicity*): If $\xi \leq \eta$ and the expectations of ξ and η exist, then $E\xi \leq E\eta$.

Proof. $\eta - \xi$ is also a discrete random variable. By Property 2. $E[\eta - \xi] = E\eta - E\xi$ exists. On the other hand, $\eta - \xi \geq 0$. It follows that $E[\eta - \xi] \geq 0$ by the definition of the expectation. So $E\eta \geq E\xi$.

Property 4 : If the expectations of ξ and η exist, and ξ and η are independent, then the expectation of $\xi\eta$ exists and

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Proof. Write $\xi = \sum_{i=1}^{\infty} x_i I\{\xi = x_i\}$ and $\eta = \sum_{j=1}^{\infty} y_j I\{\eta = y_j\}$. Let $\zeta = \xi\eta$. Then the distribution sequence of ζ is

$$\begin{aligned} P(\zeta = z_k) &= \sum_{i,j:x_i y_j = z_k} P(\xi = x_i, \eta = y_j) \\ &= \sum_{i,j:x_i y_j = z_k} P(\xi = x_i) P(\eta = y_j). \end{aligned}$$

Then

$$
\begin{aligned}
E|\zeta| &= \sum_k |z_k| P(\zeta = z_k) \\
&= \sum_k \sum_{i,j: x_i y_j = z_k} |z_k| P(\xi = x_i) P(\eta = y_j) \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |x_i y_j| P(\xi = x_i) P(\eta = y_j) \\
&= \sum_{i=1}^{\infty} |x_i| P(\xi = x_i) \cdot \sum_{j=1}^{\infty} |y_j| P(\eta = y_j) \\
&= E|\xi| \cdot E|\eta|.
\end{aligned}
$$

So $E[\zeta]$ exists.

Repeating the argument yields

$$
\begin{aligned}
E\zeta =& \sum_k z_k P(\zeta = z_k) \\
=& \sum_k \sum_{i,j: x_i y_j = z_k} z_k P(\xi = x_i) P(\eta = y_j) \\
=& \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} x_i y_j P(\xi = x_i) P(\eta = y_j) \\
=& \sum_{i=1}^{\infty} x_i P(\xi = x_i) \cdot \sum_{j=1}^{\infty} y_j P(\eta = y_j) \\
=& E\xi \cdot E\eta.
\end{aligned}
$$

3.1.2 Expectations of continuous random variables

Suppose ξ has pdf $p(x)$. First, assume that ξ takes its values only on a finite interval $[a, b]$.

Now partition $[a, b]$ into smaller intervals:

$a = x_0 < x_1 < \cdots < x_n = b$, then

$$P(x_k < \xi \leq x_{k+1}) = \int_{x_k}^{x_{k+1}} p(x)dx \approx p(x_k)\Delta x_k,$$

Define a random ξ_n as

$$\xi_n = x_k, \text{ if } x_k < \xi \le x_{k+1}.$$

The ξ_n is discrete random variable with

$$E\xi_n = \sum_k x_k P(x_k < \xi \le x_{k+1}) \approx \sum_k x_k p(x_k) \Delta x_k.$$

As $n \to \infty$,

$$|\xi_n - \xi| \le \max_k \Delta x_k \to 0$$

and

$$\sum_k x_k p(x_k) \Delta x_k \to \int_a^b x p(x) dx.$$

It is natural to define

$$E\xi = \int_a^b xp(x)dx.$$

If ξ takes its values on the real line $(-\infty, \infty)$, letting $a \to -\infty, b \to \infty$, we get the following definition.

Definition 2 Suppose that ξ is a continuous random variable with density $p(x)$, and

$$\int_{-\infty}^{+\infty} |x|p(x)dx < \infty,$$

then we call

$$E\xi = \int_{-\infty}^{+\infty} xp(x)dx$$

the mathematical expectation of ξ. If $\int_{-\infty}^{\infty} |x|p(x)dx = \infty$, we say that the expectation of ξ does not exist.

Example 7. Suppose $\xi \sim U[a, b]$. Calculate $E\xi$.

Solution.

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Solution. Since ξ has the density function

$$p(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b, \\ 0, & \text{otherwise}, \end{cases}$$

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$$p(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b, \\ 0, & \text{otherwise}, \end{cases}$$

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$$E\xi = \int_{-\infty}^{\infty} xp(x)dx = \int_a^b x\frac{1}{b-a}dx$$

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Solution. Since ξ has the density function

$$p(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b, \\ 0, & \text{otherwise}, \end{cases}$$

then we have

$$\begin{aligned} E\xi &= \int_{-\infty}^{\infty} xp(x)dx = \int_a^b x\frac{1}{b-a}dx \\ &= \frac{1}{2}\frac{b^2-a^2}{b-a} = \frac{a+b}{2}. \end{aligned}$$

Example 8. Calculate the expectation of the exponential random variable ξ with parameter λ.

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$$p(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

Example 8. Calculate the expectation of the exponential random variable ξ with parameter λ.

Solution. Since ξ has the density function

$$p(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

then we have

$$E\xi = \int_0^{\infty} x\lambda e^{-\lambda x} dx = \frac{1}{\lambda}.$$

Example 9. Calculate the expectation of the normal random variable $\xi \sim N(a, \sigma^2)$.

Solution.

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Solution. First we note

$$\int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-a)^2}{2\sigma^2}} dx < \infty,$$

which implies that ξ has expectation.

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$$\int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-a)^2}{2\sigma^2}} dx < \infty,$$

which implies that ξ has expectation. Also,

$$E\xi = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-a)^2}{2\sigma^2}} dx$$

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which implies that ξ has expectation. Also,

$$\begin{aligned} E\xi &= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-a)^2}{2\sigma^2}} dx \\ &= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-\frac{z^2}{2}} dz \end{aligned}$$

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$$\int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-a)^2}{2\sigma^2}} dx < \infty,$$

which implies that ξ has expectation. Also,

$$\begin{aligned} E\xi &= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-a)^2}{2\sigma^2}} dx \\ &= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-\frac{z^2}{2}} dz \\ &= a. \end{aligned}$$

Example 10. Show the Cauchy distribution does not have expectation.

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Proof. The Cauchy distribution has the density

$$p(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

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$$p(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

Since

$$\int_{-\infty}^{\infty} |x| p(x) dx = 2\int_0^{\infty} \frac{x}{\pi(1+x^2)} dx = \infty,$$

so the expectation does not exist.

Example 10. The expectation of the Cauchy distribution

$$p(x) = \frac{\sigma}{\pi(\sigma^2 + (x-\mu)^2)}, \quad -\infty < x < \infty.$$

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does not exist.

μ 是什么?

$$\int_{-\infty}^{\mu} p(x)dx = \frac{1}{2} - - - - \text{中位数}$$

— 分位数回归

3.1.3 General definition Suppose ξ has cdf $F(x)$. Consider $-n = x_0 < x_1 < \cdots < x_{k_n} = n$, Define a random ξ_n as

$$\xi_n = x_k, \text{ if } x_k < \xi \leq x_{k+1}.$$

The ξ_n is discrete random variable with

$$E\xi_n = \sum_k x_k P(x_k < \xi \leq x_{k+1}) = \sum_k x_k \Delta F(x_k),$$

where $\Delta F(x_k) = F(x_{k+1}) - F(x_k)$. As $n \to \infty$, $|\xi_n - \xi| \leq \max_k \Delta x_k \to 0$.

It is natural to define

$$E\xi = \lim \sum_k x_k \Delta F(x_k)$$

It is natural to define

$$\begin{aligned} E\xi &= \lim \sum_k x_k \Delta F(x_k) \\ &= \int_{-\infty}^{\infty} x dF(x) \quad \text{(Stieltjes integral).} \end{aligned}$$

Definition 3. Suppose that ξ has distribution function $F(x)$. If $\int_{-\infty}^{\infty} |x| dF(x) < \infty$, then we call

$$E\xi = \int_{-\infty}^{\infty} x dF(x)$$

the mathematical expectation of ξ. When $\int_{-\infty}^{\infty} |x| dF(x) = \infty$, we say that the expectation of ξ does not exist.

Remark 1 When ξ is a discrete r.v.,

$$\begin{aligned}\int_{-\infty}^{\infty} x dF(x) &= \sum_k x_k[F(x_k) - F(x_k - 0)] \\ &= \sum_k x_k P(\xi = x_k).\end{aligned}$$

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$$\begin{aligned}\int_{-\infty}^{\infty} x dF(x) &= \sum_k x_k[F(x_k) - F(x_k - 0)] \\ &= \sum_k x_k P(\xi = x_k).\end{aligned}$$

When ξ is a continuous r.v., then

$$\begin{aligned}\int_{-\infty}^{\infty} x dF(x) &= \int_{-\infty}^{\infty} x d\left[\int_{-\infty}^{x} p(y)dy\right] \\ &= \int_{-\infty}^{\infty} xp(x)dx.\end{aligned}$$

Remark 2. $F(x) = \int_{-\infty}^{x} dF(t)$. So for any random variable ξ, $P(\xi \in B)$ can be written as the Stieltjes integral

$$P(\xi \in B) = \int_{x \in B} dF(x).$$

注意到$\xi^+ = \max\{\xi, 0\}$, $\xi^- = \max\{-\xi, 0\}$的分布函数分别为

$$F_{\xi^+}(x) = \begin{cases} P(\xi \le x) = F(x), & \text{若} x \ge 0; \\ 0; & \text{若} x < 0; \end{cases}$$

$$F_{\xi^-}(x) = \begin{cases} P(-\xi \le x) = 1 - F(-x-0), & \text{若} x \ge 0; \\ 0; & \text{若} x < 0. \end{cases}$$

注意到$\xi^+ = \max\{\xi, 0\}$, $\xi^- = \max\{-\xi, 0\}$的分布函数分别为

$$F_{\xi^+}(x) = \begin{cases} P(\xi \le x) = F(x), & 若 x \ge 0; \\ 0; & 若 x < 0; \end{cases}$$

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容易验证$\int_0^\infty x dF(x) = \int_0^\infty x dF_{\xi^+}(x)$,

$$\int_{-\infty}^0 x dF(x) = \int_{-\infty}^0 x dF(x-0) = -\int_0^\infty x d(1 - F(-x-0))$$

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因此$E\xi$存在的充分必要条件是$E\xi^+$和$E\xi^-$存在, 并且有$E\xi = E\xi^+ - E\xi^-$, $E|\xi| = E\xi^+ + E\xi^-$.

Proposition. Let $F(x)$ be the cdf of ξ. Then

$$
\begin{aligned}
E\xi &= \int_0^\infty P(\xi > y)dy - \int_0^\infty P(-\xi > y)dy, \\
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&= -\int_{-\infty}^{0} P(\xi < y) dy
\end{aligned}
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Similarly,

$$\begin{aligned}\int_{-\infty}^{0} x dF(x) &= -\int_{-\infty}^{0}\int_{x<y\leq 0} dy dF(x)\\ &= -\int_{-\infty}^{0} dy \int_{x<y} dF(x)\\ &= -\int_{-\infty}^{0} P(\xi < y) dy\\ &= -\int_{0}^{\infty} P(-\xi > y) dy.\end{aligned}$$

The first equality is proved. The proof of the second equality is similarly.