# Basic properties of expectations

## Basic properties of expectations of discrete random variables

Property 1 (*Absolute integrability*): Suppose $\xi$ is a discrete random variable. Then $E\xi$ is finite if and only if $E|\xi| < \infty$. Further

$$E\xi = E\xi^+ - E\xi^-, \quad E|\xi| = E\xi^+ + E\xi^-.$$

Property 2 (*Linearity*): Suppose $\xi$ and $\eta$ are discrete random variables. If $E\xi$ and $E\eta$ exist, then

$$E(a\xi + b\eta) = aE\xi + bE\eta.$$

Property 3 (*Monotonicity*): Suppose $\xi$ and $\eta$ are discrete random variables. If $\xi \leq \eta$ and the expectations of $\xi$ and $\eta$ exist, then $E\xi \leq E\eta$.

## Corollary

*Suppose* $\xi$ *and* $\eta$ *are discrete random variables,* $|\xi| \leq \eta$. *If the expectation* $\eta$ *exists, then the expectation* $E\xi$ *exists, and* $|E\xi| \leq E|\xi| \leq E\eta$.

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**Proof.** Suppose $\xi = \sum_{k=1}^{\infty} x_i I\{\xi = x_i\}$. Write $\bar{\xi} = \sum_{k=1}^{N} x_i I\{\xi = x_i\}$. Then $|\bar{\xi}| = \sum_{k=1}^{N} |x_i| I\{\xi = x_i\} \leq |\xi| \leq \eta$, the expectations of $|\bar{\xi}|$ and $\eta$ exist. By Property 3, $E|\bar{\xi}| \leq E\eta$.

So,

$$\sum_{i=1}^{N} |x_i| P(\xi = x_i) \leq E\eta.$$

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Letting $N \to \infty$ yields,

$$E|\xi| = \sum_{i=1}^{\infty} |x_i| P(\xi = x_i) \le E\eta.$$

Hence, the expectation $E\xi$ exists. Note $-|\xi| \le \xi \le |\xi|$.

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Hence, the expectation $E\xi$ exists. Note $-|\xi| \leq \xi \leq |\xi|$. By Property 3, $|E\xi| \leq E|\xi|$.

Property 4 Suppose $\xi$ and $\eta$ are discrete random variables and $E\xi$ and $E\eta$ exists. Then $E[\xi\eta]$ exists and

$$E[\xi\eta] = E\xi \cdot E\eta.$$

Properties of Mathematical expectation for general random variables

For a random variable $\xi$. Define

$$\xi^{(m)} = \frac{k}{2^m} \quad \text{if} \quad \frac{k}{2^m} < \xi \leq \frac{k+1}{2^m}.$$

Then

- If $\xi \geq 0$, then $0 \leq \xi_m \nearrow \xi$ and $0 \leq \xi - \xi^{(m)} \leq \frac{1}{2^m}$.
- In general, $|\xi - \xi^{(m)}| \leq \frac{1}{2^m}$.

**Theorem**

*$E\xi$ exists if and only if $E\xi^{(m)}$ exists for one $m$ (and then all $m$). Further,*

$$E\xi = \lim_{m\to\infty} E\xi^{(m)}.$$

Suppose $\xi$ has cdf $F(x)$. Write $x_{m,k} = \frac{k}{2^m}$. Then

$$\xi^{(m)} = \sum_{k=-\infty}^{\infty} x_{m,k} I\{x_{m,k} < \xi \leq x_{m,k+1}\},$$

and

$$\begin{aligned} & \sum_{k=-\infty}^{\infty} |x_{m,k}| P(x_{m,k} < \xi \leq x_{m,k+1}) \\ = & \sum_{k=-\infty}^{\infty} |x_{m,k}| \Delta F(x_{m,k}) \\ = & \sum_{k=-\infty}^{\infty} \int_{x_{m,k} < x \leq x_{m,k+1}} |x_{m,k}| dF(x), \end{aligned}$$

where $\Delta F(x_{m,k}) = F(x_{m,k+1}) - F(x_{m,k})$.

For $x_{m,k} < x \leq x_{m,k+1}$, we have $|x_{m,k} - x| \leq \frac{1}{2^m}$. So,

$$\sum_{k=-\infty}^{\infty} |x_{m,k}| \Delta F(x_{m,k}) = \sum_{k=-\infty}^{\infty} \int_{x_{m,k} < x \leq x_{m,k+1}} |x_{m,k}| dF(x)$$

$$\leq \sum_{k=-\infty}^{\infty} \int_{x_{m,k} < x \leq x_{m,k+1}} \left(|x| + \frac{1}{2^m}\right) dF(x)$$

$$= \int_{-\infty}^{\infty} |x| dF(x) + \frac{1}{2^m}.$$

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Similarly,

$$\sum_{k=-\infty}^{\infty} |x_{m,k}| \Delta F(x_{m,k}) \geq \int_{-\infty}^{\infty} |x| dF(x) - \frac{1}{2^m}.$$

So, $E\xi$ exists if and only if $E\xi^{(m)}$ exits.

Similarly,

$$
\begin{aligned}
E\xi^{(m)} &= \sum_{k=-\infty}^{\infty} x_{m,k} P(x_{m,k} < \xi \leq x_{m,k+1}) \\
&= \sum_{k=0}^{\infty} x_{m,k} \Delta F(x_{m,k}) \\
&= \sum_{k=-\infty}^{\infty} \int_{x_{m,k} < x \leq x_{m,k+1}} |x_{m,k}| dF(x)
\end{aligned}
$$

and

$$
\int_{-\infty}^{\infty} x dF(x) - \frac{1}{2^m} \leq E\xi^{(m)} \leq \int_{-\infty}^{\infty} x dF(x) + \frac{1}{2^m}.
$$

The proof is completed.

# 3.1.5 Basic properties of expectations

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$$E\xi \geq 0.$$

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In general, if $a \leq \xi \leq b$, $a, b > 0$, then

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In fact,

$$E\xi = \int_{-\infty}^{\infty} xdF(x) = \int_{a}^{b} xdF(x).$$

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$$E\xi \geq a \int_a^b dF(x) = a \int_{-\infty}^{\infty} dF(x) = a.$$

2. (*Linearity*). $E\xi$ and $E\eta$ exist $\Longrightarrow$

$$E(a\xi + b\eta) = aE\xi + bE\eta.$$

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**Proof.** Note $|X - X^{(m)}| \leq \frac{1}{2^m}$. So

$$\begin{aligned}
&\left|(a\xi + b\eta)^{(m)} - (a\xi^{(m)} + b\eta^{(m)})\right| \\
\leq &\left|(a\xi + b\eta)^{(m)} - (a\xi + b\eta)\right| + |a|\left|\xi^{(m)} - \xi\right| + |b|\left|\eta^{(m)} - \eta\right| \\
\leq &\frac{1 + |a| + |b|}{2^m}.
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It follows that

$$\left|E\left[(a\xi + b\eta)^{(m)}\right] - (aE\xi^{(m)} + bE\eta^{(m)})\right| \le \frac{1 + |a| + |b|}{2^m}.$$

Taking the limit $m \to \infty$ completes the proof.

## Corollary

*Suppose* $\xi \leq \eta$ *and,* $E\xi$ *and* $E\eta$ *exist. Then* $E\xi \leq E\eta$.

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*Suppose $\xi \leq \eta$ and, $E\xi$ and $E\eta$ exist. Then $E\xi \leq E\eta$.*

## Proof

$$E\eta - E\xi = E[\eta - \xi] \geq 0.$$

3. Suppose that $\xi$ and $\eta$ are independent, and expectations $E\xi$ and $E\eta$ exists. Then

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$$E\xi\eta = E\xi E\eta.$$

**Proof.** Note that $E\xi^{(m)}$ and $E\eta^{(m)}$ both exist. So, $E(\xi^{(m)}\eta^{(m)})$ exists and

$$E(\xi^{(m)}\eta^{(m)}) = E\xi^{(m)}E\eta^{(m)} \to E\xi E\eta.$$

On the other hand,

$$|(\xi\eta)^{(m)} - \xi\eta| \leq \frac{1}{2^m},$$

$$\begin{aligned}\xi^{(m)}\eta^{(m)} - \xi\eta =& (\xi^{(m)} - \xi)\eta^{(m)} + \xi(\eta^{(m)} - \eta) \\ =& (\xi^{(m)} - \xi)\eta^{(m)} + \xi^{(m)}(\eta^{(m)} - \eta) \\ &+ (\xi - \xi^{(m)})(\eta^{(m)} - \eta).\end{aligned}$$

It follows that

$$|\xi^{(m)}\eta^{(m)} - (\xi\eta)^{(m)}| \leq \frac{1}{2^m}|\eta^{(m)}| + \frac{1}{2^m}|\xi^{(m)}| + \frac{2}{2^m}.$$

Note that $\xi^{(m)}\eta^{(m)}$, $(\xi\eta)^{(m)}$, $|\eta^{(m)}|$, $|\xi^{(m)}|$ are discrete random variables, and $E[\xi^{(m)}\eta^{(m)}]$, $E[|\eta^{(m)}|]$, $E[|\xi^{(m)}|]$ exist.
So $E[(\xi\eta)^{(m)}]$ exists and

$$\begin{aligned}
&|E[\xi^{(m)}\eta^{(m)}] - E[(\xi\eta)^{(m)}]| \\
\le&\frac{1}{2^m}E[|\eta^{(m)}|] + \frac{1}{2^m}E[|\xi^{(m)}|] + \frac{2}{2^m} \\
\le&\frac{1}{2^m}E[|\eta|] + \frac{1}{2^m}E[|\xi|] + \frac{4}{2^m} \to 0.
\end{aligned}$$

Hence, $E[\xi\eta]$ exists and

$$\begin{aligned} E[\xi\eta] &= \lim_{m\to\infty} E[(\xi\eta)^{(m)}] \\ &= \lim_{m\to\infty} E[\xi^{(m)}\eta^{(m)}] \\ &= \lim_{m\to\infty} E[\xi^{(m)}]E[\eta^{(m)}] \\ &= E[\xi]E[\eta]. \square \end{aligned}$$

## Integral with respect to a probability measure

The random variable $\xi(\omega)$ is a $\mathscr{F}$-measurable function on the probability space $(\Omega, \mathscr{F}, P)$. The expectation of $\xi$ is the integral of $\xi$ with respect to $P$.

$$E\xi = \int \xi(\omega) dP(\omega) = \int \xi dP.$$

- If $\xi(\omega) = \sum_i x_i I_{A_i}$, then

$$\int \xi dP = \sum_i x_i P(A_i);$$

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- $\int \xi dP = \int \xi^+ dP - \int \xi^- dP$.

For $A \in \mathscr{F}$ and $\xi$, define

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Then

$$\int_{A+B} \xi dP = \int_A \xi dP + \int_B \xi dP.$$