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Definition Suppose that the joint distribution sequence of a discrete random vector (ξ, η) satisfies

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$$i, j = 1, 2, \cdots ,$$

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then we call ξ and η mutually independent.

$$p_{ij} = p_{i\cdot} \cdot p_{\cdot j}, \qquad i, j = 1, 2, \cdots .$$

For any x and y,

$$\begin{aligned}
P(\xi \leq x, \eta \leq y) &= \sum_{x_i \leq x} \sum_{y_j \leq y} P(\xi = x_i, \eta = y_j) \\
&= \sum_{x_i \leq x} P(\xi = x_i) \sum_{y_j \leq y} P(\eta = y_j) \\
&= P(\xi \leq x) P(\eta \leq y).
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That is,

$$F(x, y) = F_\xi(x) F_\eta(y). \tag{2.62}$$

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That is,

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On the contrary,

$$(2.62) \Longrightarrow P(\xi = x_i, \eta = y_j) = P(\xi = x_i) P(\eta = y_j)$$

Definition Suppose that $F(x,y)$, $F_\xi(x)$ and $F_\eta(y)$ are the joint distribution function and marginal distribution functions of (ξ,η) respectively. If

$$F(x,y)=F_\xi(x)F_\eta(y),\quad \forall x,y,$$

$$(i.e.,\quad P(\xi\le x,\eta\le y)=P(\xi\le x)P(\eta\le y),\quad \forall x,y)$$

then we say ξ and η are independent.

Theorem Suppose that $p(x,y)$, $p_\xi(x)$ and $p_\eta(y)$ are the joint density function and marginal density functions of (ξ,η) respectively. Then ξ and η are independent if and only if

$$p(x,y)=p_\xi(x)p_\eta(y).$$

Proof. For any x, y, it follows

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\Leftrightarrow\ & \int_{-\infty}^{x}\int_{-\infty}^{y} p(u,v)\,du\,dv = \int_{-\infty}^{x} p_\xi(u)\,du \int_{-\infty}^{y} p_\eta(v)\,dv
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\Leftrightarrow\ & p(x,y) = p_\xi(x)p_\eta(y).
\end{aligned}
$$

This is the desired conclusion.

Example 2. Suppose $(\xi, \eta) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, r)$. Find out the necessary and sufficient condition for ξ, η to be independent.

Solution. Note that $\xi \sim N(\mu_1, \sigma_1^2)$ and $\eta \sim N(\mu_2, \sigma_2^2)$. By definition,

$$\xi, \eta \text{ are independent } \Leftrightarrow p(x, y) = p_\xi(x) p_\eta(y)$$

Solution. Note that $\xi \sim N(\mu_1, \sigma_1^2)$ and $\eta \sim N(\mu_2, \sigma_2^2)$. By definition,

$$
\begin{aligned}
& \xi, \eta \text{ are independent } \Leftrightarrow p(x,y) = p_\xi(x) p_\eta(y) \\
\Leftrightarrow \quad & \frac{1}{\sqrt{2\pi}\sigma_1} \exp\{-\frac{(x-\mu_1)^2}{2\sigma_1^2}\} \\
& \times \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-r^2}} \exp\{-\frac{[y-\mu_2-\frac{r\sigma_2}{\sigma_1}(x-\mu_1)]^2}{2\sigma_2^2(1-r^2)}\} \\
& = \frac{1}{2\pi\sigma_1\sigma_2} \exp\{-\frac{1}{2}[\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{(y-\mu_2)^2}{\sigma_2^2}]\} \\
\Leftrightarrow \quad & r = 0.
\end{aligned}
$$

n random variables:

Definition Suppose that $F(x_1, \cdots, x_n)$, $F_1(x_1), \cdots, F_n(x_n)$ are joint distribution function and marginal distribution functions of $\xi_1, \cdots, \xi_n$, then we call them mutually independent if

$$F(x_1, \cdots, x_n) = F_1(x_1) \cdots F_n(x_n).$$

$$\Big(i.e., \;\; P(\xi_1 \le x_1, \cdots, \xi_n \le x_n) = P(\xi_1 \le x_1) \cdots P(\xi_n \le x_n), \quad \forall x_1, \cdots, x_n \Big)$$

Corollary If $\xi_1, \cdots, \xi_n$ are mutually independent, then so are any r random variables $(2 \le r < n)$.

Proof.

Corollary If $\xi_1, \cdots, \xi_n$ are mutually independent, then so are any r random variables $(2 \le r < n)$.

Proof. By the definition of the independence of $\xi_1, \cdots, \xi_n$, we have for all $x_1, \cdots, x_n$,

$$P(\xi_1 \le x_1, \cdots, \xi_n \le x_n) = P(\xi_1 \le x_1) \cdots P(\xi_n \le x_n).$$

Corollary If $\xi_1, \cdots, \xi_n$ are mutually independent, then so are any r random variables $(2 \le r < n)$.

Proof. By the definition of the independence of $\xi_1, \cdots, \xi_n$, we have for all $x_1, \cdots, x_n$,

$$P(\xi_1 \le x_1, \cdots, \xi_n \le x_n) = P(\xi_1 \le x_1) \cdots P(\xi_n \le x_n).$$

It follows that

$$\begin{aligned} & P(\xi_{i_1} \le x_{i_1}, \cdots, \xi_{i_r} \le x_{i_r}) \\ = & P(\xi_{i_1} \le x_{i_1}) \cdots P(\xi_{i_r} \le x_{i_r}), \quad \forall x_{i_1}, \cdots, x_{i_r}. \end{aligned}$$

So, $\xi_{i_1}, \cdots, \xi_{i_r}$ are independent.

- $\xi_1, \cdots, \xi_n$ are indept. iff (if and only if)

$$P(\xi_1 \in B_1, \cdots, \xi_n \in B_n) = P(\xi_1 \in B_1) \cdots P(\xi_n \in B_n)$$

for any $B_1, \cdots, B_n \in \mathcal{B}$.

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- An n-dimensional $\boldsymbol{\xi}$ and an m-dimensional $\boldsymbol{\eta}$ are indept. iff

$$P(\boldsymbol{\xi} \in A, \boldsymbol{\eta} \in B) = P(\boldsymbol{\xi} \in A) P(\boldsymbol{\eta} \in B),$$

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for all $A \in \mathcal{B}^n, B \in \mathcal{B}^m$.

- If two random vectors are independent, then so are their sub-vectors.

Example 3. Suppose that ξ is a constant a, show ξ and η are independent for any random variable η.

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Proof Let B_1 and B_2 be two Borel sets. We want to prove

$$P(\xi \in B_1, \eta \in B_2) = P(\xi \in B_1)P(\eta \in B_2). \quad (*)$$

If $a \notin B_1$, then $P(\xi \in B_1) = 0$ and

$$P(\xi \in B_1, \eta \in B_2) \le P(\xi \in B_1) = 0.$$

$(*)$ is true.

If $a \notin B_1$, then $P(\xi \in B_1) = 0$ and

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$(*)$ is true.
If $a \in B_1$, then $P(\xi \in B_1) = 1$ and

$$\begin{aligned} P(\xi \in B_1, \eta \in B_2) =& P(\eta \in B_2) - P(\xi \notin B_1, \eta \in B_2) \\ =& P(\eta \in B_2). \end{aligned}$$

$(*)$ is also true.

2.5 Conditional distributions

I.Discrete random variables:

$P(\xi = x_i, \eta = y_j) = p_{ij}, i, j = 1, 2, \cdots .$

$$P(\eta = y_j | \xi = x_i) = \frac{P(\eta = y_j, \xi = x_i)}{P(\xi = x_i)} = \frac{p_{ij}}{p_{i\cdot}},$$

where $j = 1, 2, \cdots$. This is the conditional distribution of η conditioning on $\xi = x_i$.

定义1 称$P(\eta=y_j|\xi=x_i)$为在$\xi=x_i$的条件下η的条件概率分布列, 简称为条件分布, 记为$p_{\eta|\xi}(y_j|x_i)$. 称

$$P(\eta\le y|\xi=x_i)=\sum_{j:y_j\le y}p_{\eta|\xi}(y_j|x_i)$$

为在$\xi=x_i$的条件下η的条件分布函数.

从条件分布的定义和ξ,η的独立性的定义可知, ξ, η独立的充分必要条件是对任何$i,j\ge 1$ 有

$$P(\eta=y_j|\xi=x_i)=P(\eta=y_j).$$

例1 在独立重复伯努里试验中，记p为每次试验"成功"的概率，S_n表示第n次成功时的试验次数. 求(1) 在$S_n = t$的条件下，S_{n+1}的条件概率分布列; (2) 在$S_{n+1} = w$的条件下，S_n的条件概率分布列.

例1 在独立重复伯努里试验中，记p为每次试验"成功"的概率，S_n表示第n次成功时的试验次数. 求(1) 在$S_n = t$的条件下，S_{n+1}的条件概率分布列; (2) 在$S_{n+1} = w$的条件下，S_n的条件概率分布列.

解 对$t \le w$，事件$\{S_n = t, S_{n+1} = w\}$ 意味着在w次试验中，第t, w次出现"成功"，在第1次到第$t-1$次中出现$n-1$次"成功",其余均出现"失败". 所以

$$P(S_n = t, S_{n+1} = w) = p \cdot p \cdot \binom{t-1}{n-1} p^{n-1} q^{w-(n+1)}$$
$$= \binom{t-1}{n-1} p^{n+1} q^{w-(n+1)}.$$

而

$$P(S_n = t) = \binom{t-1}{n-1} p^n q^{t-n}.$$

而

$$P(S_n = t) = \binom{t-1}{n-1} p^n q^{t-n}.$$

从而在$S_n = t$的条件下，S_{n+1}的条件概率分布列为

$$P(S_{n+1} = w | S_n = t) = \frac{P(S_n = t, S_{n+1} = w)}{P(S_n = t)} = pq^{w-t-1}.$$

这意味着，在$S_n = t$的条件下，$S_{n+1} - S_n$服从几何分布.

而在$S_{n+1}=w$的条件下，S_n的条件概率分布列为

$$
\begin{aligned}
P(S_n=t|S_{n+1}=w) &= \frac{P(S_n=t,S_{n+1}=w)}{P(S_{n+1}=w)} \\
&= \frac{\binom{t-1}{n-1}p^{n+1}q^{w-(n+1)}}{\binom{w-1}{n}p^{n+1}q^{w-(n+1)}} \\
&= \frac{\binom{t-1}{n-1}}{\binom{w-1}{n}}, \quad t=n,\ldots,w-1.
\end{aligned}
$$

这一条件分布不依赖于p.

II.Continuous case: $P(\xi = x) = 0$. Given $\xi = x$, the conditional distribution function of η can be understood as

$$P(\eta \leq y | \xi = x)$$

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\begin{aligned}
& P(\eta \le y | \xi = x) \\
= & \lim_{\Delta x \to 0} P(\eta \le y | x < \xi \le x + \Delta x) \\
= & \lim_{\Delta x \to 0} \frac{P(x < \xi \le x + \Delta x, \eta \le y)}{P(x < \xi \le x + \Delta x)} \\
= & \lim_{\Delta x \to 0} \frac{F(x + \Delta x, y) - F(x, y)}{F_\xi(x + \Delta x) - F_\xi(x)}.
\end{aligned}
$$

$$P(\eta \leq y | \xi = x) = \frac{\partial F / \partial x}{F'_\xi(x)}$$

$$= \frac{\int_{-\infty}^{y} p(x, v) dv}{p_\xi(x)} = \int_{-\infty}^{y} \frac{p(x, v)}{p_\xi(x)} dv.$$

When $p_\xi(x) > 0$, conditioning on $\xi = x$, the density of η is

$$p_{\eta|\xi}(y|x) = \frac{p(x,y)}{p_\xi(x)}.$$

定义2 设随机向量(ξ,η)有联合密度函数$p(x,y)$, ξ有边际密度函数$p_\xi(x)=\int_{-\infty}^{\infty}p(x,y)dy$. 若在$x$处, $p_\xi(x)>0$, 则称

$$P(\eta\le y|\xi=x)=\int_{-\infty}^{y}\frac{p(x,v)}{p_\xi(x)}dv,\quad y\in \boldsymbol{R}$$

为在$\xi=x$的条件下, η的条件分布函数, 简称为条件分布, 记作$F_{\eta|\xi}(y|x)$. 称

$$p_{\eta|\xi}(y|x)=\frac{p(x,y)}{p_\xi(x)},\quad y\in \boldsymbol{R} \tag{1}$$

为在$\xi=x$的条件下, η的条件密度函数, 简称为条件密度.

若$p_\xi(x)=\int_{-\infty}^{\infty}p(x,y)dy=0$, 则对所有的$y$, $p(x,y)=0$, (1)式右边是$\frac{0}{0}$型不定式, 通常定义$p_{\eta|\xi}(y|x)$的值为0.

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同理, 若$p_\eta(y)>0$, 则在$\eta=y$ 的条件下, ξ 的密度函数为

$$p_{\xi|\eta}(x|y)=\frac{p(x,y)}{p_\eta(y)}.$$

If ξ and η are independent, then

$$p_{\eta|\xi}(y|x) = p_\eta(y), \quad p_{\xi|\eta}(x|y) = p_\xi(x).$$

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Bayesian formula:

$$\begin{aligned} p_{\xi|\eta}(x|y) =& \frac{p(x,y)}{p_\eta(y)} = \frac{p(x,y)}{\int p(u,y)du} \\ =& \frac{p_{\eta|\xi}(y|x)p_\xi(x)}{\int p_{\eta|\xi}(y|u)p_\xi(u)du}. \end{aligned}$$

Example 1. Suppose that $(\xi, \eta) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, r)$, calculate the conditional density $p_{\eta|\xi}(y|x)$.

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解 (ξ, η)的联合密度为

$$p(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \exp\left\{-\frac{1}{2(1-r^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2r\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right\}.$$

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\end{aligned}
$$

下面我们推导在$\xi=x$的条件下, η的条件密度. 为此, 我们不断把不含y的因子提出来, 用常数C_i表示. 最后的常数通过$\int_{-\infty}^{\infty}p_{\eta|\xi}(y|x)dy=1$求得.

$$
\begin{aligned}
p_{\eta|\xi}(y|x) &= \frac{p(x,y)}{\int_{-\infty}^{\infty} p(x,y)dy} = C_1 p(x,y)\\
=& C_2 \exp\left\{-\frac{1}{2(1-r^2)}\left[\frac{(y-\mu_2)^2}{\sigma_2^2} - 2r\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}\right]\right\}\\
=& C_3 \exp\left\{-\frac{1}{2(1-r^2)}\left(\frac{y-\mu_2}{\sigma_2} - r\frac{x-\mu_1}{\sigma_1}\right)^2\right\}\\
=& C_3 \exp\left\{-\frac{\left[y-\mu_2 - r\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right]^2}{2\sigma_2^2(1-r^2)}\right\}.
\end{aligned}
$$

上述过程可以简写为

$$
\begin{aligned}
p_{\eta|\xi}(y|x) &\propto_y p(x,y) \propto_y \ldots \\
&\propto_y \exp\left\{ -\frac{\left[y-\mu_2-r\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right]^2}{2\sigma_2^2(1-r^2)} \right\}.
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\end{aligned}
$$

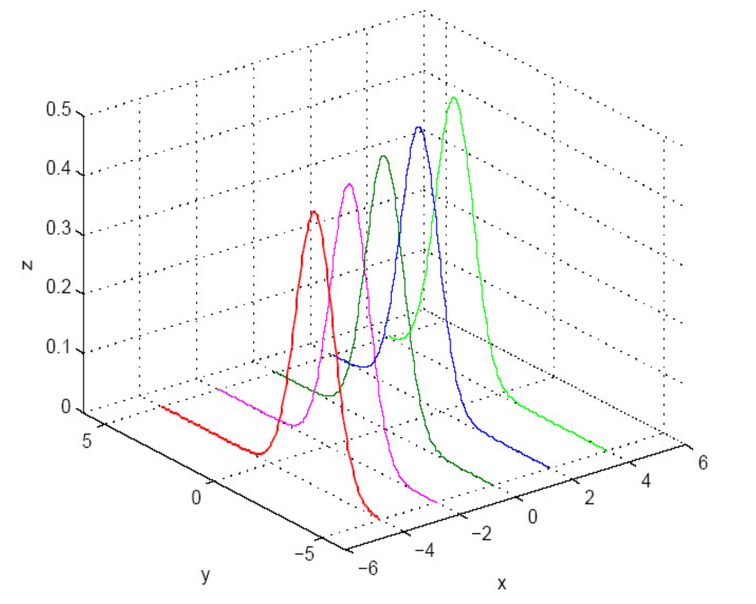
回顾正态分布的密度函数知$p_{\eta|\xi}(y|x)$为正态密度函数

$$
p_{\eta|\xi}(y|x) = \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-r^2}} \exp\left\{-\frac{\left[y-\mu_2-r\frac{\sigma_2}{\sigma_1}(x-\mu_1)\right]^2}{2\sigma_2^2(1-r^2)}\right\}. \tag{2}
$$

即在$\xi = x$的条件下，二维正态分布的条件分布是正态分布$N\Big(\mu_2 + r\frac{\sigma_2}{\sigma_1}(x-\mu_1), (1-r^2)\sigma_2^2\Big)$，记作

$$\eta\big|_{\xi=x} \sim N(\mu_2 + \frac{r\sigma_2}{\sigma_1}(x-\mu_1), (1-r^2)\sigma_2^2),$$

其中第一个参数$m = \mu_2 + r\frac{\sigma_2}{\sigma_1}(x-\mu_1)$是$x$的线性函数，第二个参数与$x$无关.


0.5
0.4
0.3
0.2
0.1
0
z
5
0
−5
y
−6
−4
−2
0
2
4
6
x

III. The general case:

In general, suppose that the joint distribution function of (ξ, η) is $F(x, y)$. If

$$\lim_{\Delta y \to 0} \frac{P(\xi \leq x, \eta \in (y, y + \Delta y])}{P(\eta \in (y, y + \Delta y])} = \lim_{\Delta y \to 0} \frac{F(x, y + \Delta y) - F(x, y)}{F_\eta(y + \Delta y) - F_\eta(y)}$$

exists for any x, we call the limit function $F_{\xi|\eta}(x|y)$ be the conditional distribution function of ξ for given $\eta = y$.

If there exists $\{x_i\}$ such that

$$F_{\xi|\eta}(x|y) = \sum_{i:x_i\le x} p_{\xi|\eta}(x_i|y), \quad x \in \boldsymbol{R},$$

the we call $p_{\xi|\eta}(x_i|y)$, $i = 1, 2, \ldots,$ the conditional mass function (条件分布列). If $F_{\xi|\eta}(x|y)$ can represented as the form

$$F_{\xi|\eta}(x|y) = \int_{-\infty}^{x} p_{\xi|\eta}(v|y)dv, \quad x \in \boldsymbol{R},$$

then we call $p_{\xi|\eta}(x|y)$ the conditional density function.

例4 设Λ服从伽玛分布$\Gamma(b,a)$, 在条件$\Lambda=\lambda$下, X服从参数为λ的泊松分布. 求在$X=x$的条件下Λ的分布.

例4 设Λ服从伽玛分布$\Gamma(b,a)$, 在条件$\Lambda=\lambda$下, X服从参数为λ的泊松分布. 求在$X=x$的条件下Λ的分布.

解 Λ为连续型随机变量, X为离散型随机变量.
对$x=0,1,\dots,$ 有

$$P(X=x|\Lambda=\lambda)=\frac{\lambda^x}{x!}e^{-\lambda}.$$

这意味着

$$P(X=x|\Lambda=\lambda)=\lim_{\Delta\lambda\to 0}\frac{P\big(X=x,\Lambda\in(\lambda,\lambda+\Delta\lambda)\big)}{P\big(\Lambda\in(\lambda,\lambda+\Delta\lambda)\big)}.$$

即

$$\begin{aligned}
&P\big(X=x,\Lambda\in(\lambda,\lambda+\Delta\lambda)\big)\\
=&P(X=x|\Lambda=\lambda)P\big(\Lambda\in(\lambda,\lambda+\Delta\lambda)\big)+o(\Delta\lambda)\\
=&P(X=x|\Lambda=\lambda)p_\Lambda(\lambda)\Delta\lambda+o(\Delta\lambda).
\end{aligned}$$

所以

$$P(X=x,\Lambda\le y)=\int_{-\infty}^{y}P(X=x|\Lambda=\lambda)p_\Lambda(\lambda)d\lambda.$$

即

$$
\begin{aligned}
&P\big(X=x,\Lambda\in(\lambda,\lambda+\Delta\lambda)\big)\\
=&P(X=x|\Lambda=\lambda)P\big(\Lambda\in(\lambda,\lambda+\Delta\lambda)\big)+o(\Delta\lambda)\\
=&P(X=x|\Lambda=\lambda)p_\Lambda(\lambda)\Delta\lambda+o(\Delta\lambda).
\end{aligned}
$$

所以

$$
P(X=x,\Lambda\le y)=\int_{-\infty}^{y}P(X=x|\Lambda=\lambda)p_\Lambda(\lambda)d\lambda.
$$

从而

$$
\begin{aligned}
P(\Lambda\le y|X=x)=&\frac{P(X=x,\Lambda\le y)}{P(X=x)}\\
=&\int_{-\infty}^{y}\frac{P(X=x|\Lambda=\lambda)p_\Lambda(\lambda)}{P(X=x)}d\lambda.
\end{aligned}
$$

因此在$X=x$的条件下, Λ的密度函数为

$$
\begin{aligned}
p_{\Lambda|\xi}(\lambda|x) =& \frac{P(X=x|\Lambda=\lambda)p_\Lambda(\lambda)}{P(X=x)} \\
& \propto_\lambda \lambda^x e^{-\lambda}\lambda^{b-1}e^{-\lambda a} = \lambda^{x+b-1}e^{-(a+1)\lambda}, \quad \lambda>0.
\end{aligned}
$$

因此在$X=x$的条件下, Λ的密度函数为

$$p_{\Lambda|\xi}(\lambda|x) = \frac{P(X=x|\Lambda=\lambda)p_{\Lambda}(\lambda)}{P(X=x)}$$
$$\propto_{\lambda} \lambda^{x} e^{-\lambda} \lambda^{b-1} e^{-\lambda a} = \lambda^{x+b-1} e^{-(a+1)\lambda}, \quad \lambda > 0.$$

将上式右边添加正则化常数因子使得其积分为1, 得

$$p_{\Lambda|\xi}(\lambda|x) = \frac{(a+1)^{x+b}}{\Gamma(x+b)} \lambda^{x+b-1} e^{-(a+1)\lambda}, \quad \lambda > 0.$$

即在$X=x$的条件下, Λ服从伽玛分布$\Gamma(x+b, a+1)$.

IV. Multi-dimensional case:

Suppose that the joint distribution function of random vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ is $F(\boldsymbol{x}, \boldsymbol{y})$. If

$$\begin{aligned} & \lim_{\Delta \boldsymbol{y} \to 0} \frac{P(\boldsymbol{\xi} \le \boldsymbol{x}, \boldsymbol{\eta} \in (\boldsymbol{y}, \boldsymbol{y} + \Delta \boldsymbol{y}])}{P(\boldsymbol{\eta} \in (\boldsymbol{y}, \boldsymbol{y} + \Delta \boldsymbol{y}]} \\ = & \lim_{\Delta \boldsymbol{y} \to 0} \frac{F(\boldsymbol{x}, \boldsymbol{y} + \Delta \boldsymbol{y}) - F(\boldsymbol{x}, \boldsymbol{y})}{F_\eta(\boldsymbol{y} + \Delta \boldsymbol{y}) - F_\eta(\boldsymbol{y})} \end{aligned}$$

exists for any $\boldsymbol{x}$, we call the limit function $F_{\xi|\eta}(\boldsymbol{x}|\boldsymbol{y})$ be the conditional distribution function of $\boldsymbol{\xi}$ for given $\boldsymbol{\eta} = \boldsymbol{y}$.

When $(\boldsymbol{\xi}, \boldsymbol{\eta})$ is a continuous random vector with probability density function $p(\boldsymbol{x}, \boldsymbol{y})$, the conditional probability density function of $\boldsymbol{\xi}$ for given $\boldsymbol{\eta} = \boldsymbol{y}$ is

$$p_{\xi|\eta}(\boldsymbol{x}|\boldsymbol{y}) = \frac{p(\boldsymbol{x}, \boldsymbol{y})}{p_\eta(\boldsymbol{y})} = \frac{p(\boldsymbol{x}, \boldsymbol{y})}{\int p(\boldsymbol{u}, \boldsymbol{y}) d\boldsymbol{u}},$$

$$\text{if } \; p_\eta(\boldsymbol{y}) > 0.$$

When $(\boldsymbol{\xi}, \boldsymbol{\eta})$ is a discrete random vector with probability mass function $P(\boldsymbol{\xi} = \boldsymbol{x}_i, \boldsymbol{\eta} = \boldsymbol{y}_j) = p(\boldsymbol{x}_i, \boldsymbol{y}_j)$, the conditional probability mass function of $\boldsymbol{\xi}$ for given $\boldsymbol{\eta} = \boldsymbol{y}_j$ is

$$p_{\xi|\eta}(\boldsymbol{x}_i|\boldsymbol{y}_j) = \frac{p(\boldsymbol{x}_i, \boldsymbol{y}_j)}{p_{\eta}(\boldsymbol{y}_j)} = \frac{P(\boldsymbol{\xi} = \boldsymbol{x}_i, \boldsymbol{\eta} = \boldsymbol{y}_j)}{P(\boldsymbol{\eta} = \boldsymbol{y}_j)},$$

$$\text{if } \; P(\boldsymbol{\eta} = \boldsymbol{y}_j) > 0.$$