1.4 Conditional probability and independent events

Conditional probability(条件概率)

Example: A dice is thrown. The sample space is

$$\Omega = \{\omega_1, \omega_2, \cdots, \omega_6\}, \quad \omega_i = \{i \text{ comes up}\}.$$

Consider the events

$$A = \{\text{an even point comes up}\} = \{\omega_2, \omega_4, \omega_6\},$$

$$B = \{\text{an odd point comes up}\} = \{\omega_1, \omega_3, \omega_5\}.$$

Then

$$P(A) = P(B) = \frac{1}{2}.$$

Now, provided that a big point (≥ 4) comes up, the probability that the point is even is just the conditional probability of A given C, written as $P(A|C)$, where $C = \{$ a big point comes up$\}$.

Now, provided that a big point (≥ 4) comes up, the probability that the point is even is just the conditional probability of A given C, written as $P(A|C)$, where $C = \{$ a big point comes up$\}$. Provided that a big point comes up, the sample space becomes

$$\Omega_2 = C = \{\omega_4, \omega_5, \omega_6\}.$$

Provided the condition that C has occurred, sample points contained by A are $\{\omega_4, \omega_6\} = AC$.

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$$P(A|C) = \frac{\#\{\omega_4, \omega_6\}}{\#\Omega_2} = \frac{2}{3}.$$

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$$P(A|C) = \frac{\#\{\omega_4, \omega_6\}}{\#\Omega_2} = \frac{2}{3}.$$

Similarly

$$P(B|C) = \frac{\#\{\omega_5\}}{\#\Omega_2} = \frac{1}{3}.$$

$$P(A|B)$$

$$= \frac{\text{Number of sample points contained by } A \text{ given } B}{\text{Number of sample points given } B}$$

$$
\begin{aligned}
& P(A|B) \\
= \; & \frac{\text{Number of sample points contained by } A \text{ given } B}{\text{Number of sample points given } B} \\
= \; & \frac{\text{Number of sample points contained by } AB}{\text{Number of sample points contained by } B}
\end{aligned}
$$

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&P(A|B)\\
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=&\ \frac{\frac{\text{Number of sample points contained by } AB}{\text{Total number of sample points}}}{\frac{\text{Number of sample points contained by } B}{\text{Total number of sample points}}}
\end{aligned}
$$

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=\ &\frac{\dfrac{\text{Number of sample points contained by } AB}{\text{Total number of sample points}}}{\dfrac{\text{Number of sample points contained by } B}{\text{Total number of sample points}}}\\
=\ &\frac{P(AB)}{P(B)}, \quad (P(B) \neq 0).
\end{aligned}
$$

In general, given that the event B has occurred, the relevant sample space is no longer Ω but consists of outcomes in B; A has occurred iff one of the outcomes in $A \cap B$ occurred, so that

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In general, given that the event B has occurred, the relevant sample space is no longer Ω but consists of outcomes in B; A has occurred iff one of the outcomes in $A \cap B$ occurred, so that

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The propotionality constant $c = 1/P(B)$ is used to ensure that the probability $P(B|B)$ of the new sample space B equals 1.

Definition 1 If A , B are two events and $P(B) \neq 0$, then the conditional probability of A given B, written as $P(A|B)$, is defined to be

$$P(A|B) = \frac{P(AB)}{P(B)}.$$

Properties of conditional probability

$P(\cdot|B) : \mathcal{F} \to [0,1]$:

1. (non-negativity) $P(A|B) \geq 0$ for all $A \in \mathcal{F}$;
2. (normalization condition) $P(\Omega|B) = 1$;
3. (countable additivity) If $A_1, \cdots, A_n, \cdots$ are mutually disjoint events ($A_iA_j = \emptyset$, $i \neq j$), then

$$P(\sum_{n=1}^{\infty} A_n|B) = \sum_{n=1}^{\infty} P(A_n|B).$$

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$$
\begin{aligned}
& P(A_1A_2\cdots A_n) \\
=& P(A_n|A_1A_2\cdots A_{n-1})P(A_1A_2\cdots A_{n-1}) \\
=& P(A_n|A_1A_2\cdots A_{n-1})P(A_{n-1}|A_1A_2\cdots A_{n-2}) \\
& \cdot P(A_1A_2\cdots A_{n-2}) \\
=& \cdots .
\end{aligned}
$$

$$
\begin{aligned}
P(A_1A_2\cdots A_n) =& P(A_1)\cdot P(A_2|A_1)\cdot P(A_3|A_1A_2) \\
& \cdots P(A_{n-1}|A_1A_2\cdots A_{n-2}) \\
& \cdot P(A_n|A_1A_2\cdots A_{n-1}).
\end{aligned}
$$

Example Two people A and B make an appointment to meet at a park between 7 o'clock and 8 o'clock and the person who first arrives at the park will keep waiting for another for 20 minutes. Find the probability that A arrives first if they meet each other.

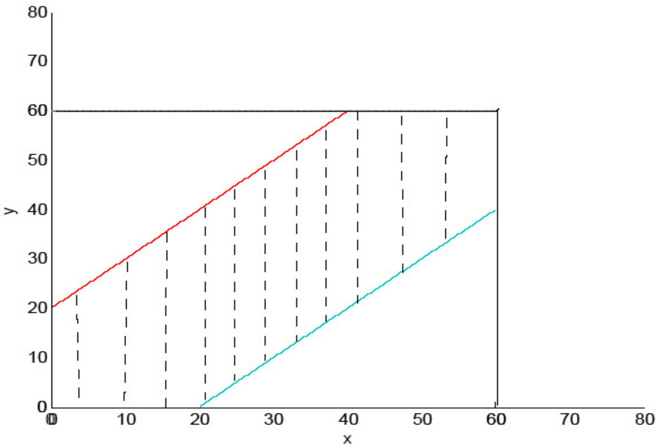
Take 7 o'clock as the beginning time and assume that A arrives at x and B arrives at y. The sample space is

$$\Omega = \{(x, y) \big| 0 \le x \le 60, 0 \le y \le 60\}$$

and

$$\begin{aligned} A =& \{\text{they meet each other}\} \\ =& \{(x, y) \big| |x - y| \le 20, 0 \le x, y \le 60\}, \end{aligned}$$

$$P(A) = \frac{5}{9}.$$


80
70
60
50
40
30
20
10
0
y
0
10
20
30
40
50
60
70
80
x

On the other hand,

$$B = \{\text{ A arrives first}\}$$
$$= \{(x, y) \big| x \le y, 0 \le x, y \le 60\}.$$

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So

$$BA = \{(x,y) \big| x \le y \le x + 20, 0 \le x, y \le 60\}.$$

On the other hand,

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So

$$BA=\{(x,y)\big|x\le y\le x+20, 0\le x,y\le 60\}.$$

$$P(BA)=\frac{\frac{1}{2}60^2-\frac{1}{2}(60-20)^2}{60^2}=\frac{5}{18}.$$

By definition, it follows that

$$P(B|A) = \frac{P(BA)}{P(A)} = \frac{1}{2}.$$

Example 2. There is a prizewinning ticket in n lottery tickets. These n lottery tickets are supposed to be sold to n different persons randomly.

(1) If the first $k-1$ customers do not get the prizewinning ticket, find the probability that the k-th customer gets the prizewinning ticket;

(2) Find the probability that the k-th customer gets the prizewinning ticket.

Solution (1). Let A_i= {the i-th customer gets the prizewinning ticket}. Then the event as condition in (1) is $\overline{A}_1\overline{A}_2\cdots\overline{A}_{k-1}$.

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$$P(A_k|\overline{A}_1\overline{A}_2\cdots\overline{A}_{k-1})=\frac{1}{n-k+1}.$$

As for (2), $A_k = \overline{A}_1\overline{A}_2\cdots\overline{A}_{k-1}A_k$ obviously holds.

So by the multiplication rule we have

$$P(A_k) = P(\overline{A}_1\overline{A}_2\cdots\overline{A}_{k-1}A_k)$$

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& P(A_k) = P(\overline{A}_1\overline{A}_2\cdots\overline{A}_{k-1}A_k) \\
= \;& P(\overline{A}_1)P(\overline{A}_2|\overline{A}_1)P(\overline{A}_3|\overline{A}_1\overline{A}_2) \\
& \quad \cdots P(A_k|\overline{A}_1\overline{A}_2\cdots\overline{A}_{k-1})
\end{aligned}
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& \quad \cdots P(A_k|\overline{A}_1\overline{A}_2\cdots\overline{A}_{k-1}) \\
= \ & \frac{n-1}{n}\cdot\frac{n-2}{n-1}\cdot\frac{n-3}{n-2}\cdots\frac{n-k+1}{n-k+2}\cdot\frac{1}{n-k+1} \\
= \ & \frac{1}{n}.
\end{aligned}
$$

Solution (2). Let A_i= {the i-th customer gets the prizewinning ticket}. The problem is equivalent to that a prizewinning ticket is assigned to one of n customers randomly. There are n assignment ways totally and only one way in A_k. So

$$P(A_k) = \frac{1}{n}.$$

Then the event $\overline{A}_1\overline{A}_2\cdots\overline{A}_{k-1}$ as condition in (1) is equivalent to that the prizewinning ticket is assigned to one of other $n-(k-1)$ customers randomly, and there are $n-k+1$ assignment ways. So

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$$P(\overline{A}_1\overline{A}_2\cdots\overline{A}_{k-1}) = \frac{n-k+1}{n}.$$

Hence, by the definition of the condition probability,

$$P(A_k|\overline{A}_1\overline{A}_2\cdots\overline{A}_{k-1}) = \frac{P(A_k\overline{A}_1\overline{A}_2\cdots\overline{A}_{k-1})}{P(\overline{A}_1\overline{A}_2\cdots\overline{A}_{k-1})}$$

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= & \frac{P(A_k)}{P(\overline{A}_1\overline{A}_2\cdots\overline{A}_{k-1})}
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= \quad & \frac{P(A_k\overline{A}_1\overline{A}_2\cdots\overline{A}_{k-1})}{P(\overline{A}_1\overline{A}_2\cdots\overline{A}_{k-1})} \\
= \quad & \frac{P(A_k)}{P(\overline{A}_1\overline{A}_2\cdots\overline{A}_{k-1})} \\
= \quad & \frac{\frac{1}{n}}{\frac{n-k+1}{n}} = \frac{1}{n-k+1}.
\end{aligned}
$$

Example

(Match problem (continued)) Suppose that a person types n letters, types the corresponding addresses on n envelopes, and then places the n letters in the n envelopes in random manner. We say that a match occurs if a letter is placed in the correct envelope. What is the probability of

(a) no matches;

(b) exactly k matches?

Solution. We denote the probability of exactly k matches by $P_k^{(n)}$.

(a) We have shown before that

$$P_0^{(n)} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + \frac{(-1)^n}{n!}.$$

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(b) To obtain the probability of exactly k matches, we consider any fixed group of k letters, say the $i_1, i_2, \cdots, i_k$-th letters.

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(b) To obtain the probability of exactly k matches, we consider any fixed group of k letters, say the $i_1, i_2, \cdots, i_k$-th letters. The probability that they, and only they, are placed in the correct envelopes is

$$
\begin{aligned}
& P(A_{i_1}\cdots A_{i_k}\overline{A}_{i_{k+1}}\cdots\overline{A}_{i_n}) \\
=& P(A_{i_1})P(A_{i_2}|A_{i_1})\cdots P(A_{i_k}|A_{i_1}\cdots A_{i_{k-1}}) \\
& \cdot P(\overline{A}_{i_{k+1}}\cdots\overline{A}_{i_n}|A_{i_1}\cdots A_{i_k}) \\
=& \frac{1}{n}\frac{1}{n-1}\cdots\frac{1}{n-(k-1)}q_{n-k} = \frac{(n-k)!}{n!}q_{n-k},
\end{aligned}
$$

where q_{n-k} is the conditional probability that the other $n-k$ letters, being placed in their own envelopes, have on matches, and so

$$
q_{n-k} = P_0^{(n-k)} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + \frac{(-1)^{n-k}}{(n-k)!}.
$$

As there are $\binom{n}{k}$ choices of a set of k letters, the desired probability of exactly k matches is

$$\begin{aligned} P_k^{(n)} &= \sum_{i_1<\cdots<i_k} P(A_{i_1}\cdots A_{i_k}\overline{A}_{i_{k+1}}\cdots\overline{A}_{i_n}) \\ &= \binom{n}{k}\cdot\frac{(n-k)!}{n!}q_{n-k} = \frac{P_0^{(n-k)}}{k!} \\ &= \frac{\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots+\frac{(-1)^{n-k}}{(n-k)!}}{k!}. \end{aligned}$$

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$$=\binom{n}{k}\cdot\frac{(n-k)!}{n!}q_{n-k}=\frac{P_0^{(n-k)}}{k!}$$

$$=\frac{\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots+\frac{(-1)^{n-k}}{(n-k)!}}{k!}.$$

It is easily seen that

$$P_k^{(n)}\to e^{-1}\frac{1}{k!}.$$

Total probability formula and Bayes' rule

Definition 2 Suppose that $\{A_1, A_2, \cdots, A_n, \cdots\}$ is a set of events satisfying: (1) $A_i, i = 1, 2, \cdots$, are mutually disjoint and $P(A_i) > 0$; (2) $\sum_{i=1}^{\infty} A_i = \Omega$. Then $\{A_1, A_2, \cdots, A_n, \cdots\}$ is called a set of *mutually exclusive* and *exhaustive* events in Ω, or a partition of Ω.

If $A_1, A_2, \cdots, A_n, \cdots$ are *mutually exclusive* and *exhaustive* events, then

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$$P(B) = P(B\sum_{i=1}^{\infty} A_i)$$

If $A_1, A_2, \cdots, A_n, \cdots$ are *mutually exclusive* and *exhaustive* events, then

$$\begin{aligned} P(B) &= P(B\sum_{i=1}^{\infty} A_i) \\ &= P(\sum_{i=1}^{\infty}(BA_i)) \end{aligned}$$

If $A_1, A_2, \cdots, A_n, \cdots$ are *mutually exclusive* and *exhaustive* events, then

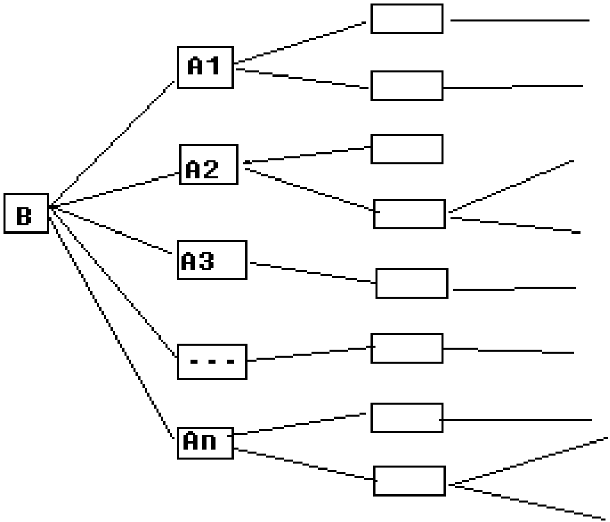
$$\begin{aligned} P(B) &= P(B\sum_{i=1}^{\infty} A_i) \\ &= P(\sum_{i=1}^{\infty}(BA_i)) \\ &= \sum_{i=1}^{\infty} P(BA_i) \end{aligned}$$

If $A_1, A_2, \cdots, A_n, \cdots$ are *mutually exclusive* and *exhaustive* events, then

$$\begin{aligned} P(B) &= P(B\sum_{i=1}^{\infty} A_i) \\ &= P(\sum_{i=1}^{\infty}(BA_i)) \\ &= \sum_{i=1}^{\infty} P(BA_i) \\ &= \sum_{i=1}^{\infty} P(A_i)P(B|A_i). \end{aligned}$$

(Total probability formula) If $A_1, A_2, \cdots, A_n, \cdots$ are *mutually exclusive* and *exhaustive* events, then for any event B

$$P(B) = \sum_{i=1}^{\infty} P(A_i)P(B|A_i).$$


B
A1
A2
A3

An

Example 4 There are 3 new balls and 2 old balls in bag. If two balls are drawn in random and in succession without replacement, find the probability that the second is a new one.

Solution Let $A = \{\text{the first ball is new}\}$, $B = \{\text{the second is new}\}$.

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$$P(B|A) = \frac{2}{4}, \quad P(B|\overline{A}) = \frac{3}{4}.$$

On the other hand

$$P(A) = \frac{3}{5}, \quad P(\overline{A}) = \frac{2}{5}.$$

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On the other hand

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So

$$P(B) = P(B|A)P(A) + P(B|\overline{A})P(\overline{A}) = \frac{3}{5}.$$

Example

(Match problem (continued)) Suppose that a person types n letters, types the corresponding addresses on n envelopes, and then places the n letters in the n envelopes in random manner. We say that a match occurs if a letter is placed in the correct envelope.

What is the probability of no matches?

Solution (2). We denote the probability of no matches by $P_0^{(n)}$. Let $E = \{$ no matches$\}$. Let A_i be the event that the i-th letter is placed its correct envelope. Then

$$P_0^{(n)} = P(E)$$

Solution (2). We denote the probability of no matches by $P_0^{(n)}$. Let $E = \{$ no matches$\}$. Let A_i be the event that the i-th letter is placed its correct envelope. Then

$$P_0^{(n)} = P(E) \;\; = \;\; P(EA_1) + P(E\overline{A}_1)$$

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$$P(E|A_1) = 0.$$

$P(E|\overline{A}_1)$ is the probability of no matches when $n-1$ letters placed in a set of $n-1$ envelopes that does not contain the envelope of one of these letters. (Say, the 1-st letter is placed to the i-th envelope)

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This can happen in either of two mutually exclusive ways.

$P(E|\overline{A}_1)$ is the probability of no matches when $n-1$ letters placed in a set of $n-1$ envelopes that does not contain the envelope of one of these letters. (Say, the 1-st letter is placed to the i-th envelope)

This can happen in either of two mutually exclusive ways.

- Either there are no matches and the extra letter is placed to the extra envelope (this being the i-th envelope that chose first letter, in the remained $(n-2)$ letters and $(n-2)$ envelopes there are no matches),
- or there are no matches and the extra letter is not placed in the extra envelope.

The probability of the first of these events is $\frac{1}{n-1}P_0^{(n-2)}$. The probability of the second event is just $P_0^{(n-1)}$, which is seen by regarding the extra envelope as "belonging" to the extra letter. So

$$P_0^{(n)} = P(E|\overline{A}_1)\frac{n-1}{n} = \left(P_0^{(n-1)} + \frac{1}{n-1}P_0^{(n-2)}\right)\frac{n-1}{n},$$

and thus,

The probability of the first of these events is $\frac{1}{n-1}P_0^{(n-2)}$. The probability of the second event is just $P_0^{(n-1)}$, which is seen by regarding the extra envelope as "belonging" to the extra letter. So

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and thus,

$$P_0^{(n)} - P_0^{(n-1)} = -\frac{1}{n}\left(P_0^{(n-1)} - P_0^{(n-2)}\right).$$

Obviously, $P_0^{(1)} = 0$, $P_0^{(2)} = \frac{1}{2}$.

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Obviously, $P_0^{(1)} = 0$, $P_0^{(2)} = \frac{1}{2}$. Hence

$$P_0^{(n)} - P_0^{(n-1)} = -\frac{1}{n}\left(P_0^{(n-1)} - P_0^{(n-2)}\right) = \ldots = \frac{(-1)^n}{n!}.$$

It follows that

$$P_0^{(n)} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + \frac{(-1)^n}{n!}.$$

Example. (The gambler's ruin problem) On each play of the game,

$p(0 < p < 1)$—-gambler A will win one dollar from gambler B

$q = (1 - p)$—-B will win one dollar from A.

The initial fortune of A is i dollars, the initial fortune of B is $k - i$ dollars.

If one loses his all money, the game is over. Find the probability that B loses all his money.

Solution. Let p_i denote the probability that gambler A will win the game (the gambler B will ruin), given that his initial fortune is i dollars. Obviously, $p_0 = 0$ and $p_k = 1$.

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Obviously, $p_0 = 0$ and $p_k = 1$.

Let $A_1(B_1)$ denote the event that gambler A wins (resp. losses) one dollar on the first play of the game; and let W denote the event that gambler A will win the game. Then

$$P(W) = P(A_1)P(W|A_1) + P(B_1)P(W|B_1)$$

$$
\begin{aligned}
P(W) &= P(A_1)P(W|A_1) + P(B_1)P(W|B_1) \\
&= pP(W|A_1) + qP(W|B_1).
\end{aligned}
$$

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That is

$$p_i = pp_{i+1} + qp_{i-1}.$$

So

$$p_i - p_{i-1} = \frac{q}{p}(p_{i-1} - p_{i-2}) = (\frac{q}{p})^{i-1} p_1, \quad i = 2, \cdots, k.$$

Taking summation on both sides yields

$$
\begin{aligned}
1 - p_1 &= p_1 \sum_{i=1}^{k-1} \left(\frac{q}{p}\right)^i \\
&= \begin{cases} p_1 \frac{(q/p)^k - q/p}{q/p - 1}, & \text{if } q \neq p, \\ (k-1)p_1, & \text{if } q = p. \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
1-p_1 &= p_1\sum_{i=1}^{k-1}(\frac{q}{p})^i \\
&= \begin{cases} p_1\frac{(q/p)^k-q/p}{q/p-1}, & \text{if } q\neq p, \\ (k-1)p_1, & \text{if } q=p. \end{cases}
\end{aligned}
$$

Hence

$$
p_1=\begin{cases} \frac{q/p-1}{(q/p)^k-1}, & \text{if } p\neq 1/2, \\ 1/k, & \text{if } p=1/2. \end{cases}
$$

So, if $p \neq 1/2$, then

$$p_i = \frac{(q/p)^i - 1}{(q/p)^k - 1}, \qquad \text{for } i = 1, \cdots, k-1;$$

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if $p = 1/2$, then

$$p_i = \frac{i}{k}, \qquad \text{for } i = 1, \cdots, k-1.$$

(Bayes's rule) If $A_1, A_2, \cdots, A_n, \cdots$ are *mutually exclusive* and *exhaustive* events, then for any event B with $P(B) > 0$ we have

$$P(A_i|B) = \frac{P(BA_i)}{P(B)} = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{\infty} P(B|A_j)P(A_j)}.$$

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$P(A_i)$ — priori probability(先验概率),

$P(A_i|B)$ — a posteriori probability(后验概率).

例: 医生: 如果患者患A病的可能性$\geq 85\%$, 则建议立即做手续; 否则就建议做一些(昂贵)的检查;

Jonhson: 开始时, 得A病的可能性为60%, 做了一项检查B呈阳性;

但同时得知他患有糖尿病, 糖尿病导致检查B呈阳性的可能性为25%.

问医生是建议Jonhson立即做手术? 还是做更多昂贵的检查?

解: 用A表示Jonhson患有A病, B表示检查B 呈阳性, 已知$P(A)=0.6$. 如果患A病,则B呈阳性, 即$P(B|A)=1$.

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$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B|A)}{P(B|A)P(A)+P(B|\overline{A})P(\overline{A})}$$

解: 用A表示Jonhson患有A病，B表示检查B 呈阳性，已知$P(A)=0.6$. 如果患A病,则B呈阳性，即$P(B|A)=1$. 在通常情况下，如果病人没有患A 病，检查不会呈阳性，即$P(B|\overline{A})=0$，但Jonhson患有糖尿病，也可能导致检查呈阳性，这时$P(B|\overline{A})=0.25$. 从而

$$\begin{aligned} P(A|B) &= \frac{P(AB)}{P(B)} = \frac{P(A)P(B|A)}{P(B|A)P(A)+P(B|\overline{A})P(\overline{A})} \\ &= \frac{1\times 0.6}{1\times 0.6+0.25\times 0.4} = 0.857. \end{aligned}$$

解: 用A表示Jonhson患有A病, B表示检查B 呈阳性, 已知$P(A)=0.6$. 如果患A病,则B呈阳性, 即$P(B|A)=1$. 在通常情况下, 如果病人没有患A 病, 检查不会呈阳性, 即$P(B|\overline{A})=0$, 但Jonhson患有糖尿病, 也可能导致检查呈阳性, 这时$P(B|\overline{A})=0.25$. 从而

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因此医生应该建议Jonhson立即做手术.

Example 7 A doctor uses to diagnose patients in order to see whether they suffer from liver cancer. Let C be the event that a patient suffers from liver cancer, A the event that a patient is diagnosed suffering from liver cancer (阳性). Suppose

$$P(A|C) = 0.95,\ P(A|\overline{C}) = 0.01(\text{假阳性}),$$

$P(C) = 0.0001$, find the probability that one patient diagnosed suffering from liver cancer suffers truly from liver cancer.

Solution: According to Bayes' formula, we have

$$P(C|A) =$$

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$$P(C|A) = \frac{P(C)\cdot P(A|C)}{P(C)\cdot P(A|C) + P(\overline{C})\cdot P(A|\overline{C})}.$$

Solution: According to Bayes' formula, we have

$$P(C|A) = \frac{P(C)\cdot P(A|C)}{P(C)\cdot P(A|C) + P(\overline{C})\cdot P(A|\overline{C})}.$$

In addition,

$$P(\overline{C}) = 1 - P(C) = 0.9999,$$

$$P(A|\overline{C}) = 0.01.$$

Solution: According to Bayes' formula, we have

$$P(C|A) = \frac{P(C) \cdot P(A|C)}{P(C) \cdot P(A|C) + P(\overline{C}) \cdot P(A|\overline{C})}.$$

In addition,

$$P(\overline{C}) = 1 - P(C) = 0.9999,$$

$$P(A|\overline{C}) = 0.01.$$

Substituting these numerical values into Bayes's formula

$$P(C|A) = 0.0094.$$

例8 某工厂有四条流水线生产同一种产品,其中每条流水线产量分别占总产量的12%,25%,25%和38%. 根据经验,每条流水线的不合格率分别为0.06, 0.05, 0.04, 0.03. 某客户够买该产品后, 发现是不合格品, 向厂家提出索赔10000元. 按规定, 工厂要求四条流水线共同承担责任. 问每条流水线应该各赔付多少?

解: 用B表示“任取一件产品为不合格产品”，A_i表示“任取一件产品是第i流水线生产的”，$i = 1, 2, 3, 4$.

解: 用B表示“任取一件产品为不合格产品”，A_i表示“任取一件产品是第i流水线生产的”，$i=1,2,3,4.$由题意得

$$P(B)=\sum_{i=1}^{4}P(B|A_i)P(A_i)$$

$$=0.12\times0.06+0.25\times0.05+0.25\times0.04+0.38\times0.03$$

$$=0.0411.$$

上式表明该工厂产品不合格率为4.11%.

现在客户发现所购买产品为不合格品, 即B发生了,我们要分析其发生的原因,计算条件概率$P(A_i|B)$, 并按其大小比例赔付客户.

现在客户发现所购买产品为不合格品，即B发生了,我们要分析其发生的原因,计算条件概率$P(A_i|B)$，并按其大小比例赔付客户. 由Bayes公式得

$$P(A_1|B) = \frac{P(B|A_1)P(A_1)}{P(B)} = \frac{0.12 \times 0.06}{0.0411} \simeq 0.175.$$

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$$P(A_1|B) = \frac{P(B|A_1)P(A_1)}{P(B)} = \frac{0.12 \times 0.06}{0.0411} \simeq 0.175.$$

类似地

$$P(A_2|B) \simeq 0.304, \quad P(A_3|B) \simeq 0.243, \quad P(A_4|B) \simeq 0.278.$$

现在客户发现所购买产品为不合格品，即B发生了,我们要分析其发生的原因,计算条件概率$P(A_i|B)$，并按其大小比例赔付客户． 由Bayes公式得

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类似地

$$P(A_2|B) \simeq 0.304, \quad P(A_3|B) \simeq 0.243, \quad P(A_4|B) \simeq 0.278.$$

这样，每条生产线应分别赔付1750元，3040元，2430元和2780元．

Independent events

1. Independence of two events:

$$P(A|B) = P(A)$$

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$$P(AB) = P(A)P(B)$$

$$\Longrightarrow \; P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A).$$

Definition. Two events A and B are independent if

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$$P(AB) = P(A)P(B).$$

Any event is independent of $\emptyset$.

Example 9. An urn contains a black balls and b white balls. If two balls are drawn in succession and we denote by A the event that the first ball drawn is black, B the event that the second ball drawn is black. Are A and B independent of each other? Consider two different situations: (1) with replacement, (2) without replacement.

Solution. For case (1),

$$P(B|A) = P(B|\overline{A}) = \frac{a}{a+b}.$$

So

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So

$$P(B) = P(B|A)P(A) + P(B|\overline{A})P(\overline{A})$$

Solution. For case (1),

$$P(B|A) = P(B|\overline{A}) = \frac{a}{a+b}.$$

So

$$\begin{aligned} P(B) &= P(B|A)P(A) + P(B|\overline{A})P(\overline{A}) \\ &= \frac{a}{a+b}(P(A) + P(\overline{A})) = \frac{a}{a+b} \end{aligned}$$

Solution. For case (1),

$$P(B|A) = P(B|\overline{A}) = \frac{a}{a+b}.$$

So

$$\begin{aligned} P(B) &= P(B|A)P(A) + P(B|\overline{A})P(\overline{A}) \\ &= \frac{a}{a+b}(P(A) + P(\overline{A})) = \frac{a}{a+b} \\ &= P(B|A), \end{aligned}$$

which shows that A and B are independent.

For case (2), we have

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$$P(B|A) = \frac{a-1}{a+b-1}, \quad P(B|\overline{A}) = \frac{a}{a+b-1}.$$

By the formula for total probability, we have

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$$P(B|A) = \frac{a-1}{a+b-1}, \quad P(B|\overline{A}) = \frac{a}{a+b-1}.$$

By the formula for total probability, we have

$$\begin{aligned} P(B) &= \frac{a}{a+b} \cdot \frac{a-1}{a+b-1} + \frac{b}{a+b} \cdot \frac{a}{a+b-1} \\ &= \frac{a}{a+b} \neq P(B|A), \end{aligned}$$

which shows that A and B are not independent.

Example 11. Suppose A and B are two events independent of each other, show that so are A and $\overline{B}$, $\overline{A}$ and B, $\overline{A}$ and $\overline{B}$.

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Proof. Since A and B are indept.,

$P(AB) = P(A)P(B)$, then

$$P(A\overline{B}) \;=\; P(A - AB)$$

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$$\begin{aligned} P(A\overline{B}) &= P(A-AB) = P(A) - P(AB) \\ &= P(A) - P(A)P(B) \end{aligned}$$

Example 11. Suppose A and B are two events independent of each other, show that so are A and $\overline{B}$, $\overline{A}$ and B, $\overline{A}$ and $\overline{B}$.

Proof. Since A and B are indept.,

$P(AB) = P(A)P(B)$, then

$$\begin{aligned} P(A\overline{B}) &= P(A-AB) = P(A) - P(AB) \\ &= P(A) - P(A)P(B) = P(A)(1-P(B)) \end{aligned}$$

Example 11. Suppose A and B are two events independent of each other, show that so are A and $\overline{B}$, $\overline{A}$ and B, $\overline{A}$ and $\overline{B}$.

Proof. Since A and B are indept.,

$P(AB) = P(A)P(B)$, then

$$\begin{aligned} P(A\overline{B}) &= P(A - AB) = P(A) - P(AB) \\ &= P(A) - P(A)P(B) = P(A)(1 - P(B)) \\ &= P(A)P(\overline{B}). \end{aligned}$$

Definition Two σ-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ are said to independent with regard to P , if

$$P(A_1A_2) = P(A_1)P(A_2).$$

holds for arbitrary A_1, A_2 such that $A_1 \in \mathcal{F}_1$, and $A_2 \in \mathcal{F}_2$.

2. Independence of several events

Definition Events A, B and C are said to be independent if

$$\left.\begin{aligned} P(AB) &= P(A)\cdot P(B) \\ P(AC) &= P(A)\cdot P(C) \\ P(BC) &= P(B)\cdot P(C) \end{aligned}\right\} \tag{9}$$

and

$$P(ABC) = P(A)\cdot P(B)\cdot P(C).$$

Definition Suppose that $A_1, A_2, \cdots, A_n$ are n events. If for $1 \leq i < j < k < \cdots \leq n$,

$$\left.\begin{aligned} &P(A_iA_j) = P(A_i)P(A_j), \\ &P(A_iA_jA_k) = P(A_i)P(A_j)P(A_k), \\ &\cdots\cdots \\ &P(A_1A_2\cdots A_n) = P(A_1)P(A_2)\cdots P(A_n) \end{aligned}\right\} \quad (11)$$

hold, then $A_1, A_2, \cdots, A_n$ are said to be independent.

Example 13. Suppose that A_1, A_2, $\cdots$, A_n are independent, and $P(A_i) = p_i, i = 1, 2, \cdots, n$. Find the probabilities that

(1) neither of them occurs;

(2) at least one of them occurs;

(3) only one of them occurs.

Solution:

(1) {neither of them occurs}$= \overline{A}_1\ \overline{A}_2 \cdots\ \overline{A}_n$. We have

$$P(\overline{A}_1\overline{A}_2\cdots\overline{A}_n)$$

Solution:

(1) {neither of them occurs}$= \overline{A}_1\ \overline{A}_2 \cdots \overline{A}_n$. We have

$$P(\overline{A}_1\overline{A}_2\cdots\overline{A}_n) = P(\overline{A}_1)P(\overline{A}_2)\cdots P(\overline{A}_n)$$

Solution:

(1) {neither of them occurs}$= \overline{A}_1\ \overline{A}_2 \cdots \overline{A}_n$. We have

$$\begin{aligned} P(\overline{A}_1\overline{A}_2\cdots\overline{A}_n) &= P(\overline{A}_1)P(\overline{A}_2)\cdots P(\overline{A}_n) \\ &= \prod_{i=1}^{n}(1-p_i). \end{aligned}$$

(2) {at least one of them occurs}=

$A_1 \cup A_2 \cup \cdots \cup A_n = \overline{\overline{A}_1 \overline{A}_2 \cdots \overline{A}_n}$. So

(2) {at least one of them occurs}=
$A_1 \cup A_2 \cup \cdots \cup A_n = \overline{\overline{A}_1 \overline{A}_2 \cdots \overline{A}_n}$. So

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) \; = \; 1 - P(\overline{A}_1 \overline{A}_2 \cdots \overline{A}_n)$$

(2) {at least one of them occurs}$=$
$A_1 \cup A_2 \cup \cdots \cup A_n = \overline{\overline{A}_1 \overline{A}_2 \cdots \overline{A}_n}$. So

$$\begin{aligned} P(A_1 \cup A_2 \cup \cdots \cup A_n) &= 1 - P(\overline{A}_1 \overline{A}_2 \cdots \overline{A}_n) \\ &= 1 - \prod_{i=1}^{n} (1 - p_i). \end{aligned}$$

(3) {only one of them occurs}
$=\overline{A}_1\overline{A}_2\cdots\overline{A}_{n-1}A_n+\overline{A}_1\overline{A}_2\cdots A_{n-1}\overline{A}_n+\cdots+A_1\overline{A}_2\cdots\overline{A}_n$. Therefore, the desired probability is

$$P(\sum_{k=1}^{n}\overline{A}_1\overline{A}_2\cdots\overline{A}_{k-1}A_k\overline{A}_{k+1}\cdots\overline{A}_n)$$

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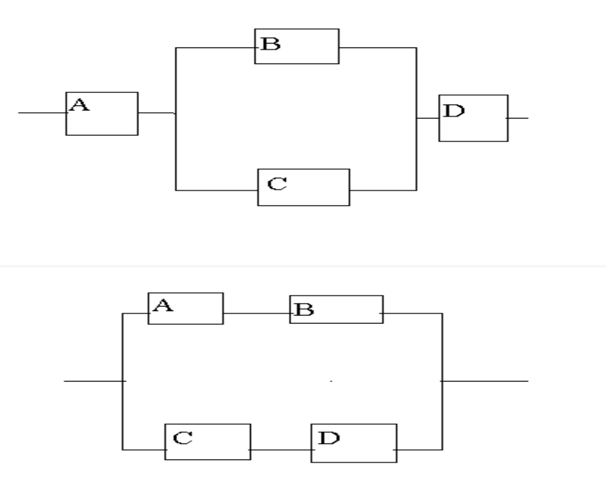
$=\overline{A}_1\overline{A}_2\cdots\overline{A}_{n-1}A_n+\overline{A}_1\overline{A}_2\cdots A_{n-1}\overline{A}_n+\cdots+A_1\overline{A}_2\cdots\overline{A}_n$. Therefore, the desired probability is

$$
\begin{aligned}
& P(\sum_{k=1}^{n}\overline{A}_1\overline{A}_2\cdots\overline{A}_{k-1}A_k\overline{A}_{k+1}\cdots\overline{A}_n) \\
= & \sum_{k=1}^{n}P(\overline{A}_1\overline{A}_2\cdots\overline{A}_{k-1}A_k\overline{A}_{k+1}\cdots\overline{A}_n)
\end{aligned}
$$

(3) $\{$only one of them occurs$\}$
$=\overline{A}_1\overline{A}_2\cdots\overline{A}_{n-1}A_n+\overline{A}_1\overline{A}_2\cdots A_{n-1}\overline{A}_n+\cdots+A_1\overline{A}_2\cdots\overline{A}_n$. Therefore, the desired probability is

$$\begin{aligned}
& P(\sum_{k=1}^{n}\overline{A}_1\overline{A}_2\cdots\overline{A}_{k-1}A_k\overline{A}_{k+1}\cdots\overline{A}_n) \\
= & \sum_{k=1}^{n}P(\overline{A}_1\overline{A}_2\cdots\overline{A}_{k-1}A_k\overline{A}_{k+1}\cdots\overline{A}_n) \\
= & \sum_{k=1}^{n}P(\overline{A}_1)P(\overline{A}_2)\cdots P(\overline{A}_{k-1})P(A_k)P(\overline{A}_{k+1})\cdots P(\overline{A}_n) \\
= & \sum_{k=1}^{n}p_k\prod_{i=1,i\neq k}^{n}(1-p_i).
\end{aligned}$$

Exmaple 14. The reliability of each component is p, find the reliability of both systems.



Solution.

$$\begin{aligned} R_1 &= P\big(A \cap (B \cup C) \cap D\big) \\ &= P\big(ABD \cup ACD\big) \end{aligned}$$

Solution.

$$\begin{aligned} R_1 &= P\big(A \cap (B \cup C) \cap D\big) \\ &= P\big(ABD \cup ACD\big) \\ &= P(ABD) + P(ACD) - P(ABCD) \end{aligned}$$

Solution.

$$\begin{aligned} R_1 &= P\big(A \cap (B \cup C) \cap D\big) \\ &= P\big(ABD \cup ACD\big) \\ &= P(ABD) + P(ACD) - P(ABCD) \\ &= P(A)P(B)P(D) + P(A)P(C)P(D) \\ &\quad - P(A)P(B)P(C)P(D) \end{aligned}$$

Solution.

$$
\begin{aligned}
R_1 &= P\big(A \cap (B \cup C) \cap D\big) \\
&= P\big(ABD \cup ACD\big) \\
&= P(ABD) + P(ACD) - P(ABCD) \\
&= P(A)P(B)P(D) + P(A)P(C)P(D) \\
&\quad -P(A)P(B)P(C)P(D) \\
&= 2p^3 - p^4.
\end{aligned}
$$

$$R_2 = P(AB \cup CD)$$

$$
\begin{aligned}
R_2 &= P(AB \cup CD) \\
&= P(AB) + P(CD) - P(ABCD)
\end{aligned}
$$

$$
\begin{aligned}
R_2 &= P(AB \cup CD) \\
&= P(AB) + P(CD) - P(ABCD) \\
&= P(A)P(B) + P(C)P(D) \\
&\quad -P(A)P(B)P(C)P(D) \\
&= 2p^2 - p^4.
\end{aligned}
$$

Example (分支过程) 设某种单性繁殖的生物群(如果是两性繁殖的生物, 只考虑男性及其男性的后代)中每个个体进行独立繁衍, 每个个体产生k个下一代个体的概率为p_k, $k = 0, 1, 2 \ldots,$ 记$m = \sum_{k=1}^{\infty} kp_k$. 设该生物群开始时(即第0代)只有一个个体. 证明: 如果$m \le 1$, $p_1 < 1$, 则这一生物群灭绝(即到某一代时个体数为0) 的概率为1.

证. 记A为该生物群灭绝这一事件, B_k表示第一代有k个个体(即第0产生的k个子代), 由全概率公式知所求的概率为

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在事件B_k的条件下, 生物群有k个个体, 而以其中任意一个个体及其后代构成的生物子群灭绝的概率仍然为q. 故$P(A|B_k) = q^k$.

所以

$$q = \sum_{k=0}^{\infty} q^k p_k.$$

即q是方程$g(s) = s$的解，其中$g(s) = \sum_{k=0}^{\infty} s^k p_k$ $(0 \le s \le 1)$. 显然，$g(1) = 1$.

所以

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即q是方程$g(s)=s$的解，其中$g(s)=\sum_{k=0}^{\infty} s^k p_k$ $(0\le s\le 1)$. 显然，$g(1)=1$. 而当$0\le s<1$时，函数$g(s)$的导数为

$$g'(s)=\sum_{k=1}^{\infty} s^{k-1} k p_k = p_1+\sum_{k=2}^{\infty} s^{k-1} k p_k.$$

如果$p_0 + p_1 < 1$, 则必有一个$p_k > 0$, $k \geq 2$, 这时

$$g'(s) = p_1 + \sum_{k=2}^{\infty} s^{k-1} k p_k < p_1 + \sum_{k=2}^{\infty} k p_k = m \leq 1;$$

如果$p_0+p_1<1$, 则必有一个$p_k>0,\ k\geq 2$, 这时

$$g'(s)=p_1+\sum_{k=2}^{\infty}s^{k-1}kp_k<p_1+\sum_{k=2}^{\infty}kp_k=m\leq 1;$$

如果$p_0+p_1=1$, 这时

$$g'(s)=p_1<1.$$

所以总是有$(g(s)-s)'<0,\ 0\leq s<1$. 从而$g(s)-s$在$[0,1]$上严格单调递减, 故$q=1$是方程$g(s)=s$的唯一解. 结论得证.

The independence of experiments

Suppose E_1, E_2,$\cdots$, E_n are n experiments, then each possible outcome of each experiment can be treated as an event. E_1, E_2, $\cdots$, E_n are said to be independent if A_1, A_2, $\cdots$, A_n are independent for any $A_1 \in E_1$, $A_2 \in E_2$, $\cdots$, $A_n \in E_n$.

Ω_i—E_i. To describe these n experiments, we construct a compound experiment

$E = (E_1, E_2, \cdots, E_n)$ with $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$,

and let sample points $\omega = (\omega^1, \cdots, \omega^n)$, where $\omega^i \in \Omega_i$.

In a compound sample space, event A^i can be represented as $\Omega_1 \times \cdots \times A^i \times \cdots \times \Omega_n$, which we still denote by A^i.

Then the independence of E_1, E_2, $\cdots$, E_n can be expressed in terms of

$$P(A^1 A^2 \cdots A^n) = P(A^1)P(A^2)\cdots P(A^n),$$

for all A^i of E_i, $i = 1, 2, \cdots, n$.

Repeated independent experiments.

4. The Bernoulli model

A trial is called Bernoulli trial if there are only two possible outcomes for each trial.

Let A denote "success" and $\overline{A}$ "failure" in a Bernoulli trial, then

$$\Omega = \{\omega_1, \omega_2\}, \quad \omega_1 = A, \omega_2 = \overline{A},$$

$$\mathcal{F} = \{\emptyset, A, \overline{A}, \Omega\}.$$

Given $P(A) = p$, $(0 < p < 1)$, $P(\overline{A}) = 1 - p$.

Repeated independent Bernoulli trials are widely studied. We call this probability model the Bernoulli model.

Its sample points are $\omega = (\omega^1, \cdots, \omega^n)$, where ω^i is A or $\overline{A}$ and the total number of its sample points is 2^n.

The Bernoulli model is not a classical probability model since the probabilities of its sample points is not necessarily equal.

Example 16. Consider a Bernoulli model of n repeated independent trials. Let A_k={A occurs only in the first k trials}, B_k ={A occurs exactly k times}. Find (1) $P(A_k)$, (2) $P(B_k)$.

Solution. (1) It is easy to see

$$A_k = \underbrace{AA\cdots A}_{k}\underbrace{\overline{A}\overline{A}\cdots\overline{A}}_{n-k}.$$

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Since A occurs independently with probability p each trial, then $P(A_k) = p^k q^{n-k}$.

Solution. (1) It is easy to see

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Since A occurs independently with probability p each trial, then $P(A_k) = p^k q^{n-k}$.

(2) Note that

$$B_k = \underbrace{AA\cdots A}_{k}\underbrace{\overline{A}\,\overline{A}\cdots\overline{A}}_{n-k} + A\overline{A}\underbrace{A\cdots A}_{k-1}\underbrace{\overline{A}\,\overline{A}\cdots\overline{A}}_{n-k-1}$$

$$+\cdots+\underbrace{\overline{A}\,\overline{A}\cdots\overline{A}}_{n-k}\underbrace{AA\cdots A}_{k}.$$

So

$$P(B_k) = b(k,n,p) \triangleq \binom{n}{k} p^k q^{n-k} = \frac{n!}{k!(n-k)!} p^k q^{n-k},$$

$k = 0, 1, 2, \cdots, n$, which appear in the expansion $(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$ with the total sum 1.

So call $b(k,n,p)$ the binomial distribution.

例 考察由投掷两个均匀的骰子组成的独立重复试验，问两个骰子点数之和为5的结果出现在它们的点数之和为7的结果之前的概率是多少?

解法1: 令E_n表示前$n-1$次试验5点和7点都没有出现而在第n次试验出现了5点这一事件,

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$$P(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} P(E_n).$$

每次试验中5点的出现的概率为$P(F)=\frac{4}{36}$，而7点的出现的概率为$P(S)=\frac{6}{36}$.

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每次试验中5点的出现的概率为$P(F) = \frac{4}{36}$，而7点的出现的概率为$P(S) = \frac{6}{36}$. 所以

$$P(E_n) = (1 - P(F) - P(S))^{n-1} P(F).$$

从而有

$$P(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} (1 - P(F) - P(S))^{n-1} P(F)$$
$$= \frac{P(F)}{P(F) + P(S)} = \frac{2}{5}.$$

解法2: 令E表示5点出现在7点之前这一事件, F表示第一次试验结果为5点, S表示第一次试验结果为7点, O表示第一次试验结果为其它的点.

解法2: 令E表示5点出现在7点之前这一事件, F表示第一次试验结果为5点, S表示第一次试验结果为7点, O表示第一次试验结果为其它的点. 那么

$$P(E) = P(E|F)P(F)+P(E|S)P(S)+P(E|O)P(O).$$

显然,

$$P(E|F) = 1, \quad P(E|S) = 0, \quad P(E|O) = P(E).$$

所以

$$P(E) = P(F) + P(E)(1 - P(F) - P(S)).$$

因此

$$P(E) = \frac{P(F)}{P(F) + P(S)}.$$

Example 10. One has two boxes of matches, each having n matches, in his pocket. Each time he wants to use match, he will randomly take out a box and draw one match from it. When he finds the box he takes out is empty, find the probability that the other box has just m matches.

Solution. The desired probability is

$$\begin{aligned} P =&P(\{\text{box A is empty, box B has } m \text{ matches}\}) \\ &+ P(\{\text{ box B is empty, box A has } m \text{ matches}\}) \\ \triangleq& P_1 + P_2. \end{aligned}$$

Consider P_1 first. When one box is empty, $2n+1-m$ drawings are considered. So

$$
\begin{aligned}
& \{ \text{ box A is empty, box B has } m \text{ matches} \} \\
= & \{ \text{ in the first } 2n+1-m \text{ drawings,} \\
& \text{box A is drawn at the } (2n+1-m)\text{-th draw} \\
& \text{and, in the first } 2n-m \text{ drawings,} \\
& \text{ box A is drawn } n \text{ times,} \\
& \text{ box B is drawn } n-m \text{ times } \}
\end{aligned}
$$

Consider it as a Bernoulli model of $2n-m+1$ repeated independent trials, where A={box A is drawn} and $\overline{A}$={box B is drawn}, and $p=P(A)=1/2$.

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$$P_1=\binom{2n-m}{n}(\frac{1}{2})^n(\frac{1}{2})^{n-m}\frac{1}{2}.$$

Consider it as a Bernoulli model of $2n-m+1$ repeated independent trials, where A={box A is drawn} and $\overline{A}$={box B is drawn}, and $p=P(A)=1/2$. Thus

$$P_1=\binom{2n-m}{n}(\frac{1}{2})^n(\frac{1}{2})^{n-m}\frac{1}{2}.$$

Similarly,

$$P_2=\binom{2n-m}{n}(\frac{1}{2})^n(\frac{1}{2})^{n-m}\frac{1}{2}.$$

Hence, the desired probability is

$$\begin{aligned} P &= 2\binom{2n-m}{n}\left(\frac{1}{2}\right)^{n}\left(\frac{1}{2}\right)^{n-m}\frac{1}{2} \\ &= \binom{2n-m}{n}\left(\frac{1}{2}\right)^{2n-m}. \end{aligned}$$