

SUPPLEMENTARY MATERIAL OF "ASYMPTOTIC PROPERTIES OF MULTI-COLOR RANDOMLY REINFORCED PÓLYA URNS"

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Appendix A. Detail proofs of the main results

The proofs are arranged as follows. We first consider the homogeneous case and prove Theorems 2.2 and 2.3 in Section 2 for the first order convergence. We then prove Theorems 4.1 and 4.2, Corollaries 4.1 and 4.2 in Section 4 for second order convergence. Finally, we consider the results in Section 5 for non-homogeneous cases. Now, let us denote $\mathcal{F}_n = \sigma(\mathbf{Y}_m, \mathbf{X}_m, U_{m,k}, m = 1, \dots, n, k = 1, \dots, K)$, which is the sigma-field that contain the history of the urn process.

A.1. Proofs of the first order asymptotic properties

Before the proofs, we need two lemmas. The first lemma can be proved using the same argument as Lemma A.4 of Hu and Zhang (2004).

Lemma A.1. *With a probability of one, on the event $\{N_{n,k} \rightarrow \infty\}$, we have*

$$Y_{n,k} = \sum_{l=1}^n X_{l,k} U_{l,k} \sim N_{n,k}, \text{ if } E[U_{1,k}] < \infty;$$

$$\sum_{l=1}^n X_{l,k} (U_{l,k} - m_k) = \begin{cases} o(N_{n,k}/\log N_{n,k}), & \text{if } E[U_{1,k} \log U_{1,k}] < \infty; \\ o(N_{n,k}^{1/p}), & \text{if } E[U_{1,k}^p] < \infty, 1 \leq p < 2; \\ o(\sqrt{N_{n,k} \log \log N_{n,k}}), & \text{if } E[U_{1,k}^2] < \infty. \end{cases}$$

The following is the key lemma for proving Theorems 2.2 and 2.3.

Lemma A.2. *Suppose that for each k , $U_{n,k}$, $n = 1, 2, \dots$, are i.i.d. nonnegative random variables with $0 < m_k = \mathbb{E}U_{n,k} < \infty$. Then*

$$\log Y_{n,k} \sim \frac{m_k}{m_{\max}} \log n \quad a.s. \quad k = 1, \dots, K, \quad (\text{A.1})$$

and there is random variable ϖ_k such that

$$Y_{n,k} \exp \left\{ - \sum_{l=1}^n \frac{m_k}{|\mathbf{Y}_{l-1}|} \right\} \rightarrow \varpi_k \quad a.s. \quad (\text{A.2})$$

Further,

$$\text{either } P(\varpi_k > 0) = 0 \quad \text{or} \quad P(\varpi_k > 0) = 1, \quad (\text{A.3})$$

$$\varpi_k > 0 \quad a.s. \iff \mathbb{E}[U_{1,k} \log U_{1,k}] < \infty. \quad (\text{A.4})$$

Proof. First, it is trivial that $|\mathbf{Y}_n| \leq \sum_{l=1}^n \sum_{k=1}^K U_{l,k} = O(n)$, hence

$$\sum_{n=1}^{\infty} \mathbb{P}(X_{n,k} = 1 | \mathcal{F}_{n-1}) \geq c \sum_{n=1}^{\infty} 1/|\mathbf{Y}_{n-1}| = \infty \quad a.s.,$$

which implies that $\mathbb{P}(X_{n,k} = 1 \text{ i.o.}) = 1$. Further, $Y_{n,k} \sim m_k N_{n,k} \rightarrow \infty$ a.s. by Lemma A.1. Write $q_{n-1} = \sum_{l=1}^n 1/|\mathbf{Y}_{l-1}|$. It is obvious that

$$\mathbb{E}[Y_{n,k} | \mathcal{F}_{n-1}] = Y_{n-1,k} \left(1 + \frac{m_k}{|\mathbf{Y}_{n-1}|} \right) \leq Y_{n-1,k} \exp \left\{ \frac{m_k}{|\mathbf{Y}_{n-1}|} \right\}.$$

It follows that $Y_{n,k} \exp\{-m_k q_{n-1}\}$ is a non-negative supermartingale and hence it converges almost surely to a finite limit according to the fundamental convergence theorem for supermartingales. Therefore, (A.2) is proved.

However, if we let $H_k(x) = \mathbb{E} \left[\frac{U_{1,k}^2}{x+U_{1,k}} \right]$, then

$$\begin{aligned} \mathbb{E} \left[\frac{Y_{n-1,k}}{Y_{n,k}} \middle| \mathcal{F}_{n-1} \right] &= \mathbb{E} \left[1 - \frac{U_{n,k} X_{n,k}}{Y_{n-1,k}} + \frac{X_{n,k}}{Y_{n-1,k}} \frac{U_{n,k}^2}{Y_{n-1,k} + U_{n-1,k}} \middle| \mathcal{F}_{n-1} \right] \\ &= 1 - \frac{m_k}{|\mathbf{Y}_{n-1}|} + \frac{H_k(Y_{n-1,k})}{|\mathbf{Y}_{n-1}|} \leq \exp \left\{ - \frac{m_k}{|\mathbf{Y}_{n-1}|} + \frac{H_k(Y_{n-1,k})}{|\mathbf{Y}_{n-1}|} \right\}. \end{aligned}$$

It follows that

$$Y_{n,k}^{-1} \exp \left\{ m_k q_{n-1} - \sum_{l=1}^n \frac{H_k(Y_{l-1,k})}{|\mathbf{Y}_{l-1}|} \right\} \text{ converges to a finite limit } a.s., \quad (\text{A.5})$$

because it is also a non-negative supermartingale. In addition, $H_k(Y_{l-1,k}) \rightarrow 0$ a.s. because $\mathbb{E}U_{1,k} < \infty$ and $Y_{l-1,k} \rightarrow \infty$ a.s. By combining (A.2) and (A.5) we conclude that

$$\log Y_{n,k} \sim m_k q_{n-1} \quad \text{a.s. } k = 1, \dots, K, \quad (\text{A.6})$$

From (A.6), it is obvious that $Y_{n,k}/|\mathbf{Y}_n| \rightarrow 0$ a.s. if $m_k < m_{\max}$. As $Y_{n,k} \sim m_k N_{n,k}$ a.s., we have

$$|\mathbf{Y}_n| \sim m_{\max} \sum_{i:m_i=m_{\max}} N_{n,i} \sim m_{\max} \sum_{i=1}^K N_{n,i} = m_{\max} n \quad \text{a.s.}, \quad (\text{A.7})$$

which together with (A.6) implies that

$$\log Y_{n,k} \sim \frac{m_k}{m_{\max}} \sum_{l=1}^n \frac{1}{l} \sim \frac{m_k}{m_{\max}} \log n \quad \text{a.s. } k = 1, \dots, K.$$

Hence, (A.1) is proven.

Finally, we show (A.3) and (A.4). Assume that $\mathbb{E}[U_{1,k} \log U_{1,k}] < \infty$. Then

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E} \left[\frac{U_{1,k}^2}{n^{m_k/(2m_{\max})} + U_{1,k}} \right] \\ & \leq \sum_{n=1}^{\infty} \frac{1}{n} \frac{n^{m_k/(4m_{\max})} \mathbb{E}U_{1,k}}{n^{m_k/(2m_{\max})}} + \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E} \left[U_{1,k} I\{U_{1,k} \geq n^{m_k/(4m_{\max})}\} \right] < \infty, \end{aligned}$$

which together with (A.7) and (A.1) implies that

$$\sum_{n=1}^{\infty} \frac{1}{|\mathbf{Y}_{n-1}|} H_k(Y_{n-1,k}) < \infty \quad \text{a.s.} \quad (\text{A.8})$$

From (A.8), (A.2) and (A.5), it follows that both $Y_{n,k} \exp\{-m_k q_{n-1}\}$ and $Y_{n,k}^{-1} \exp\{m_k q_{n-1}\}$ have finite limits, and so $\varpi_k > 0$ a.s.

Now, suppose that $m_k = \mathbb{E}[U_{1,k}] < \infty$ and $\mathbb{E}[U_{1,k} \log U_{1,k}] = \infty$. We will show that $\varpi_k = 0$ a.s. Define $U_{n,k}^{(1)} = U_{n,k} I\{U_{n,k} < n\}$,

$$Y_{n+1,k}^{(1)} = Y_{n,k}^{(1)} + X_{n+1,k} U_{n+1,k}^{(1)} \quad \text{with } Y_{0,k}^{(1)} = Y_{0,k},$$

$U_{n,k}^{(2)} = U_{n,k} - U_{n,k}^{(1)}$ and $Y_{n,k}^{(2)} = Y_{n,k} - Y_{n,k}^{(1)}$. Then $Y_{n+1,k}^{(2)} = Y_{n,k}^{(2)} + X_{n+1,k} U_{n+1,k}^{(2)}$ with $Y_{0,k}^{(2)} = 0$. Denote $m_{n,k}^{(1)} = \mathbb{E}[U_{1,k} I\{U_{1,k} < n\}]$. Then

$$\mathbb{E}[Y_{n,k}^{(1)} | \mathcal{F}_{n-1}] = Y_{n-1,k}^{(1)} + \frac{Y_{n-1,k}^{(1)}}{|\mathbf{Y}_{n-1}|} m_{n,k}^{(1)} = Y_{n-1,k}^{(1)} \left(1 + \frac{m_{n,k}^{(1)}}{|\mathbf{Y}_{n-1}|} + \frac{m_{n,k}^{(1)} Y_{n-1,k}^{(2)}}{|\mathbf{Y}_{n-1}| Y_{n-1,k}^{(1)}} \right).$$

Following the same argument as in the proof of (A.2), we find that

$$Y_{n,k}^{(1)} \exp \left\{ - \sum_{l=1}^n \frac{m_{l,k}^{(1)}}{|\mathbf{Y}_{l-1}|} - \sum_{l=1}^n \frac{m_{l,k}^{(1)} Y_{l-1,k}^{(2)}}{|\mathbf{Y}_{l-1}| Y_{l-1,k}^{(1)}} \right\}$$

converges to a finite limit almost surely. However, by $\mathbf{E}[U_{1,k}] < \infty$, $\sum_{n=1}^{\infty} \mathbf{P}(U_{n,k} \geq n) < \infty$. According to the Borel-Cantelli lemma, $\mathbf{P}(U_{n,k}^{(2)} \neq 0 \text{ i.o.}) = 0$. It follows that $Y_{n,k}^{(2)} = O(1)$ and $Y_{n,k} = Y_{n,k}^{(1)} + O(1)$ a.s. Hence

$$\sum_{l=1}^{\infty} \frac{m_{l,k}^{(1)} Y_{l-1,k}^{(2)}}{|\mathbf{Y}_{l-1}| Y_{l-1,k}^{(1)}} \leq C \sum_{l=1}^{\infty} \frac{m_k}{|\mathbf{Y}_{l-1}| Y_{l-1,k}} \leq C \sum_{l=1}^{\infty} \frac{1}{l^{1+m_k/(2m_{\max})}} < \infty \text{ a.s.}$$

by (A.7) and (A.1). It follows that

$$Y_{n,k} \exp \left\{ -m_k q_{n-1} + \sum_{l=1}^n \frac{m_{l,k}^{(2)}}{|\mathbf{Y}_{l-1}|} \right\} = Y_{n,k} \exp \left\{ - \sum_{l=1}^n \frac{m_{l,k}^{(1)}}{|\mathbf{Y}_{l-1}|} \right\} \rightarrow \zeta$$

for some $0 \leq \zeta < \infty$, where $m_{l,k}^{(2)} = \mathbf{E}[U_{1,k} I\{U_{1,k} \geq l\}]$. It can be shown that

$$\begin{aligned} \sum_{l=1}^{\infty} \frac{m_{l,k}^{(2)}}{|\mathbf{Y}_{l-1}|} &\geq c \sum_{l=1}^{\infty} \frac{m_{l,k}^{(2)}}{l} \geq c \mathbf{E} \left[\int_e^{\infty} \frac{U_{1,k} I\{U_{1,k} \geq x\}}{x} dx \right] \\ &= c \mathbf{E} [U_{1,k} (\log U_{1,k} - 1)] = \infty \text{ a.s.} \end{aligned}$$

Hence $Y_{n,k} \exp \{-m_k q_{n-1}\} \rightarrow 0$ a.s. The proof is now complete. \square

Proof of Theorem 2.2. (2.4) follows immediately from (A.2) and (A.4). By noting that $Y_{n,k} = \sum_{m=1}^n X_{m,k} U_{m,k} \sim m_k N_{n,k}$ a.s. due to Lemma A.1, (2.5) is also proven. To prove (2.6) and (2.7), without loss of generality, we suppose $m_1 = m_2 = \dots = m_{k_0} = m_{\max} > m_k > 0$, $k = k_0 + 1, \dots, K$. Due to (2.5),

$$\frac{N_{n,k}}{N_{n,1}} \rightarrow \frac{\varpi_k}{\varpi_1} \text{ a.s. } k \leq k_0 \text{ and } \frac{N_{n,k}}{N_{n,1}} \rightarrow 0 \text{ a.s. } k > k_0 + 1.$$

Notice $N_{n,1} + \dots + N_{n,K} = n$. It follows that

$$\frac{N_{n,k}}{n} \rightarrow 0 \text{ a.s. } k > k_0 + 1 \text{ and } \frac{N_{n,k}}{n} \rightarrow \frac{\varpi_k}{\varpi_1 + \dots + \varpi_{k_0}} \text{ a.s. } k \leq k_0,$$

which, together with (2.5), imply (2.7). Finally, (2.6) follows from (2.7) because $Y_{n,k} \sim m_k N_{n,k}$ a.s. \square

Proof of Theorem 2.3. $\log Y_{n,k} \sim m_k/m_{\max} \log n$ a.s. due to Lemma A.2. Hence, if $Y_{n,k}/n^{\delta_k}$ converges in distribution to a finite limit ϖ_k^* with $\mathbb{P}(\varpi_k^* > 0) > 0$, then $\delta_k = m_k/m_{\max}$. Whereas if $N_{n,k}/n^{\delta_k}$ converges in distribution to a finite limit φ_k^* , then $Y_{n,k}/n^{\delta_k}$ converges in distribution to $m_k\varphi_k^*$ by the fact that $Y_{n,k} \sim m_k N_{n,k}$ a.s. The first part of the theorem is proven.

Now, suppose that

$$Y_{n,k}/n^{m_k/m_{\max}} \xrightarrow{D} \varpi_k^* \text{ with } \mathbb{P}(\varpi_k^* > 0) > 0. \quad (\text{A.9})$$

By (A.2), we have

$$\sum_{j:m_j=m_{\max}} Y_{n,j} \exp \left\{ - \sum_{l=1}^n \frac{m_{\max}}{|\mathbf{Y}_{l-1}|} \right\} \rightarrow \sum_{j:m_j=m_{\max}} \varpi_k \text{ a.s.}$$

and $\sum_{j:m_j=m_{\max}} Y_{n,j} \sim m_{\max} \sum_{j:m_j=m_{\max}} N_{n,j} \sim m_{\max} n$ a.s. We conclude that

$$n^{1/m_{\max}} \exp \left\{ - \sum_{l=1}^n \frac{1}{|\mathbf{Y}_{l-1}|} \right\} \rightarrow \left(\frac{\sum_{j:m_j=m_{\max}} \varpi_k}{m_{\max}} \right)^{1/m_{\max}} \hat{=} \tilde{\varpi}^* \text{ a.s.}, \quad (\text{A.10})$$

where $\mathbb{P}(\tilde{\varpi}^* > 0) = 0$ or $\mathbb{P}(\tilde{\varpi}^* > 0) = 1$ by (A.3).

If $\mathbb{P}(\tilde{\varpi}^* > 0) = 0$, then by (A.9) and (A.10), we have $Y_{n,k} \exp \left\{ - \sum_{l=1}^n \frac{m_k}{|\mathbf{Y}_{l-1}|} \right\} \xrightarrow{D} 0$. It follows that $\mathbb{P}(\varpi_k > 0) = 0$ by (A.2).

If $\mathbb{P}(\tilde{\varpi}^* > 0) = 1$, then (A.10) and (A.2) imply that $Y_{n,k}/n^{m_k/m_{\max}} \rightarrow \varpi_k/(\tilde{\varpi}^*)^{m_k}$ a.s. It follows that $\mathbb{P}(\varpi_k > 0) = \mathbb{P}(\varpi_k^* > 0) > 0$ by (A.9). Hence $\mathbb{P}(\varpi_k > 0) = 1$ by (A.3). We conclude that if one of the ϖ_k , $k = 1, \dots, K$ is positive, all of them are positive, while, if one of ϖ_k , $k = 1, \dots, K$, is zero, then all of them are zero. By (A.4), the proof is complete. \square

A.2. Proofs of the limit distribution

In this subsection, we derive the limit distributions given in Theorem 3.1 and Example 3.1.

Proof of Theorem 3.1. As the number of balls is an integer, we can use the embedding method of Athreya and Ney (1972) to derive the limit. Let $\{Z(t) =$

$(Z_1(t), \dots, Z_K(t)); t \geq 0\}$ be a K -type Markov branching process with $Z_k(t)$, $k = 1, 2, \dots, K$ (the K branching processes are independent) and $\mathbf{Z}(0) = \mathbf{Y}_0$. Assume that (i) each particle lives for a unit exponential random time, and (ii) when a type k ($k = 1, 2, \dots, K$) particle dies, new type k particles are born according to the probability generating function $sf_k(s)$, i.e., the random number of born particles has the same distribution as $U_{1,k} + 1$. Let $\tau_0 = 0$ and τ_n be the time of the n -th death in the system. Then following the same argument as in Theorem 9.2 of Athreya and Ney (1972), $\{\mathbf{Z}(\tau_n); n \geq 0\}$ is equivalent to $\{\mathbf{Y}_n; n \geq 0\}$, in other words, these two random sequences have the same distribution. In fact, it is obvious that $\mathbf{Z}(0) \stackrel{D}{=} \mathbf{Y}_0$. Suppose

$$\{\mathbf{Z}(\tau_0), \dots, \mathbf{Z}(\tau_n)\} \stackrel{D}{=} \{\mathbf{Y}_0, \dots, \mathbf{Y}_n\}.$$

Given $\{\mathbf{Z}(\tau_0), \dots, \mathbf{Z}(\tau_n)\}$, the probability that the $(n+1)$ -th death is a type k particle is

$$\frac{Z_k(\tau_n)}{|\mathbf{Z}(\tau_n)|}$$

and when a type k particle dies, $1 + U_{n,k}^*$ new type k particles are born, and the distribution of $U_{n,k}^*$ is the same as that of $U_{n,k}$. Thus, for a given $\{\mathbf{Z}(\tau_0), \dots, \mathbf{Z}(\tau_n)\}$, the distribution of $\mathbf{Z}(\tau_{n+1}) - \mathbf{Z}(\tau_n)$ is the same as that of $\mathbf{Y}_{n+1} - \mathbf{Y}_n$ for a given $\{\mathbf{Y}_0, \dots, \mathbf{Y}_n\}$. Hence,

$$\{\mathbf{Z}(\tau_0), \dots, \mathbf{Z}(\tau_n), \mathbf{Z}(\tau_{n+1})\} \stackrel{D}{=} \{\mathbf{Y}_0, \dots, \mathbf{Y}_n, \mathbf{Y}_{n+1}\}.$$

Therefore, these two random sequences have the same distribution by induction.

Furthermore, it is obvious that the extinction probability of each $Z_k(t)$ is zero because $Z_k(t) \geq Z_i(0)$. By the additive property of the branching process, $Z_k(t)$ can be written as $\sum_{j=1}^{Z_k(0)} Z_k^{(j)}(t)$, where $Z_k^{(j)}(t)$, $j = 1, \dots, Z_k(0)$, are i.i.d. branching processes with $Z_k^{(j)}(0) = 1$ and the same generating function as that of $Z_k(t)$. By Theorem 8.3 of Athreya and Ney (1972) and the assumption that $E[U_{1,k} \log U_{1,k}] < \infty$, for each $Z_k^{(j)}$ there exists a positive continuous random

variable $\tilde{\varpi}_k^{(j)}$, with $\mathbb{E}[e^{-u\tilde{\varpi}_k^{(j)}}] = g_k(u)$ satisfying

$$\text{inv}g_k(u) = (1-u) \exp \left\{ \int_1^u \left(\frac{m_k}{s(f_k(s)-1)} + \frac{1}{1-s} \right) ds \right\}, \quad 0 < u \leq 1.$$

such that

$$e^{-m_k t} Z_k^{(j)}(t) \rightarrow \tilde{\varpi}_k^{(j)} \text{ a.s.}$$

It is obvious that $\tilde{\varpi}_k^{(j)}$, $j = 1, \dots, Z_k(0)$, are independent. Hence,

$$e^{-m_k t} Z_k(t) \rightarrow \tilde{\varpi}_k =: \sum_{j=1}^{Z_k(0)} \tilde{\varpi}_k^{(j)} \text{ a.s.},$$

and $\tilde{\varpi}_k$ is a positive continuous random variable with $\mathbb{E}[e^{-u\tilde{\varpi}_k}] = [g_k(u)]^{Z_k(0)} = [g_k(u)]^{Y_{0,k}}$. Further, $\tilde{\varpi}_k$, $k = 1, \dots, K$, are independent because $\{Z_k(t)\}$, $k = 1, \dots, K$, are K independent processes.

Now, it is obvious that $\tau_n \rightarrow \infty$ a.s. as $n \rightarrow \infty$. We conclude that

$$e^{-m_k \tau_n} Z_k(\tau_n) \rightarrow \tilde{\varpi}_k \text{ a.s.}$$

Hence,

$$\frac{Z_i^{1/m_i}(\tau_n)}{Z_j^{1/m_j}(\tau_n)} \rightarrow \frac{\tilde{\varpi}_i^{1/m_i}}{\tilde{\varpi}_j^{1/m_j}} \text{ a.s.}$$

By (2.4) of Theorem 2.2, it follows that

$$\frac{\tilde{\varpi}_i^{1/m_i}}{\tilde{\varpi}_j^{1/m_j}} = \frac{\varpi_i^{1/m_i}}{\varpi_j^{1/m_j}} \text{ a.s.}$$

So, without loss of generality we can assume that $\tilde{\varpi}_k = \varpi_k$. \square

Proof of Example 3.1. Notice $m_k = \alpha_k p_k$, $f_k(s) = 1 - p_k + s^{\alpha_k} p_k$. From (3.1) it follows that

$$g_k(u) = (1 + \alpha_k u)^{-1/\alpha_k},$$

which is the Laplace transform of a gamma distribution with parameters $1/\alpha_k$ and $1/\alpha_k$. It follows that the distribution of ϖ_k is gamma with the parameters $Y_{0,k}/\alpha_k$ and $1/\alpha_k$. And so, the distribution of ϖ_k/α_k is gamma with the parameters $Y_{0,k}/\alpha_k$ and 1. (3.2) and (3.3) are proved.

If $\alpha_k \equiv \alpha$, then $m_k = \alpha p_k$. Notice that $\sum_{j=1}^m \Gamma_j(Y_{0,j}/\alpha, 1)$ is distributed as $\Gamma(\sum_{j=1}^m Y_{0,j}/\alpha, 1)$. So for $k > m$, the random variable in (3.2) is distributed as

$$\frac{p_1^{p_k/p_1} \Gamma_k(Y_{0,k}/\alpha, 1)}{p_k [\Gamma(\sum_{j=1}^m Y_{0,j}/\alpha, 1)]^{p_k/p_1}},$$

and for $k = 1, \dots, m$, the random variables in (3.3) is

$$\frac{\Gamma_k(Y_{0,k}/\alpha, 1)}{\Gamma_k(Y_{0,k}/\alpha, 1) + \sum_{j \neq k} \Gamma_j(Y_{0,j}/\alpha, 1)},$$

which is distributed as $Beta(Y_{0,k}/\alpha, \sum_{j \neq k} Y_{0,j}/\alpha)$.

If $p_k \equiv p$, then $m_k = \alpha_k p$. Notice that $\sum_{j=1}^m \alpha_j \Gamma_j(Y_{0,j}/\alpha_j, 1) = \alpha_1 \sum_{j=1}^m \Gamma_j(Y_{0,j}/\alpha_1, 1)$ is distributed as $\alpha_1 \Gamma(\sum_{j=1}^m Y_{0,j}/\alpha_1, 1)$. So for $k > m$, the random variable in (3.2) is distributed as

$$p^{\alpha_k/\alpha_1 - 1} \frac{\Gamma_k(Y_{0,k}/\alpha_k, 1)}{[\Gamma(\sum_{j=1}^m Y_{0,j}/\alpha_1, 1)]^{\alpha_k/\alpha_1}},$$

and for $k = 1, \dots, m$, the random variables in (3.3) is

$$\frac{\Gamma_k(Y_{0,k}/\alpha_1, 1)}{\Gamma_k(Y_{0,k}/\alpha_1, 1) + \sum_{j \neq k} \Gamma_j(Y_{0,j}/\alpha_1, 1)},$$

which is distributed as $Beta(Y_{0,k}/\alpha_1, \sum_{j \neq k} Y_{0,j}/\alpha_1)$.

Finally in the two-treatment case, when $\alpha_1 p_1 = \alpha_2 p_2$, for $k = 1$ the random variable in (3.3) is

$$\frac{\alpha_1 Beta\left(\frac{Y_{0,1}}{\alpha_1}, \frac{Y_{0,2}}{\alpha_2}\right)}{\alpha_1 Beta\left(\frac{Y_{0,1}}{\alpha_1}, \frac{Y_{0,2}}{\alpha_2}\right) + \alpha_2 \left(1 - Beta\left(\frac{Y_{0,1}}{\alpha_1}, \frac{Y_{0,2}}{\alpha_2}\right)\right)},$$

where

$$Beta\left(\frac{Y_{0,1}}{\alpha_1}, \frac{Y_{0,2}}{\alpha_2}\right) = \frac{\Gamma_1(Y_{0,1}/\alpha_1, 1)}{\Gamma_1(Y_{0,1}/\alpha_1, 1) + \Gamma_2(Y_{0,2}/\alpha_2, 1)}$$

is distributed as the beta distribution with the parameters given in the parentheses. This result is consistent with the new distribution in §6.2 of Aletti, May and Secchi (2012).

A.3. Proofs of the second order asymptotic properties

To prove the second order convergence we need the following central limit theorem for martingale vectors which is a multi-dimensional version of Corollary 3.1 of Hall and Heyde (1980, p.58) and can be obtained using the Cramér-Wold device.

Lemma A.3. *Let $\{\zeta_{n,i} = (\zeta_{n,i}^{(1)}, \dots, \zeta_{n,i}^{(K)}), \mathcal{A}_{n,0}, \mathcal{A}_{n,i}; 1 \leq i \leq k_n\}$ be an array of martingale differences with $\mathcal{A}_{n,i} \subset \mathcal{A}_{n+1,i}$, $0 \leq i \leq k_n$, $n \geq 1$,*

$$\sum_i E[\|\zeta_{n,i}\|^2 I\{\|\zeta_{n,i}\| \geq \epsilon\} | \mathcal{A}_{n,i-1}] \xrightarrow{P} 0, \quad \forall \epsilon > 0,$$

$$\mathbf{V}_n =: \sum_i E[(\zeta_{n,i})' \zeta_{n,i} | \mathcal{A}_{n,i-1}] \xrightarrow{P} \mathbf{V} := (V_{ij}).$$

Then $\sum_{i=1}^{k_n} \zeta_{n,i} \xrightarrow{D} N(\mathbf{0}, \mathbf{V})$ stably, where $N(\mathbf{0}, \mathbf{V})$ is a multi-dimensional mixing normal distribution with the characteristic function $E[\exp\{-\frac{1}{2} \sum_{i,j} t_i t_j V_{ij}\}]$.

Proof of Theorem 4.1. Recall $\delta_k = m_k/m_{\max}$. Let

$$q_{n-1} = \sum_{l=1}^n \frac{1}{|\mathbf{Y}_{l-1}|} \quad \text{and} \quad Q_{n,k} = \frac{1}{m_k} \log Y_{n,k} - q_{n-1}.$$

In addition, $N_{n,k} \approx Y_{n,k} \approx n^{\delta_k}$ a.s. due to Theorem 2.2. Also,

$$\begin{aligned} \frac{1}{m_k} \log(m_k N_{n,k}) - \frac{1}{m_k} \log(Y_{n,k}) &= -\frac{1}{m_k} \log\left(1 + \frac{\sum_{l=1}^n X_{l,k}(U_{l,k} - m_k)}{m_k N_{n,k}}\right) \\ &= o(N_{n,k}^{1/p-1}) = o(Y_{n-1,k}^{1/p-1}) \quad a.s. \end{aligned}$$

due to Lemma A.1. So, according to the Taylor-expansion, it is sufficient to prove that

$$Q_{n,k} - \log \varpi_k = o(Y_{n-1,k}^{1/p-1}) \quad a.s. \quad (\text{A.11})$$

Now, we let $U_{n,k}^{(\delta_k)} = U_{n,k} I\{U_{n,k} \leq n^{\delta_k/p}\}$, $\bar{U}_{n,k}^{(\delta_k)} = U_{n,k} - U_{n,k}^{(\delta_k)}$, $f(x) = x - \log(1+x)$. Then $0 \leq f(x) \leq x^2/(1+x)$ ($x \geq 0$), $Q_{n,k} - \log \varpi_k = -\sum_{l=n+1}^{\infty} \Delta Q_{l,k}$

and

$$\begin{aligned}
\Delta Q_{n,k} &=: Q_{n,k} - Q_{n-1,k} = \frac{1}{m_k} X_{n,k} \log \left(1 + \frac{U_{n,k}}{Y_{n-1,k}} \right) - \frac{1}{|\mathbf{Y}_{n-1}|} \\
&= \frac{1}{m_k} \left[\frac{U_{n,k}}{Y_{n-1,k}} X_{n,k} - \frac{m_k}{|\mathbf{Y}_{n-1}|} - X_{n,k} f \left(\frac{U_{n,k}}{Y_{n-1,k}} \right) \right] \\
&= \frac{1}{m_k} \left[\left(\frac{U_{n,k}^{(\delta_k)}}{Y_{n-1,k}} X_{n,k} - \frac{\mathbb{E}[U_{n,k}^{(\delta_k)}]}{|\mathbf{Y}_{n-1}|} \right) + \left(\frac{\bar{U}_{n,k}^{(\delta_k)}}{Y_{n-1,k}} X_{n,k} - \frac{\mathbb{E}[\bar{U}_{n,k}^{(\delta_k)}]}{|\mathbf{Y}_{n-1}|} \right) \right. \\
&\quad \left. - X_{n,k} f \left(\frac{U_{n,k}}{Y_{n-1,k}} \right) \right] := (\Delta Q_{n,k}^{(11)} + \Delta Q_{n,k}^{(12)} - \Delta Q_{n,k}^{(2)})/m_k.
\end{aligned}$$

It is obvious that

$$\begin{aligned}
\sum_{l=1}^{\infty} Y_{l-1,k}^{1-1/p} \mathbb{E}[|\Delta Q_{l,k}^{(2)}| | \mathcal{F}_{l-1}] &\leq \sum_{l=1}^{\infty} \frac{Y_{l-1,k}^{1-1/p}}{|\mathbf{Y}_{l-1}|} \mathbb{E} \left[\frac{U_{l,k}^2}{Y_{l-1,k} + U_{l,k}} | \mathcal{F}_{l-1} \right] \\
&\leq \sum_{l=1}^{\infty} \frac{Y_{l-1,k}^{1-1/p}}{|\mathbf{Y}_{l-1}|} \frac{\mathbb{E}[U_{l,k}^p]}{Y_{l-1,k}^{p-1}} \leq C \sum_{l=1}^{\infty} l^{-1-\delta_k(p+1/p-2)} < \infty \quad a.s. \quad (A.12)
\end{aligned}$$

It follows that $\sum_{l=1}^{\infty} Y_{l-1,k}^{1-1/p} |\Delta Q_{l,k}^{(2)}| < \infty$ a.s., and hence

$$\sum_{l=n+1}^{\infty} |\Delta Q_{l,k}^{(2)}| = o(Y_{n-1,k}^{1/p-1}) \quad a.s. \quad (A.13)$$

On the other hand, $\{\Delta Q_{n,k}^{(11)}\}$ and $\{\Delta Q_{n,k}^{(12)}\}$ are both martingale differences. It is easily seen that

$$\begin{aligned}
&\sum_{l=1}^{\infty} Y_{l-1,k}^{(1-1/p)} \mathbb{E}[|\Delta Q_{l,k}^{(12)}| | \mathcal{F}_{l-1}] \\
&\leq C \sum_{l=1}^{\infty} Y_{l-1,k}^{(1-1/p)} \mathbb{E}[U_{1,k} I\{U_{1,k} \geq l^{\delta_k/p}\} | \mathbf{Y}_{l-1}] \\
&\leq C \sum_{l=1}^{\infty} l^{(1-1/p)\delta_k-1} \mathbb{E}[U_{1,k} I\{U_{1,k} \geq l^{\delta_k/p}\}] \\
&\leq C \mathbb{E} U_{1,k}^p < \infty,
\end{aligned}$$

by noticing $|\mathbf{Y}_l| \approx l$ a.s. and $Y_{l,k} \approx l^{\delta_k}$ a.s.; and also

$$\begin{aligned}
 & \sum_{l=1}^{\infty} Y_{l-1,k}^{2(1-1/p)} \mathbb{E}[|\Delta Q_{l,k}^{(11)}|^2 | \mathcal{F}_{l-1}] \\
 & \leq C \sum_{l=1}^{\infty} Y_{l-1,k}^{2(1-1/p)} \frac{\mathbb{E}[U_{1,k}^2 I\{U_{1,k} \leq l^{\delta_k/p}\}]}{|\mathbf{Y}_{l-1}| Y_{l-1,k}} \\
 & \leq C \sum_{l=1}^{\infty} l^{(1-2/p)\delta_k-1} \mathbb{E}[U_{1,k}^2 I\{U_{1,k} \leq l^{\delta_k/p}\}] \\
 & \leq CEU_{1,k}^p < \infty.
 \end{aligned}$$

We conclude that

$$\sum_{l=1}^{\infty} Y_{l-1,k}^{(1-1/p)} \mathbb{E}[|\Delta Q_{l,k}^{(12)}| | \mathcal{F}_{l-1}] < \infty, \quad \sum_{l=1}^{\infty} Y_{l-1,k}^{2(1-1/p)} \mathbb{E}[|\Delta Q_{l,k}^{(11)}|^2 | \mathcal{F}_{l-1}] < \infty.$$

It follows that $\sum_{l=1}^{\infty} Y_{l-1}^{(1-1/p)} \Delta Q_{l,k}^{(i)}$ converges almost surely, $i = 1, 2$, and hence

$$\sum_{l=n+1}^{\infty} (\Delta Q_{l,k}^{(11)} + \Delta Q_{l,k}^{(12)}) = o(Y_{n-1}^{1/p-1}) \text{ a.s.}$$

Therefore $\log \varpi_k - Q_{n,k} = \sum_{l=n+1}^{\infty} \Delta Q_{l,k} = o(Y_{n-1}^{1/p-1})$ a.s. The proof is complete. \square

Proof of Theorem 4.2. Let δ_k , q_n and $Q_{n,k}$ be defined as in the proof of Theorem 4.1. Without loss of generality, we assume that $m_1 = \dots = m_{k_0} > m_k$, $k = k_0 + 1, \dots, K$. Define $I_k = I\{m_k = m_{\max}\}$. Let $N_0(0, 1)$ be a standard normal variable which is independent of all other variables, and $\varsigma = \sum_{k=1}^{k_0} \sqrt{\eta_k/(1 + \sigma_{U,k}^2)} N_{k1}(0, 1) + \sqrt{1 - \sum_{k=1}^{k_0} \eta_k/(1 + \sigma_{U,k}^2)} N_0(0, 1)$. Then ς is a standard normal variable such that $\mathbb{E}[\varsigma N_{k1}(0, 1) | \eta_k] = \sqrt{\eta_k/(1 + \sigma_{U,k}^2)}$ for a given η_k , $k = 1, \dots, k_0$.

According to the delta-method, it is sufficient to show that

$$\begin{aligned}
 \sqrt{n^{\delta_k}} (Q_{n,k} - \log \varpi_k) & \xrightarrow{D} \frac{1}{m_k \sqrt{\eta_k}} \sqrt{1 + \sigma_{U,k}^2} N_{k1}(0, 1) - \frac{\varsigma I_k}{m_k} \text{ (stably)}, \\
 \sqrt{n^{\delta_k}} (\log(m_k N_{n,k}) - q_{n-1} - Q_{n,k}) & \xrightarrow{D} \frac{1}{m_k \sqrt{\eta_k}} \sigma_{U,k} N_{k2}(0, 1) \text{ (stably)},
 \end{aligned}$$

for $k = 1, \dots, K$. Note that (A.13) and (A.12) also hold for $p = 2$. It follows that

$$Q_{n,k} - \log \varpi_k = - \sum_{l=n+1}^{\infty} \frac{1}{m_k} \Delta Q_{l,k}^{(1)} + o(n^{-\delta_k/2}) \text{ a.s.},$$

where $\Delta Q_{n,k}^{(1)} = \Delta Q_{n,k}^{(11)} + \Delta Q_{n,k}^{(12)} = X_{n,k} U_{n,k} / Y_{n-1,k} - m_k / |\mathbf{Y}_{n-1}|$, and

$$\begin{aligned} \frac{1}{m_k} \log(m_k N_{n,k}) - q_{n-1} &= Q_{n,k} - \frac{1}{m_k} \log \left(1 + \frac{\sum_{l=1}^n X_{l,k} (U_{l,k} - m_k)}{m_k N_{n,k}} \right) \\ &= Q_{n,k} - \frac{1}{m_k} \frac{\sum_{l=1}^n X_{l,k} (U_{l,k} / m_k - 1)}{\eta_k n^{\delta_k}} + o(n^{-\delta_k/2}) \text{ a.s.} \end{aligned}$$

Hence, it is sufficient to show that

$$\frac{\sum_{l=1}^n X_{l,k} (U_{l,k} - m_k)}{\sqrt{n^{\delta_k}}} \xrightarrow{D} \sqrt{\eta_k} m_k \sigma_{U,k} N_{k2}(0, 1) \text{ (stably)}, \quad (\text{A.14})$$

$$\sqrt{n^{\delta_k}} \sum_{l=n+1}^{\infty} \Delta Q_{l,k}^{(1)} \xrightarrow{D} \frac{1}{\sqrt{\eta_k}} \sqrt{1 + \sigma_{U,k}^2} N_{k1}(0, 1) - \varsigma I_k \text{ (stably)}. \quad (\text{A.15})$$

For the martingale differences $\{\Delta Q_{n,k}^{(1)}\}$, we have

$$\begin{aligned} & n^{\delta_k} \sum_{l=n+1}^{\infty} \mathbb{E} \left[(\Delta Q_{n,k}^{(1)})^2 \mid \mathcal{F}_{l-1} \right] \\ &= n^{\delta_k} \sum_{l=n+1}^{\infty} \left[\frac{\mathbb{E} U_{1,k}^2}{|\mathbf{Y}_{l-1}| Y_{l-1,k}} - \frac{m_k^2}{|\mathbf{Y}_{l-1}|^2} \right] \\ &= n^{\delta_k} \sum_{l=n+1}^{\infty} (1 + o(1)) \left[\frac{\mathbb{E} U_{1,k}^2}{m_{\max} \psi_k l \cdot l^{\delta_k}} - \frac{m_k^2}{m_{\max}^2 l^2} \right] \\ &= (1 + o(1)) \frac{\mathbb{E}[U_{1,k}^2]}{m_{\max} \psi_k \delta_k} - (1 + o(1)) \frac{m_k^2 n^{\delta_k}}{m_{\max}^2 n} \\ &= (1 + o(1)) \frac{\mathbb{E}[U_{1,k}^2]}{\eta_k m_k^2} - (1 + o(1)) \frac{m_k^2 n^{\delta_k}}{m_{\max}^2 n} \\ &\rightarrow \begin{cases} (\sigma_{U,k}^2 + 1) / \eta_k \text{ a.s.}, & \text{if } m_k \neq m_{\max}, \\ (\sigma_{U,k}^2 + 1) / \eta_k - 1 \text{ a.s.}, & \text{if } m_k = m_{\max}. \end{cases} \end{aligned}$$

and

$$\begin{aligned}
 & \sqrt{n^{\delta_k} n^{\delta_j}} \sum_{l=n+1}^{\infty} \mathbb{E} \left[(\Delta Q_{n,k}^{(1)}) (\Delta Q_{n,j}^{(1)}) | \mathcal{F}_{l-1} \right] \\
 &= \sqrt{n^{\delta_k + \delta_j}} \sum_{l=n+1}^{\infty} \left[-\frac{m_k m_j}{|\mathbf{Y}_{l-1}|^2} \right] \\
 &= -\sqrt{n^{\delta_k + \delta_j}} \sum_{l=n+1}^{\infty} (1 + o(1)) \frac{m_k m_j}{m_{\max}^2 l^2} \\
 &= -(1 + o(1)) \frac{m_k m_j}{m_{\max}^2} n^{\delta_k/2 + \delta_j/2 - 1} \\
 &\rightarrow \begin{cases} -1 \text{ a.s.}, & \text{if } m_k = m_j = m_{\max}, k \neq j, \\ 0 \text{ a.s.}, & \text{otherwise.} \end{cases}
 \end{aligned}$$

On the other hand, the martingales $\sum_{l=1}^n X_{l,k}(U_{l,k} - m_k)$, $k = 1, \dots, K$, are uncorrelated among themselves, and also uncorrelated with all $\sum_{l=n+1}^{\infty} \Delta Q_{l,j}^{(1)}$, $j = 1, \dots, K$. Further,

$$\begin{aligned}
 n^{-\delta_k} \sum_{l=1}^n \mathbb{E}[X_{l,k}(U_{l,k} - m_k)^2 | \mathcal{F}_{l-1}] &= n^{-\delta_k} \sum_{l=1}^n \frac{Y_{l-1,k}}{|\mathbf{Y}_{l-1}|} \sigma_{U,k}^2 m_k^2 \\
 &= n^{-\delta_k} \sum_{l=1}^n (1 + o(1)) \frac{\psi_k l^{\delta_k}}{m_{\max} l} \sigma_{U,k}^2 m_k^2 \\
 &\rightarrow \frac{\psi_k}{m_{\max} \delta_k} \sigma_{U,k}^2 m_k^2 = \eta_k \sigma_{U,k}^2 m_k^2 \text{ a.s.}
 \end{aligned}$$

Next, we check the Linderberg condition. First, we have

$$\begin{aligned}
 & n^{-\delta_k} \sum_{l=1}^n \mathbb{E}[X_{l,k}(U_{l,k} - m_k)^2 I\{X_{l,k}(U_{l,k} - m_k)^2 \geq \epsilon n^{\delta_k} | \mathcal{F}_{l-1}\}] \\
 &= n^{-\delta_k} \sum_{l=1}^n \frac{Y_{l-1,k}}{|\mathbf{Y}_{l-1}|} \mathbb{E} \left[(U_{l,k} - m_k)^2 I \left\{ \frac{Y_{l-1,k}}{|\mathbf{Y}_{l-1}|} (U_{l,k} - m_k)^2 \geq \epsilon n^{\delta_k} \right\} \middle| \mathcal{F}_{l-1} \right] \\
 &\leq C n^{-\delta_k} \sum_{l=1}^n \frac{l^{\delta_k}}{l} \mathbb{E} \left[(U_{l,k} - m_k)^2 I \left\{ (U_{l,k} - m_k)^2 \geq \epsilon c l \right\} \middle| \mathcal{F}_{l-1} \right] \rightarrow 0 \text{ a.s.}
 \end{aligned}$$

Also,

$$\begin{aligned}
& \sum_{l=n+1}^{\infty} \mathbb{E} \left[(\sqrt{n^{\delta_k}} X_{l,k} U_{l,k} / Y_{l-1,k})^2 I \left\{ (\sqrt{n^{\delta_k}} X_{l,k} U_{l,k} / Y_{l-1,k})^2 \geq \epsilon \right\} \middle| \mathcal{F}_{l-1} \right] \\
&= n^{\delta_k} \sum_{l=n+1}^{\infty} \frac{Y_{l-1,k}}{|\mathbf{Y}_{l-1}| Y_{l-1,k}^2} \mathbb{E} \left[(U_{l,k})^2 I \left\{ \frac{Y_{l-1,k}}{|\mathbf{Y}_{l-1}|} (\sqrt{n^{\delta_k}} U_{l,k} / Y_{l-1,k})^2 \geq \epsilon \right\} \middle| \mathcal{F}_{l-1} \right] \\
&\leq C n^{\delta_k} \sum_{l=n+1}^{\infty} \frac{1}{l^{1+\delta_k}} \mathbb{E} \left[U_{l,k}^2 I \{ U_{l,k}^2 \geq \epsilon c l \} \middle| \mathcal{F}_{l-1} \right] \rightarrow 0 \text{ a.s.}
\end{aligned}$$

We conclude the Linderberg condition:

$$\begin{aligned}
& \sum_{l=n+1}^{\infty} \mathbb{E} \left[(\sqrt{n^{\delta_k}} \Delta Q_{l,k}^{(1)})^2 I \left\{ (\sqrt{n^{\delta_k}} \Delta Q_{l,k}^{(1)})^2 \geq \epsilon \right\} \middle| \mathcal{F}_{l-1} \right] \rightarrow 0 \text{ a.s.}, \\
& n^{-\delta_k} \sum_{l=1}^n \mathbb{E} [X_{l,k} (U_{l,k} - m_k)^2 I \{ X_{l,k} (U_{l,k} - m_k)^2 \geq \epsilon n^{\delta_k} \middle| \mathcal{F}_{l-1} \}] \rightarrow 0 \text{ a.s.}
\end{aligned}$$

To apply Lemma 4.3 to the central limit theorem for martingale vectors, it is sufficient to let $\mathcal{A}_{n,i} \equiv \mathcal{F}_i$ and define $\zeta_{n,i} = (\zeta_{n,i,1}, \dots, \zeta_{n,i,2K})$ as follows. For $k = 1, \dots, K$, let $\zeta_{n,i,k} = n^{-\delta_k/2} X_{i,k} (U_{i,k} - m_k)$ if $1 \leq i \leq n$, and $\zeta_{n,i,k} = 0$ if $i \geq n+1$. For $k = K+1, \dots, 2K$, let $\zeta_{n,i,k} = n^{\delta_k - \kappa/2} \Delta Q_{i,k-K}$ if $i \geq n+1$, and $\zeta_{n,i,k} = 0$ if $1 \leq i \leq n$. Then, by Lemma A.3, (A.14) and (A.15) hold. \square

Next we give the proofs of Corollaries 4.1 and 4.2.

Proof of Corollary 4.1. Let $f(x_i : i \in \Omega_j \setminus \{j\}) = 1 / (1 + \sum_{i \in \Omega_j \setminus \{j\}} x_i^{m_j} \eta_i / \eta_j)$.

Then

$$\begin{aligned}
\frac{Y_{n,j}}{\sum_{i \in \Omega_j} Y_{n,i}} &= f \left(\frac{(Y_{n,i} / \eta_i)^{1/m_i}}{(Y_{n,j} / \eta_j)^{1/m_j}} : i \in \Omega_j \setminus \{j\} \right), \\
\frac{N_{n,j}}{\sum_{i \in \Omega_j} N_{n,i}} &= f \left(\frac{(N_{n,i} / \eta_i)^{1/m_i}}{(N_{n,j} / \eta_j)^{1/m_j}} : i \in \Omega_j \setminus \{j\} \right), \\
\frac{\partial f}{\partial x_i} \Big|_{x_k=1, k \in \Omega_j \setminus \{j\}} &= - \frac{m_j \eta_i \eta_j}{(\sum_{k \in \Omega_j} \eta_k)^2}.
\end{aligned}$$

The results follow from Theorem 4.2 using the delta-method and some elementary calculations of variability. \square

Proof of Corollary 4.2. We only show the proof of (4.4) and (4.5) because the proofs of the remaining results are similar. Suppose $m_1 = \dots = m_{k_0} > m_k$

for $k > k_0$. For $i = 1, \dots, k_0$, write

$$\zeta_{n,i,j} = n^{\delta_j/2} \left(\frac{(N_{n,i}/\eta_i)^{1/m_1}}{(N_{n,j}/\eta_j)^{1/m_j}} - 1 \right).$$

Then, for $m_i = m_{\max}$,

$$\begin{aligned} \frac{N_{n,i}}{(N_{n,j}/\eta_j)^{1/\delta_j}} - \eta_i &= \eta_i \left[(\zeta_{n,i,j} n^{-\delta_j/2} + 1)^{m_1} - 1 \right] \\ &= m_{\max} \eta_i \zeta_{n,i,j} n^{-\delta_j/2} + o((\zeta_{n,i,j} n^{-\delta_j/2})^2) \\ &= m_{\max} \eta_i \zeta_{n,i,j} n^{-\delta_j/2} + o(n^{-\delta_j/2}) \quad a.s., \end{aligned}$$

due to (4.2). Note that $N_{n,1} + \dots + N_{n,K} = n$. It follows that

$$\begin{aligned} \frac{n}{(N_{n,j}/\eta_j)^{1/\delta_j}} - 1 &= \sum_{i:m_i=m_{\max}} \left(\frac{N_{n,i}}{(N_{n,j}/\eta_j)^{1/\delta_j}} - \eta_i \right) + \frac{\sum_{i:m_i < m_{\max}} N_{n,i}}{(N_{n,j}/\eta_j)^{1/\delta_j}} \\ &= m_{\max} n^{-\delta_j/2} \sum_{i:m_i=m_{\max}} \eta_i \zeta_{n,i,j} + \sum_{i:m_i=m_{\text{sec}}} \eta_i n^{\delta_{\text{sec}}-1} (1 + o(1)) + o(n^{-\delta_j/2}) \quad a.s., \end{aligned}$$

because $N_{n,i} \sim \eta_i n^{\delta_i}$ a.s. by Theorem 2.2. Hence,

$$\begin{aligned} \frac{N_{n,j}/\eta_j}{n^{\delta_j}} - 1 &= -m_{\max} \delta_j n^{-\delta_j/2} \sum_{i:m_i=m_{\max}} \eta_i \zeta_{n,i,j} - \delta_j n^{\delta_{\text{sec}}-1} \sum_{i:m_i=m_{\text{sec}}} \eta_i \\ &\quad + o(n^{\delta_{\text{sec}}-1}) + o(n^{-\delta_j/2}) \quad a.s. \end{aligned}$$

Recall the definitions of A_{ij} , B_{ij} in Theorem 4.2. Notice $\sum_{i:m_i=m_{\max}} \eta_i = 1$.

For $m_i = m_{\max} > m_j$, we have

$$\sum_{i:m_i=m_{\max}} \eta_i (A_{ij} + B_{ij}) = -\frac{1}{m_j \sqrt{\eta_j}} \left(\sqrt{1 + \sigma_{U,j}^2} N_{j1}(0, 1) + \sigma_{U,j} N_{j2}(0, 1) \right).$$

(4.5) follows from (4.3).

For (4.6), it is sufficient to notice that $Y_{n,j} = (Y_{n,j} - m_j N_{n,j}) + m_j N_{n,j}$ and

$$\frac{Y_{n,j} - m_j N_{n,j}}{\sqrt{n^{\delta_j}}} \xrightarrow{D} \sqrt{\eta_j} \sigma_{U,j} m_j N_{j2}(0, 1) \quad (\text{stably}). \quad \square$$

A.4. Proofs for the non-homogeneous case

To prove Theorem 5.1, we need another lemma.

Lemma A.4. *Suppose $\sup_n E[U_{n,k} \log^p U_{n,k} | \mathbf{Y}_0, \dots, \mathbf{Y}_{n-1}] < \infty$ a.s. for some $p > 1$. Under (5.1) or (5.3), we have that $N_{n,k} \rightarrow \infty$ a.s., $Y_{n,k} \sim m_k N_{n,k}$ a.s.,*

$$\min_k m_k \leq \liminf_{n \rightarrow \infty} \frac{|\mathbf{Y}_n|}{n} \leq \limsup_{n \rightarrow \infty} \frac{|\mathbf{Y}_n|}{n} \leq \max_k m_k \quad \text{a.s.} \quad (\text{A.16})$$

and

$$Y_{n,k} \exp \left\{ - \sum_{l=1}^n \frac{m_{l,k}}{|\mathbf{Y}_{l-1}|} \right\} \text{ converges a.s. to a positive finite limit.} \quad (\text{A.17})$$

Proof. We prove (A.16) first. Note that

$$\begin{aligned} Y_{n,k} &= \sum_{m=1}^n X_{m,k} m_{m,k} + \sum_{m=1}^n X_{m,k} (U_{m,k} - m_{m,k}) \\ &= m_k N_{n,k} + \sum_{m=1}^n X_{m,k} (m_{m,k} - m_k) + \sum_{m=1}^n X_{m,k} (U_{m,k} - m_{m,k}). \end{aligned}$$

Let $\mathcal{A}_n = \sigma(\mathbf{Y}_0, \dots, \mathbf{Y}_n, \mathbf{X}_1, \dots, \mathbf{X}_n, \mathbf{X}_{n+1})$. It is easily seen that $\{X_{n,k}(U_{n,k} - m_{n,k})\}$ is a sequence of martingale differences. Let $f(x) = x \log^p(e+x)$. Then

$$\begin{aligned} & \sum_{m=m_0}^{\infty} \mathbb{E} \left[\frac{f(|X_{n,k}(U_{n,k} - m_{n,k})|)}{f(N_{m,k})} \middle| \mathcal{A}_{m-1} \right] \\ & \leq C \sup_m \mathbb{E} [f(U_{m,k}) | \mathcal{A}_{m-1}] \sum_{m=m_0}^{\infty} \frac{X_{m,k}}{f(N_{m,k})} \\ & \leq C \sum_{m=m_0}^{\infty} \frac{N_{m,k} - N_{m-1,k}}{f(N_{m,k})} \leq C \int_{N_{m_0-1,k}}^{\infty} \frac{1}{f(x)} dx < \infty, \end{aligned}$$

if $N_{m_0-1,k} > 0$. By the law of large numbers for martingales (c.f., Stout (1974, P157)),

$$\sum_{m=1}^n X_{m,k} (U_{m,k} - m_{m,k}) = o(N_{n,k}) \quad \text{a.s. on } \{N_{n,k} \rightarrow \infty\}.$$

It follows that

$$\begin{aligned} Y_{n,k} &= m_k N_{n,k} + \sum_{m=1}^n X_{m,k} (m_{m,k} - m_k) + o(N_{n,k}) \quad \text{a.s.} \quad (\text{A.18}) \\ & \quad \text{on } \{N_{n,k} \rightarrow \infty\}. \end{aligned}$$

If the condition (5.1) (i.e., $m_{n,k} \rightarrow m_k$ a.s., $k = 1, \dots, K$) is satisfied, it is obvious that $\sum_{m=1}^n X_{m,k}(m_{m,k} - m_k) = o(n)$ a.s. If (5.3) (i.e., $\sum_n |m_{n,k} - m_k|/n < \infty$ a.s. $k = 1, \dots, K$) is satisfied, then we also have by Kronecker's Lemma

$$\left| \sum_{m=1}^n X_{m,k}(m_{m,k} - m_k) \right| \leq n \sum_{m=1}^n \frac{|m_{m,k} - m_k|}{n} = o(n) \text{ a.s.}$$

In either case, we have $Y_{n,k} = m_k N_{n,k} + o(n)$ a.s. We conclude that

$$|\mathbf{Y}_n| = \sum_{k=1}^K m_k N_{n,k} + o(n) \text{ a.s.},$$

which, together with $\sum_{k=1}^K N_{n,k} = n$, implies (A.16).

From (A.16), it follows that

$$\sum_n \mathbb{P}(X_{n+1,k} = 1 | \mathcal{F}_n) = \sum_n \frac{Y_{n,k}}{|\mathbf{Y}_n|} \geq \sum_n \frac{Y_{0,k}}{|\mathbf{Y}_n|} = \infty.$$

It follows that $\mathbb{P}(X_{n+1,k} = 1, i.o.) = 1$ which is equivalent to $N_{n,k} \rightarrow \infty$ a.s.

Next, we prove $Y_{n,k} \sim m_k N_{n,k}$ a.s. If (5.1) is satisfied, $Y_{n,k} \sim m_k N_{n,k}$ is obvious by combining $N_{n,k} \rightarrow \infty$ a.s. and (A.18). Suppose (5.3) is satisfied. Then by (A.16),

$$\sum_{l=1}^{\infty} \mathbb{E} \left[\frac{X_{l,k} |m_{l,k} - m_k|}{Y_{l-1,k}} \middle| \mathcal{F}_{l-1} \right] = \sum_{l=1}^{\infty} \frac{|m_{l,k} - m_k|}{|\mathbf{Y}_{l-1}|} < \infty \text{ a.s.}$$

It follows that

$$\sum_{m=1}^n X_{m,k}(m_{m,k} - m_k) = o(Y_{n,k}) \text{ a.s. on } \{Y_{n,k} \rightarrow \infty\}$$

and

$$\sum_{m=1}^n X_{m,k}(m_{m,k} - m_k) = O(1) \text{ a.s. on } \{\sup_n Y_{n,k} < \infty\}.$$

Combining the above argument with (A.18) and $N_{n,k} \rightarrow \infty$, we have

$$Y_{n,k} = m_k N_{n,k} + o(Y_{n,k}) + o(N_{n,k}) \text{ a.s.},$$

which implies $Y_{n,k} \sim m_k N_{n,k}$ a.s.

Finally, we show (A.17). Following the same argument given in proving (A.2) and (A.5),

$$Y_{n,k} \exp \left\{ - \sum_{l=1}^n \frac{m_{l,k}}{|\mathbf{Y}_{l-1}|} \right\} \text{ converges a.s. to a finite limit,} \quad (\text{A.19})$$

and

$$Y_{n,k}^{-1} \exp \left\{ \sum_{l=1}^n b_{l-1,k} \right\} \text{ converges a.s. to a finite limit,} \quad (\text{A.20})$$

where

$$b_{n,k} = \frac{m_{n,k}}{|\mathbf{Y}_{n-1}|} - \frac{1}{|\mathbf{Y}_{n-1}|} \mathbb{E} \left[\frac{U_{n,k}^2}{Y_{n-1,k} + U_{n,k}} \middle| \mathcal{F}_{n-1} \right].$$

It is sufficient to prove that the limit in (A.19) is strictly positive. Note that

$$\begin{aligned} \mathbb{E} \left[\frac{U_{n,k}^2}{Y_{n-1,k} + U_{n,k}} \middle| \mathcal{F}_{n-1} \right] &\leq C \frac{1}{\log^p Y_{n-1,k}} \sup_n \mathbb{E} \left[U_{n,k} \log^p U_{n,k} \middle| \mathcal{F}_{n-1} \right] \\ &\rightarrow 0 \text{ a.s.,} \end{aligned} \quad (\text{A.21})$$

which implies that

$$b_{n,k} = \frac{m_{n,k}}{|\mathbf{Y}_{n-1}|} - \frac{o(1)}{|\mathbf{Y}_{n-1}|} \text{ a.s.}$$

By (A.16), it follows that for any $0 < \delta < \min_i m_i / \max_i m_i$,

$$\sum_{l=1}^{n-1} b_{l,k} = \sum_{l=1}^{n-1} \frac{m_k}{|\mathbf{Y}_{l-1}|} (1 - o(1)) \geq \left(\frac{m_k}{\max_i m_i} - o(1) \right) \sum_{l=1}^{n-1} \frac{1}{l} > \delta \log n \text{ a.s.}$$

under either (5.1) or (5.3), so, $Y_{n,k} \geq Cn^\delta$ a.s. by (A.20). Hence, by (A.21) we have

$$\sum_{n=1}^{\infty} \frac{1}{|\mathbf{Y}_{n-1}|} \mathbb{E} \left[\frac{U_{n,k}^2}{Y_{n-1,k} + U_{n,k}} \middle| \mathcal{F}_{n-1} \right] \leq C \sum_{n=1}^{\infty} \frac{1}{n \log^p(Cn^\delta)} < \infty \text{ a.s.,}$$

which, together with (A.19) and (A.20), implies that the limit in (A.19) is a.s. positive. (A.17) is proven. \square

Proof of Theorem 5.1. If (5.1) is satisfied, then

$$\log Y_{n,k} \sim \sum_{l=1}^n \frac{m_{l,k}}{|\mathbf{Y}_{l-1}|} \sim \sum_{l=1}^n \frac{m_k}{|\mathbf{Y}_{l-1}|} \text{ a.s.} \quad (\text{A.22})$$

That is, (A.6) remains true, which together with $Y_{n,k} \sim m_k N_{n,k}$, implies (A.1). So (5.2) is proven. Finally, (5.3) and the fact that $|\mathbf{Y}_n| \approx n$ imply that $\sum_{l=1}^{\infty} \frac{|m_{l,k} - m_k|}{|\mathbf{Y}_{l-1}|} < \infty$ a.s. It follows that $Y_{n,k} \exp \left\{ - \sum_{l=1}^n \frac{m_k}{|\mathbf{Y}_{l-1}|} \right\}$ converges a.s. to a positive finite limit by (A.17). Then, (2.4)-(2.7) follow by the same argument as in the proof of Theorem 2.2. \square

Proof of Theorem 5.2. First, the conditions in the theorem imply

$$\sup_n \mathbf{E}[U_{n,k}^2 | \mathbf{Y}_0, \dots, \mathbf{Y}_{n-1}] < \infty \text{ a.s.}$$

and (5.3), and so (2.4)-(2.7) follow by Theorem 5.1.

Similar to (A.13) (where $p = 2$), we have

$$\begin{aligned} Q_{n,k} - \log \varpi_k &= - \sum_{l=n+1}^{\infty} \frac{1}{m_k} \left[X_{l,k} \frac{U_{l,k}}{Y_{l-1,k}} - \frac{m_k}{|\mathbf{Y}_{l-1}|} \right] + o(n^{-\delta_k/2}) \\ &= - \sum_{l=n+1}^{\infty} \frac{1}{m_k} \left[X_{l,k} \frac{U_{l,k}}{Y_{l-1,k}} - \frac{m_{l,k}}{|\mathbf{Y}_{l-1}|} \right] - \sum_{l=n+1}^{\infty} \frac{1}{m_k} \frac{m_{l,k} - m_k}{|\mathbf{Y}_{l-1}|} + o(n^{-\delta_k/2}) \\ &= - \sum_{l=n+1}^{\infty} \frac{1}{m_k} \left[X_{l,k} \frac{U_{l,k}}{Y_{l-1,k}} - \frac{m_{l,k}}{|\mathbf{Y}_{l-1}|} \right] + o(n^{-\delta_k/2}) \text{ a.s.} \end{aligned}$$

Also

$$\begin{aligned} \frac{1}{m_k} \log(m_k N_{n,k}) - q_{n-1} - Q_{n,k} &= - \frac{1}{m_k} \log \left(1 + \frac{\sum_{l=1}^n X_{l,k} (U_{l,k} - m_k)}{m_k N_{n,k}} \right) \\ &= - \frac{1}{m_k} \frac{\sum_{l=1}^n X_{l,k} (U_{l,k} - m_k)}{\eta_k m_k n^{\delta_k}} (1 + o(1)) \\ &= - \frac{1}{m_k} \frac{\sum_{l=1}^n X_{l,k} (U_{l,k} - m_{l,k})}{\eta_k m_k n^{\delta_k}} (1 + o(1)) - \frac{1}{m_k} \frac{\sum_{l=1}^n X_{l,k} (m_{l,k} - m_k)}{\eta_k m_k n^{\delta_k}} (1 + o(1)). \end{aligned}$$

Notice

$$\begin{aligned} \sum_{l=1}^{\infty} \mathbf{E} \left[\frac{X_{l,k} |m_{l,k} - m_k|}{l^{\delta_k/2}} \middle| \mathcal{F}_{l-1} \right] &= \sum_{l=1}^{\infty} \frac{Y_{l-1,k} |m_{l,k} - m_k|}{|\mathbf{Y}_{l-1}| l^{\delta_k/2}} \\ &\leq C \sum_{l=1}^{\infty} \frac{l^{\delta_k} |m_{l,k} - m_k|}{l \cdot l^{\delta_k/2}} \leq \sum_{l=1}^{\infty} \frac{|m_{l,k} - m_k|}{\sqrt{l}} < \infty \text{ a.s.}, \end{aligned}$$

which implies

$$\frac{\sum_{l=1}^n X_{l,k} (m_{l,k} - m_k)}{n^{\delta_k/2}} \rightarrow 0 \text{ a.s.}$$

So, it is sufficient to show that

$$\frac{\sum_{l=1}^n X_{l,k}(U_{l,k} - m_{l,k})}{\sqrt{n^{\delta_k}}} \xrightarrow{D} \sqrt{\eta_k} \sigma_{U,k} m_k N_{k2}(0, 1) \quad (\text{stably}), \quad (\text{A.23})$$

$$\sqrt{n^{\delta_k}} \sum_{l=n+1}^{\infty} \Delta \tilde{Q}_{l,k}^{(1)} \xrightarrow{D} \frac{1}{\sqrt{\eta_k}} \sqrt{1 + \sigma_{U,k}^2} N_{k1}(0, 1) - \varsigma I_k \quad (\text{stably}), \quad (\text{A.24})$$

where $\Delta \tilde{Q}_{l,k}^{(1)} = X_{l,k} U_{l,k} / Y_{l-1,k} - m_{l,k} / |\mathbf{Y}_{l-1}|$.

For the martingale differences $\{\Delta Q_{n,k}^{(1)}\}$ and $X_{l,k}(U_{l,k} - m_{l,k})$, we have

$$\begin{aligned} & n^{\delta_k} \sum_{l=n+1}^{\infty} \mathbb{E} \left[(\Delta \tilde{Q}_{n,k}^{(1)})^2 | \mathcal{F}_{l-1} \right] = n^{\delta_k} \sum_{l=n+1}^{\infty} \left[\frac{\mathbb{E}[U_{l,k}^2 | \mathcal{F}_{l-1}]}{|\mathbf{Y}_{l-1}| Y_{l-1,k}} - \frac{m_{l,k}^2}{|\mathbf{Y}_{l-1}|^2} \right] \\ &= n^{\delta_k} \sum_{l=n+1}^{\infty} \left[\frac{\mathbb{E}[(U_{1,k} - m_{l,k})^2 | \mathcal{F}_{l-1}] + m_{l,k}^2}{|\mathbf{Y}_{l-1}| Y_{l-1,k}} - \frac{m_{l,k}^2}{|\mathbf{Y}_{l-1}|^2} \right] \\ &= n^{\delta_k} \sum_{l=n+1}^{\infty} (1 + o(1)) \left[\frac{m_k^2 \sigma_{U,k}^2 + m_k^2}{m_{\max} \psi_k l \cdot l^{\delta_k}} - \frac{m_k^2}{m_{\max}^2 l^2} \right] \\ &\quad + n^{\delta_k} \sum_{l=n+1}^{\infty} \left[\frac{m_{l,k}^2 - m_k^2}{|\mathbf{Y}_{l-1}| Y_{l-1,k}} - \frac{m_{l,k}^2 - m_k^2}{|\mathbf{Y}_{l-1}|^2} \right] \\ &= (1 + o(1)) \frac{\sigma_{U,k}^2 + 1}{\eta_k} - (1 + o(1)) \frac{m_k^2 n^{\delta_k}}{m_{\max}^2 n} + o(1) \\ &\rightarrow \begin{cases} (\sigma_{U,k}^2 + 1) / \eta_k \quad a.s., & \text{if } m_k \neq m_{\max}, \\ (\sigma_{U,k}^2 + 1) / \eta_k - 1 \quad a.s., & \text{if } m_k = m_{\max}; \end{cases} \\ \\ & \sqrt{n^{\delta_k} n^{\delta_j}} \sum_{l=n+1}^{\infty} \mathbb{E} \left[(\Delta \tilde{Q}_{n,k}^{(1)}) (\Delta \tilde{Q}_{n,j}^{(1)}) | \mathcal{F}_{l-1} \right] \\ &= \sqrt{n^{\delta_k + \delta_j}} \sum_{l=n+1}^{\infty} \left[-\frac{m_{l,k} m_{l,j}}{|\mathbf{Y}_{l-1}|^2} \right] \\ &= -\sqrt{n^{\delta_k + \delta_j}} \sum_{l=n+1}^{\infty} (1 + o(1)) \frac{m_k m_j}{m_{\max}^2 l^2} + \sqrt{n^{\delta_k + \delta_j}} \sum_{l=n+1}^{\infty} \left[\frac{m_k m_j - m_{l,k} m_{l,j}}{|\mathbf{Y}_{l-1}|^2} \right] \\ &= -(1 + o(1)) \frac{m_k m_j}{m_{\max}^2} n^{\delta_k/2 + \delta_j/2 - 1} + o(1) \quad a.s. \\ &\rightarrow \begin{cases} -1 \quad a.s., & \text{if } m_k = m_j = m_{\max}, k \neq j, \\ 0 \quad a.s., & \text{otherwise;} \end{cases} \end{aligned}$$

and

$$\begin{aligned} & n^{-\delta_k} \sum_{l=1}^n \mathbb{E}[X_{l,k}(U_{l,k} - m_{l,k})^2 | \mathcal{F}_{l-1}] \\ &= n^{-\delta_k} \sum_{l=1}^n \frac{Y_{l-1,k}}{|\mathbf{Y}_{l-1}|} (\sigma_{U,k}^2 m_k^2 + o(1)) \rightarrow \eta_k \sigma_{U,k}^2 m_k^2 \text{ a.s.} \end{aligned}$$

Further, by the similar arguments as in the proof of Theorem 4.2, the Linderberg conditions can be verified:

$$\begin{aligned} & \sum_{l=n+1}^{\infty} \mathbb{E} \left[(\sqrt{n^{\delta_k}} \Delta \tilde{Q}_{l,k}^{(1)})^2 I \left\{ (\sqrt{n^{\delta_k}} \Delta \tilde{Q}_{l,k}^{(1)})^2 \geq \epsilon \right\} | \mathcal{F}_{l-1} \right] \rightarrow 0 \text{ a.s.}, \\ & n^{-\delta_k} \sum_{l=1}^n \mathbb{E}[X_{l,k}(U_{l,k} - m_{l,k})^2 I \{X_{l,k}(U_{l,k} - m_{l,k})^2 \geq \epsilon n^{\delta_k} | \mathcal{F}_{l-1}\}] \rightarrow 0 \text{ a.s.} \end{aligned}$$

Then applying Lemma A.3 we conclude (A.23) and (A.24). The proof is completed. \square

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