

Precise rates in the law of the iterated logarithm *

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ABSTRACT. Let X, X_1, X_2, \dots be i.i.d. random variables, and set $S_n = X_1 + \dots + X_n$, $M_n = \max_{k \leq n} |S_k|$, $n \geq 1$. Let $a_n = o(\sqrt{n/\log \log n})$. By using the strong approximation, we prove that, if $\mathbb{E}X^2 I\{|X| \geq t\} = o((\log \log t)^{-1})$ as $t \rightarrow \infty$, then for $a > -1$ and $b > -1$,

$$\begin{aligned} & \lim_{\epsilon \searrow \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \mathbb{P}\{M_n \geq \sigma \phi(n)\epsilon + a_n\} \\ &= 2 \sqrt{\frac{1}{\pi(a+1)}} \Gamma(b+1/2) \end{aligned}$$

holds if and only if

$$\mathbb{E}X = 0, \quad \mathbb{E}X^2 = \sigma^2 < \infty \quad \text{and} \quad \mathbb{E}[X^2(\log |X|)^a (\log \log |X|)^{b-1}] < \infty.$$

We also show that the condition $\mathbb{E}X^2 I\{|X| \geq t\} = o((\log \log t)^{-1})$ is sharp. The results of Gut and Spätaru (2000) are special cases of ours.

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1 Introduction and main results.

Let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d random variables with common distribution function F , mean 0 and positive, finite variance σ^2 , and set $S_n = \sum_{k=1}^n X_k$, $M_n = \max_{k \leq n} |S_k|$, $n \geq 1$. Also let $\log x = \ln(x \vee e)$, $\log \log x = \log(\log x)$ and $\phi(x) = \sqrt{2x \log \log x}$. Then by the well-known law of the iterated logarithm (LIL) we have

$$\limsup_{n \rightarrow \infty} \frac{M_n}{\phi(n)} = \limsup_{n \rightarrow \infty} \frac{|S_n|}{\phi(n)} = \sigma \quad a.s.. \quad (1.1)$$

Gut and Spătaru (2000) proved the following two results on its precise asymptotics.

Theorem A *Suppose that $EX = 0$, $EX^2 = \sigma^2$ and $E[X^2(\log \log |X|)^{1+\delta}] < \infty$ for some $\delta > 0$, and let $a_n = O(\sqrt{n}/(\log \log n)^\gamma)$ for some $\gamma > 1/2$. Then*

$$\lim_{\epsilon \searrow 1} \sqrt{\epsilon^2 - 1} \sum_{n=1}^{\infty} \frac{1}{n} P(|S_n| \geq \epsilon \sigma \phi(n) + a_n) = 1.$$

Theorem B *Suppose that $EX = 0$ and $EX^2 = \sigma^2 < \infty$. Then*

$$\lim_{\epsilon \searrow 0} \epsilon^2 \sum_{n=1}^{\infty} \frac{1}{n \log n} P(|S_n| \geq \epsilon \sqrt{n \log \log n}) = \sigma^2.$$

The main purpose of this paper is to show general results under the *minimal* conditions by using an Feller's (1945) and Einmahl's (1989) truncation method. The following two theorems are our main results.

Theorem 1.1 *Let $a > -1$ and $b > -1/2$ and let $a_n(\epsilon)$ be a function of ϵ such that*

$$a_n(\epsilon) \log \log n \rightarrow \tau \quad \text{as } n \rightarrow \infty \text{ and } \epsilon \searrow \sqrt{1+a}. \quad (1.2)$$

Suppose that

$$EX = 0, \quad EX^2 = \sigma^2 < \infty \quad \text{and} \quad E[X^2(\log |X|)^a (\log \log |X|)^{b-1}] < \infty \quad (1.3)$$

and

$$EX^2 I\{|X| \geq t\} = o((\log \log t)^{-1}) \quad \text{as } t \rightarrow \infty. \quad (1.4)$$

Then

$$\begin{aligned} & \lim_{\epsilon \searrow \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} P\{M_n \geq \sigma \phi(n)(\epsilon + a_n(\epsilon))\} \\ & = 2 \sqrt{\frac{1}{\pi(a+1)}} \exp\{-2\tau \sqrt{1+a}\} \Gamma(b+1/2) \end{aligned} \quad (1.5)$$

and

$$\begin{aligned} & \lim_{\epsilon \searrow \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} P\{|S_n| \geq \sigma \phi(n)(\epsilon + a_n(\epsilon))\} \\ &= \sqrt{\frac{1}{\pi(a+1)}} \exp\{-2\tau\sqrt{1+a}\} \Gamma(b+1/2). \end{aligned} \quad (1.6)$$

Here, $\Gamma(\cdot)$ is a gamma function. Conversely, if (1.5) or (1.6) holds for $a > -1$, $b > -1/2$ and some $0 < \sigma < \infty$, then (1.3) holds and

$$\liminf_{t \rightarrow \infty} (\log \log t) EX^2 I\{|X| \geq t\} = 0. \quad (1.7)$$

Theorem 1.2 Suppose that $EX = 0$ and $EX^2 = \sigma^2 < \infty$, and let $a_n = O(1/\log \log n)$. For $b > -1$, we have

$$\begin{aligned} & \lim_{\epsilon \searrow 0} \epsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^b}{n \log n} P\{M_n \geq \sigma \phi(n)(\epsilon + a_n)\} \\ &= \frac{2}{(b+1)\sqrt{\pi}} \Gamma(b+3/2) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+2}} \end{aligned} \quad (1.8)$$

and

$$\begin{aligned} & \lim_{\epsilon \searrow 0} \epsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^b}{n \log n} P\{|S_n| \geq \sigma \phi(n)(\epsilon + a_n)\} \\ &= \frac{1}{(b+1)\sqrt{\pi}} \Gamma(b+3/2). \end{aligned} \quad (1.9)$$

Conversely, if (1.8) or (1.9) holds for some $b > -1$ and $0 < \sigma < \infty$, then $EX = 0$ and $EX^2 = \sigma^2$.

Remark 1.1 Note that the condition (1.4) is sharp. A sufficient condition for it is given by

$$EX^2 \log \log |X| < \infty.$$

So, (1.4) is weaker than Gut and Spătaru's condition in Theorem A (see also their Remark 1.1).

When $a > 0$ (or $a = 0$ and $b \geq 2$), the condition (1.4) is implied by (1.3).

Remark 1.2 The condition that $EX = 0$ and $EX^2 < \infty$ is obviously sufficient and necessary for the conclusion of Theorem B to hold, by Theorem 1.2. (see also Remark 1.2 of Gut and Spătaru, 2000).

The proofs of Theorem 1.1 and 1.2 are given in Section 4. Before that, we first verify (1.5), (1.6), (1.8) and (1.9) under the assumption that F is the normal distribution in Section 2, after which, by using the truncation and approximation method, we then show that the probabilities in (1.5), (1.6), (1.8) and (1.9) can be replaced by those for normal random variables in Section 3. Throughout this paper, we let $K(\alpha, \beta, \dots)$, $C(\alpha, \beta, \dots)$ etc denote positive constants which depend on α, β, \dots only, whose values can differ in different places. $a_n \sim b_n$ means that $a_n/b_n \rightarrow 1$.

2 Normal cases.

In this section, we prove Theorems 1.1 and 1.2 in the case that $\{X, X_n; n \geq 1\}$ are normal random variables. Let $\{W(t); t \geq 0\}$ be a standard Wiener process and N a standard normal variable. Our results are as follows.

Proposition 2.1 *Let $a > -1$ and $b > -1/2$ and let $a_n(\epsilon)$ be a function of ϵ satisfying (1.2). Then*

$$\begin{aligned} & \lim_{\epsilon \searrow \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \\ & \quad \cdot \mathcal{P} \left\{ \sup_{0 \leq s \leq 1} |W(s)| \geq \sqrt{2 \log \log n} (\epsilon + a_n(\epsilon)) \right\} \\ & = 2 \sqrt{\frac{1}{\pi(a+1)}} \exp\{-2\tau\sqrt{1+a}\} \Gamma(b+1/2) \end{aligned} \quad (2.1)$$

and

$$\begin{aligned} & \lim_{\epsilon \searrow \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \mathcal{P} \left\{ |N| \geq \sqrt{2 \log \log n} (\epsilon + a_n(\epsilon)) \right\} \\ & = \sqrt{\frac{1}{\pi(a+1)}} \exp\{-2\tau\sqrt{1+a}\} \Gamma(b+1/2) \end{aligned} \quad (2.2)$$

Proposition 2.2 *Let $a_n = O(1/\log \log n)$. For any $b > -1$, we have*

$$\begin{aligned} & \lim_{\epsilon \searrow 0} \epsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^b}{n \log n} \mathcal{P} \left\{ \sup_{0 \leq s \leq 1} |W(s)| \geq (\epsilon + a_n) \sqrt{2 \log \log n} \right\} \\ & = \frac{2}{(b+1)\sqrt{\pi}} \Gamma(b+3/2) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+2}} \end{aligned}$$

and

$$\begin{aligned} & \lim_{\epsilon \searrow 0} \epsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^b}{n \log n} \mathcal{P} \left\{ |N| \geq (\epsilon + a_n) \sqrt{2 \log \log n} \right\} \\ & = \frac{1}{(b+1)\sqrt{\pi}} \Gamma(b+3/2). \end{aligned}$$

The following lemma will be used in the proofs.

Lemma 2.1 *Let $\{W(t); t \geq 0\}$ be a standard Wiener process. Then for all $x > 0$,*

$$\begin{aligned} \mathcal{P} \left(\sup_{0 \leq s \leq 1} |W(s)| \geq x \right) & = 1 - \sum_{k=-\infty}^{\infty} (-1)^k \mathcal{P}((2k-1)x \leq N \leq (2k+1)x) \\ & = 4 \sum_{k=0}^{\infty} (-1)^k \mathcal{P}(N \geq (2k+1)x) \\ & = 2 \sum_{k=0}^{\infty} (-1)^k \mathcal{P}(|N| \geq (2k+1)x). \end{aligned} \quad (2.3)$$

In particular,

$$\mathcal{P} \left(\sup_{0 \leq s \leq 1} |W(s)| \geq x \right) \sim 2\mathcal{P}(|N| \geq x) \sim \frac{4}{\sqrt{2\pi}x} e^{-x^2/2} \text{ as } x \rightarrow +\infty.$$

Proof. It is well known. See Billingsley (1968).

Now, we turn to prove the propositions.

Proof Proposition 2.1: First, note that the limit in (2.1) does not depend on any finite terms of the infinite series. Secondly, by Lemma 2.1 and the condition (1.2) we have

$$\begin{aligned} & \mathbb{P}\left\{\sup_{0 \leq s \leq 1} |W(s)| \geq \sqrt{2 \log \log n}(\epsilon + a_n(\epsilon))\right\} \sim 2\mathbb{P}\left\{|N| \geq \sqrt{2 \log \log n}(\epsilon + a_n(\epsilon))\right\} \\ & \sim \frac{4}{\sqrt{2\pi}(\epsilon + a_n(\epsilon))\sqrt{2 \log \log n}} \exp\left\{-\frac{(\epsilon + a_n(\epsilon))^2 \log \log n}{2}\right\} \\ & \sim \frac{2}{\sqrt{\pi}\epsilon} \frac{1}{\sqrt{\log \log n}} \exp\left\{-\epsilon^2 \log \log n\right\} \exp\left\{-2\epsilon a_n(\epsilon) \log \log n\right\} \end{aligned}$$

as $n \rightarrow \infty$, uniformly in $\epsilon \in (\sqrt{1+a}, \sqrt{1+a} + \delta)$ for some $\delta > 0$. So, for any $0 < \theta < 1$, there exist $\delta > 0$ and n_0 such that for all $n \geq n_0$ and $\epsilon \in (\sqrt{1+a}, \sqrt{1+a} + \delta)$,

$$\begin{aligned} & \frac{2}{\sqrt{\pi(1+a)}} \frac{1}{\sqrt{\log \log n}} \exp\left\{-\epsilon^2 \log \log n\right\} \exp\left\{-2\tau\sqrt{1+a} - \theta\right\} \\ & \leq \mathbb{P}\left\{\sup_{0 \leq s \leq 1} |W(s)| \geq \sqrt{2 \log \log n}(\epsilon + a_n(\epsilon))\right\} \\ & \leq \frac{2}{\sqrt{\pi(1+a)}} \frac{1}{\sqrt{\log \log n}} \exp\left\{-\epsilon^2 \log \log n\right\} \exp\left\{-2\tau\sqrt{1+a} + \theta\right\} \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{\sqrt{\pi(1+a)}} \frac{1}{\sqrt{\log \log n}} \exp\left\{-\epsilon^2 \log \log n\right\} \exp\left\{-2\tau\sqrt{1+a} - \theta\right\} \\ & \leq \mathbb{P}\left\{|N| \geq \sqrt{2 \log \log n}(\epsilon + a_n(\epsilon))\right\} \\ & \leq \frac{1}{\sqrt{\pi(1+a)}} \frac{1}{\sqrt{\log \log n}} \exp\left\{-\epsilon^2 \log \log n\right\} \exp\left\{-2\tau\sqrt{1+a} + \theta\right\}, \end{aligned}$$

by the condition (1.2) again. Also,

$$\begin{aligned} & \lim_{\epsilon \searrow \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \frac{1}{\sqrt{\log \log n}} \exp\left\{-\epsilon^2 \log \log n\right\} \\ & = \lim_{\epsilon \searrow \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \int_{e^\epsilon}^{\infty} \frac{(\log x)^a (\log \log x)^{b-1/2}}{x} \exp\left\{-\epsilon^2 \log \log x\right\} dx \\ & = \lim_{\epsilon \searrow \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \int_1^{\infty} y^{b-1/2} \exp\left\{-y(\epsilon^2 - 1 - a)\right\} dy \\ & = \lim_{\epsilon \searrow \sqrt{1+a}} \int_{\epsilon^2 - 1 - a}^{\infty} y^{b-1/2} e^{-y} dy = \int_0^{\infty} y^{b-1/2} e^{-y} dy = \Gamma(b + 1/2). \end{aligned}$$

(2.1) and (2.2) are now proved.

Proof Proposition 2.2: Observe (2.3),

$$\mathbb{P}(|N| \geq x) = 2\mathbb{P}(N \geq x), \quad \forall x > 0,$$

and for any $m \geq 1$ and $x > 0$,

$$\begin{aligned} 4 \sum_{k=0}^{2m+1} (-1)^k \mathbb{P}(N \geq (2k+1)x) & \leq \mathbb{P}\left(\sup_{0 \leq s \leq 1} |W(s)| \geq x\right) \\ & \leq 4 \sum_{k=0}^{2m} (-1)^k \mathbb{P}(N \geq (2k+1)x). \end{aligned}$$

It is sufficient to show that for any $q > 0$,

$$\begin{aligned} & \lim_{\epsilon \searrow 0} \epsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^b}{n \log n} \mathbf{P}\left(N \geq q(\epsilon + a_n) \sqrt{2 \log \log n}\right) \\ &= q^{-2(b+1)} \frac{1}{2(b+1)\sqrt{\pi}} \Gamma(b+3/2). \end{aligned}$$

Obviously,

$$\begin{aligned} & \lim_{\epsilon \searrow 0} \epsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^b}{n \log n} \mathbf{P}\left(N \geq q(\epsilon + a_n) \sqrt{2 \log \log n}\right) \\ &= q^{-2(b+1)} \lim_{\epsilon \searrow 0} \epsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^b}{n \log n} \mathbf{P}\left(N \geq (\epsilon + qa_n) \sqrt{2 \log \log n}\right). \end{aligned}$$

So, it is sufficient to show that

$$\lim_{\epsilon \searrow 0} \epsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^b}{n \log n} \mathbf{P}\left(N \geq (\epsilon + a_n) \sqrt{2 \log \log n}\right) = \frac{1}{2(b+1)\sqrt{\pi}} \Gamma(b+3/2).$$

Without losing of generality, we assume that $|a_n| \leq \tau / \log \log n$. Notice that

$$\begin{aligned} & \left| \mathbf{P}\left(N \geq (\epsilon + a_n) \sqrt{2 \log \log n}\right) - \mathbf{P}\left(N \geq \epsilon \sqrt{2 \log \log n}\right) \right| \\ & \leq \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{2 \log \log n (\epsilon - \tau / \log \log n)^2}{2}\right\} |a_n| \sqrt{2 \log \log n} \\ & \leq \frac{\tau}{\sqrt{\log \log n}} \exp\{-\epsilon^2 \log \log n + 2\epsilon\tau\} \end{aligned}$$

and

$$\begin{aligned} & \lim_{\epsilon \searrow 0} \epsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^b}{n \log n} \cdot \frac{1}{\sqrt{\log \log n}} \exp\{-\epsilon^2 \log \log n\} \\ &= \lim_{\epsilon \searrow 0} \epsilon^{2(b+1)} \int_{e^\epsilon}^{\infty} \frac{(\log \log x)^{b-1/2}}{x \log x} \exp\{-\epsilon^2 \log \log x\} dx \\ &= \lim_{\epsilon \searrow 0} \epsilon^{2(b+1)} \int_1^{\infty} y^{b-1/2} \exp\{-\epsilon^2 y\} dy = \lim_{\epsilon \searrow 0} \epsilon \int_{\epsilon^2}^{\infty} y^{b-1/2} e^{-y} dy \\ &= \lim_{\epsilon \searrow 0} \epsilon \int_{\epsilon^2}^1 y^{b-1/2} e^{-y} dy + \lim_{\epsilon \searrow 0} \epsilon \int_1^{\infty} y^{b-1/2} e^{-y} dy \\ & \leq \lim_{\epsilon \searrow 0} \epsilon \int_{\epsilon^2}^1 y^{b-1/2} dy = 0. \end{aligned}$$

Thus, it follows that

$$\begin{aligned} & \lim_{\epsilon \searrow 0} \epsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^b}{n \log n} \mathbf{P}\left(N \geq (\epsilon + a_n) \sqrt{2 \log \log n}\right) \\ &= \lim_{\epsilon \searrow 0} \epsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^b}{n \log n} \mathbf{P}\left(N \geq \epsilon \sqrt{2 \log \log n}\right) \\ &= \lim_{\epsilon \searrow 0} \epsilon^{2(b+1)} \int_{e^\epsilon}^{\infty} \frac{(\log \log x)^b}{x \log x} \mathbf{P}\left(N \geq \epsilon \sqrt{2 \log \log x}\right) dx \\ &= \lim_{\epsilon \searrow 0} \int_{\epsilon^2}^{\infty} y^b \mathbf{P}\left(N \geq \sqrt{2y}\right) dy = \frac{1}{b+1} \int_0^{\infty} \mathbf{P}\left(N \geq \sqrt{2y}\right) dy^{b+1} \\ &= \frac{1}{b+1} \mathbf{P}\left(N \geq \sqrt{2y}\right) y^{b+1} \Big|_0^{\infty} + \frac{1}{2(b+1)\sqrt{\pi}} \int_0^{\infty} y^{b+1/2} e^{-y} dy \\ &= \frac{1}{2(b+1)\sqrt{\pi}} \Gamma(b+3/2). \end{aligned}$$

The proposition is now proved.

3 Truncation and Approximation.

The purpose of this section is to use Feller's (1945) and Einmahl's (1989) truncation methods to show that the probabilities in (1.5), (1.8) for M_n can be approximated by those for $\sqrt{n} \sup_{0 \leq s \leq 1} |W(s)|$ and the probabilities in (1.6), (1.9) for S_n can be approximated by those for $\sqrt{n}N$.

Suppose that $\mathbf{E}X = 0$ and $\mathbf{E}X^2 = \sigma^2 < \infty$. Without losing of generality, we assume that $\sigma = 1$ throughout this section. Let $p > 1/2$. For each n and $1 \leq j \leq n$, we let

$$\begin{aligned} X'_{nj} &= X_{nj}I\{|X_j| \leq \sqrt{n}/(\log \log n)^p\}, & \bar{X}'_{nj} &= X'_{nj} - \mathbf{E}[X'_{nj}], \\ S'_{nj} &= \sum_{i=1}^j X'_{ni}, & \bar{S}'_{nj} &= \sum_{i=1}^j \bar{X}'_{ni}, \\ \bar{M}'_n &= \max_{k \leq n} |\bar{S}'_{nk}|, & B_n &= \sum_{k=1}^n \text{Var}(\bar{X}'_{nk}) \end{aligned}$$

and

$$\begin{aligned} X''_{nj} &= X_{nj}I\{\sqrt{n}/(\log \log n)^p < |X_j| \leq \phi(n)\}, & \bar{X}''_{nj} &= X''_{nj} - \mathbf{E}[X''_{nj}], \\ X'''_{nj} &= X_{nj}I\{|X_j| > \phi(n)\}, & \bar{X}'''_{nj} &= X'''_{nj} - \mathbf{E}[X'''_{nj}]. \end{aligned}$$

And also define S''_{nj} , S'''_{nj} , \bar{S}''_{nj} , \bar{S}'''_{nj} , \bar{M}''_n and \bar{M}'''_n similarly.

The following two propositions are the main results of this section.

Proposition 3.1 *Let $a > -1$, $b > -1$ and $2 \geq p > p' > 1/2$. Suppose that the condition (1.3) is satisfied. Then there exist $\delta > 0$ and a sequence of positive numbers $\{q_n\}$ such that*

$$\begin{aligned} & P\left\{ \sup_{0 \leq s \leq 1} |W(s)| \geq \epsilon \sqrt{2 \log \log n} + \frac{3}{(\log \log n)^{p'}} \right\} - q_n \\ & \leq P\left\{ M_n \geq \epsilon \sqrt{2B_n \log \log n} \right\} \\ & \leq P\left\{ \sup_{0 \leq s \leq 1} |W(s)| \geq \epsilon \sqrt{2 \log \log n} - \frac{3}{(\log \log n)^{p'}} \right\} + q_n, \end{aligned} \tag{3.1}$$

$$\begin{aligned} & P\left\{ |N| \geq \epsilon \sqrt{2 \log \log n} + \frac{3}{(\log \log n)^{p'}} \right\} - q_n \\ & \leq P\left\{ |S_n| \geq \epsilon \sqrt{2B_n \log \log n} \right\} \\ & \leq P\left\{ |N| \geq \epsilon \sqrt{2 \log \log n} - \frac{3}{(\log \log n)^{p'}} \right\} + q_n, \\ & \quad \forall \epsilon \in (\sqrt{1+a} - \delta, \sqrt{1+a} + \delta), \quad n \geq 1 \end{aligned} \tag{3.2}$$

and

$$\sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} q_n \leq K(a, b, p, p', \delta) < \infty. \tag{3.3}$$

Proposition 3.2 Let b be a real number and $2 \geq p > p' > 1/2$. Suppose that $EX = 0$ and $EX^2 = 1$.

Then

$$\begin{aligned} & P\left(\sup_{0 \leq s \leq 1} |W(s)| \geq x + 3/(\log \log n)^{p'}\right) - q_n^* \leq P(M_n \geq x\sqrt{B_n}) \\ & \leq P\left(\sup_{0 \leq s \leq 1} |W(s)| \geq x - 3/(\log \log n)^{p'}\right) + q_n^*, \quad \forall x > 0, \end{aligned} \quad (3.4)$$

$$\begin{aligned} & P(|N| \geq x + 3/(\log \log n)^{p'}) - q_n^* \leq P(|S_n| \geq x\sqrt{B_n}) \\ & \leq P(|N| \geq x - 3/(\log \log n)^{p'}) + q_n^*, \quad \forall x > 0, \end{aligned} \quad (3.5)$$

where $q_n^* \geq 0$ satisfies

$$\sum_{n=1}^{\infty} \frac{(\log \log n)^b}{n \log n} q_n^* \leq K(b, p, p') < \infty. \quad (3.6)$$

To show this two results, we need some lemmas.

Lemma 3.1 For any sequence of independent random variables $\{\xi_n; n \geq 1\}$ with mean zero and finite variance, there exists a sequence of independent normal variables $\{\eta_n; n \geq 1\}$ with $E\eta_n = 0$ and $E\eta_n^2 = E\xi_n^2$ such that, for all $Q > 2$ and $y > 0$,

$$P\left(\max_{k \leq n} \left| \sum_{i=1}^k \xi_i - \sum_{i=1}^k \eta_i \right| \geq y\right) \leq (AQ)^Q y^{-Q} \sum_{i=1}^n E|\xi_i|^Q,$$

whenever $E|\xi_i|^Q < \infty$, $i = 1, \dots, n$. Here, A is a universal constant.

Proof. See Sakhaneko (1980,1984, 1985).

Lemma 3.2 Let $Q \geq 2$, $\xi_1, \xi_2, \dots, \xi_n$ be independent random variables with $E\xi_k = 0$ and $E|\xi_k|^Q < \infty$, $k = 1, \dots, n$. Then for all $y > 0$,

$$P\left(\max_{k \leq n} \left| \sum_{i=1}^k \xi_i \right| \geq y\right) \leq 2 \exp\left\{-\frac{y^2}{8 \sum_{k=1}^n \text{Var}(\xi_k)}\right\} + (2AQ)^Q y^{-Q} \sum_{i=1}^n E|\xi_i|^Q,$$

where A is a universal constant as in Lemma 3.1.

Proof. It follows from Lemma 3.1 easily. See also Petrov (1995, Page 78).

Lemma 3.3 Define $\Delta_n = \max_{k \leq n} |\bar{S}'_{nk} - S_k|$. Let $a > -1$, $b > -1$ and $p > 1/2$. Suppose that the condition (1.3) is satisfied and $EX^2 = 1$. Then for any $\lambda > 0$ there exist a constant $K = K(a, b, p, \lambda)$ such that

$$\sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} I_n \leq KE \left[X^2 (\log |X|)^a (\log \log |X|)^{b-1} \right] < \infty, \quad (3.7)$$

where

$$I_n = P\left(\Delta_n \geq \sqrt{n}/(\log \log n)^2, \bar{M}'_n \geq \lambda \phi(n)\right).$$

Proof. Let $\beta_n = n\mathbb{E}[|X|I\{|X| > \sqrt{n}/(\log \log n)^p\}]$. Then $|\mathbb{E}\sum_{i=1}^j X'_{ni}| \leq \beta_n$, $1 \leq j \leq n$. Setting

$$\mathcal{L} = \{n : \beta_n \leq \frac{1}{8}\sqrt{n}/(\log \log n)^2\},$$

we have

$$\{\Delta_n \geq \sqrt{n}/(\log \log n)^2\} \subset \bigcup_{j=1}^n \{X_j \neq X'_{nj}\}, \quad n \in \mathcal{L}.$$

So for $n \in \mathcal{L}$,

$$I_n \leq \sum_{j=1}^n \mathbb{P}(X_j \neq X'_{nj}, \bar{M}'_n \geq \lambda\phi(n)).$$

Observe that $X'_{nj} = 0$ whenever $X_j \neq X'_{nj}$, $j \leq n$, so that we have for n large enough and all $1 \leq j \leq n$,

$$\begin{aligned} & \mathbb{P}(X_j \neq X'_{nj}, \bar{M}'_n \geq \lambda\phi(n)) \\ &= \mathbb{P}(X_j \neq X'_{nj}, \max_{k \leq j-1} |\bar{S}'_{nk}| \vee \max_{j < k \leq n} |\bar{S}'_{nk} - X'_{nj}| \geq \lambda\phi(n)) \\ &= \mathbb{P}(X_j \neq X'_{nj}) \mathbb{P}(\max_{k \leq j-1} |\bar{S}'_{nk}| \vee \max_{j < k \leq n} |\bar{S}'_{nk} - X'_{nj}| \geq \lambda\phi(n)) \\ &\leq \mathbb{P}(X_j \neq X'_{nj}) \mathbb{P}(\bar{M}'_n \geq \lambda\phi(n) - |X'_{nj}|) \\ &\leq \mathbb{P}(|X| > \sqrt{n}/(\log \log n)^p) \mathbb{P}(\bar{M}'_n \geq \lambda\phi(n) - \sqrt{n}/(\log \log n)^p) \\ &\leq \mathbb{P}(|X| > \sqrt{n}/(\log \log n)^p) \mathbb{P}(\bar{M}'_n \geq \frac{\lambda}{2}\phi(n)). \end{aligned}$$

A straightforward application of the inequalities of Ottaviani and Bernstein yields:

$$\begin{aligned} \mathbb{P}(\bar{M}'_n \geq \frac{\lambda}{2}\phi(n)) &\leq 2\mathbb{P}(|\bar{S}'_n| \geq \frac{\lambda}{4}\phi(n)) \leq (\log n)^{-\eta} \\ &\text{for some } \eta = \eta(\lambda) > 0. \end{aligned}$$

So,

$$\begin{aligned} & \sum_{n \in \mathcal{L}} \frac{(\log n)^a (\log \log n)^b}{n} I_n \\ &\leq C \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \cdot n \mathbb{P}(|X| > \frac{\sqrt{n}}{(\log \log n)^p}) (\log n)^{-\eta} \\ &\leq \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} \mathbb{P}\left(\frac{\sqrt{j}}{(\log \log j)^p} < |X| \leq \frac{\sqrt{j+1}}{(\log \log(j+1))^p}\right) (\log n)^{a-\eta} (\log \log n)^b \\ &\leq \sum_{j=1}^{\infty} \mathbb{P}\left(\frac{\sqrt{j}}{(\log \log j)^p} < |X| \leq \frac{\sqrt{j+1}}{(\log \log(j+1))^p}\right) \sum_{n=1}^j (\log n)^{a-\eta} (\log \log n)^b \\ &\leq \sum_{j=1}^{\infty} \mathbb{P}\left(\frac{\sqrt{j}}{(\log \log j)^p} < |X| \leq \frac{\sqrt{j+1}}{(\log \log(j+1))^p}\right) j (\log j)^{a-\eta} (\log \log j)^b \\ &\leq C\mathbb{E}\left[X^2 (\log |X|)^{a-\eta} (\log \log |X|)^{b+2p}\right] \leq C\mathbb{E}\left[X^2 (\log |X|)^a (\log \log |X|)^{b-1}\right]. \end{aligned}$$

If $n \notin \mathcal{L}$, then we have

$$I_n \leq \mathbb{P}\left(\overline{M}'_n \geq \lambda\phi(n)\right) \leq (\log n)^{-\eta}.$$

It follows that

$$\begin{aligned} & \sum_{n \notin \mathcal{L}} \frac{(\log n)^a (\log \log n)^b}{n} I_n \leq \sum_{n \notin \mathcal{L}} \frac{(\log n)^{a-\eta} (\log \log n)^b}{n} \\ & \leq 8 \sum_{n \notin \mathcal{L}} \frac{(\log n)^{a-\eta} (\log \log n)^{b+2}}{n^{3/2}} \beta_n \\ & \leq 8 \sum_{n=1}^{\infty} \frac{(\log n)^{a-\eta} (\log \log n)^{b+2}}{n^{1/2}} \\ & \quad \cdot \sum_{j=n}^{\infty} \mathbb{E}\left[|X| I\left\{\frac{\sqrt{j}}{(\log \log j)^p} < |X| \leq \frac{\sqrt{j+1}}{(\log \log(j+1))^p}\right\}\right] \\ & = 8 \sum_{j=1}^{\infty} \mathbb{E}\left[|X| I\left\{\frac{\sqrt{j}}{(\log \log j)^p} < |X| \leq \frac{\sqrt{j+1}}{(\log \log(j+1))^p}\right\}\right] \\ & \quad \cdot \sum_{n=1}^j \frac{(\log n)^{a-\eta} (\log \log n)^{b+2}}{n^{1/2}} \\ & \leq C \sum_{j=1}^{\infty} \mathbb{E}\left[|X| I\left\{\frac{\sqrt{j}}{(\log \log j)^p} < |X| \leq \frac{\sqrt{j+1}}{(\log \log(j+1))^p}\right\}\right] \\ & \quad \cdot \sqrt{j} (\log j)^{a-\eta} (\log \log j)^{b+2} \\ & \leq C \mathbb{E}\left[X^2 (\log |X|)^{a-\eta} (\log \log |X|)^{b+2+p}\right] \leq C \mathbb{E}\left[X^2 (\log |X|)^a (\log \log |X|)^{b-1}\right]. \end{aligned}$$

(3.7) is proved.

Lemma 3.4 *Let $a > -1$, $b > -1$ and $p > 1/2$. Suppose the condition (1.3) is satisfied and $EX^2 = 1$.*

Then for any $\lambda > 0$ there exist a constant $K = K(a, b, p, \lambda)$ such that

$$\sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} II_n \leq K \mathbb{E}\left[X^2 (\log |X|)^a (\log \log |X|)^{b-1}\right] < \infty,$$

where

$$II_n = \mathbb{P}\left(\Delta_n \geq \sqrt{n}/(\log \log n)^2, M_n \geq \lambda\phi(n)\right).$$

Proof. Obviously,

$$II_n \leq \mathbb{P}\left(\Delta_n \geq \sqrt{n}/(\log \log n)^2, \overline{M}'_n \geq \frac{\lambda}{3}\phi(n)\right) + \mathbb{P}\left(\overline{M}''_n \geq \frac{\lambda}{3}\phi(n)\right) + \mathbb{P}\left(\overline{M}'''_n \geq \frac{\lambda}{3}\phi(n)\right).$$

Observe that $\max_{k \leq n} |\mathbf{E}S_{nk}''''| \leq n\mathbf{E}X^2/\phi(n) = o(\sqrt{n})$. We have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \mathbf{P}\left(\overline{M}_n'''' \geq \frac{\lambda}{3} \phi(n)\right) \\
& \leq C \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \sum_{j=1}^n \mathbf{P}\left(X_j'''' \neq 0\right) \\
& \leq \sum_{n=1}^{\infty} (\log n)^a (\log \log n)^b \mathbf{P}(|X| \geq \phi(n)) \\
& \leq K\mathbf{E}\left[X^2 (\log |X|)^a (\log \log |X|)^{b-1}\right].
\end{aligned}$$

Also, notice that $\sum_{k=1}^n \text{Var}(\overline{X}_{nk}'') \leq n\mathbf{E}\left[X^2 I\{\sqrt{n}/(\log \log n)^p < |X| \leq \phi(n)\}\right] = o(n)$. By Lemma 3.2 we have for $Q > 2$,

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \mathbf{P}\left(\overline{M}_n'' \geq \frac{\lambda}{3} \phi(n)\right) \\
& \leq C \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \exp\left\{-\frac{\lambda^2 \phi^2(n)}{328 \cdot o(n)}\right\} \\
& \quad + C \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \cdot \frac{3^Q}{\lambda^Q \phi^Q(n)} n \mathbf{E}[|X|^Q I\{|X| \leq \phi(n)\}] \\
& \leq K + C \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{\phi^Q(n)} \sum_{j=1}^n \mathbf{E}[|X|^Q I\{\phi(j-1) < |X| \leq \phi(j)\}] \\
& \leq K + C \sum_{j=1}^{\infty} \mathbf{E}[|X|^Q I\{\phi(j-1) < |X| \leq \phi(j)\}] \sum_{n=j}^{\infty} \frac{(\log n)^a (\log \log n)^b}{\phi^Q(n)} \\
& \leq K + C \sum_{j=1}^{\infty} \mathbf{E}[|X|^Q I\{\phi(j-1) < |X| \leq \phi(j)\}] j \frac{(\log j)^a (\log \log j)^b}{\phi^Q(j)} \\
& \leq K + C \sum_{j=1}^{\infty} \mathbf{E}[|X|^2 I\{\phi(j-1) < |X| \leq \phi(j)\}] (\log j)^a (\log \log j)^{b-1} \\
& \leq K + C\mathbf{E}\left[X^2 (\log |X|)^a (\log \log |X|)^{b-1}\right] < \infty.
\end{aligned}$$

Finally, by noticing Lemma 3.3, we complete the proof of Lemma 3.4.

Lemma 3.5 *Suppose that the condition (1.3) is satisfied. Then for any $1/2 < p' < p$ we have*

$$\begin{aligned}
& \mathbf{P}\left(\sup_{0 \leq s \leq 1} |W(s)| \geq x + 1/(\log \log n)^{p'}\right) - p_n \leq \mathbf{P}(\overline{M}_n' \geq x\sqrt{B_n}) \\
& \leq \mathbf{P}\left(\sup_{0 \leq s \leq 1} |W(s)| \geq x - 1/(\log \log n)^{p'}\right) + p_n, \quad \forall x > 0
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
& \mathbf{P}(|N| \geq x + 1/(\log \log n)^{p'}) - p_n \leq \mathbf{P}(|\overline{S}_n'| \geq x\sqrt{B_n}) \\
& \leq \mathbf{P}(|N| \geq x - 1/(\log \log n)^{p'}) + p_n, \quad \forall x > 0,
\end{aligned} \tag{3.9}$$

where $p_n \geq 0$ satisfies

$$\sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} p_n \leq K(a, b, p, p') < \infty. \tag{3.10}$$

Proof. By Lemma 3.1, there exist a universal constant $A > 0$ and a sequence of standard Wiener processes $\{W_n(\cdot)\}$ such that for all $Q > 2$,

$$\begin{aligned} & \mathbb{P}\left(\max_{k \leq n} |\bar{S}'_{nk} - W_n(\frac{k}{n}B_n)| \geq \frac{1}{2}\sqrt{B_n}/(\log \log n)^{p'}\right) \\ & \leq (AQ)^Q \left(\frac{(\log \log n)^{p'}}{\sqrt{B_n}}\right)^Q \sum_{k=1}^n \mathbb{E}|\bar{X}'_{nk}|^Q \\ & \leq Cn \left(\frac{(\log \log n)^{p'}}{\sqrt{n}}\right)^Q \mathbb{E}[|X|^Q I\{|X| \leq \sqrt{n}/(\log \log n)^p\}]. \end{aligned}$$

On the other hand, by Lemma 1.1.1 of Csörgő and Révész (1981),

$$\begin{aligned} & \mathbb{P}\left(\max_{0 \leq s \leq 1} |W_n(sB_n) - W_n(\frac{[ns]}{n}B_n)| \geq \frac{1}{2}\sqrt{B_n}/(\log \log n)^{p'}\right) \\ & = \mathbb{P}\left(\max_{0 \leq s \leq 1} |W_n(s) - W_n(\frac{[ns]}{n})| \geq \frac{1}{2}\sqrt{\frac{1}{n}} \frac{\sqrt{n}}{(\log \log n)^{p'}}\right) \\ & \leq Cn \exp\left\{-\frac{(\frac{1}{2}\sqrt{n}/(\log \log n)^{p'})^2}{3}\right\} \leq Cn \exp\left\{-\frac{1}{12}n/(\log \log n)^{2p'}\right\}. \end{aligned}$$

Let

$$p_n = \mathbb{P}\left(\sup_{0 \leq s \leq 1} \left|\frac{\bar{S}'_{n,[ns]}}{\sqrt{B_n}} - \frac{W_n(sB_n)}{\sqrt{B_n}}\right| \geq \frac{1}{(\log \log n)^{p'}}\right). \quad (3.11)$$

Then p_n satisfies (3.8) and (3.9), since $\{W_n(tB_n)/\sqrt{B_n}; t \geq 0\} \stackrel{D}{=} \{W(t); t \geq 0\}$ for each n . And also,

$$p_n \leq Cn \left(\frac{(\log \log n)^{p'}}{\sqrt{n}}\right)^Q \mathbb{E}[|X|^Q I\{|X| \leq \sqrt{n}/(\log \log n)^p\}] + Cn \exp\left\{-\frac{1}{12}n/(\log \log n)^{2p'}\right\}.$$

It follows that

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} p_n \\ & \leq K_1 + C \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^{b+p'Q}}{n^{Q/2}} \mathbb{E}[|X|^Q I\{|X| \leq \sqrt{n}/(\log \log n)^p\}] \\ & \leq K_1 + C \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^{b+p'Q}}{n^{Q/2}} \\ & \quad \cdot \sum_{j=1}^n \mathbb{E}[|X|^Q I\{\frac{\sqrt{j-1}}{(\log \log(j-1))^p} < |X| \leq \frac{\sqrt{j}}{(\log \log j)^p}\}] \\ & \leq K_1 + C \sum_{j=1}^{\infty} \mathbb{E}[|X|^Q I\{\frac{\sqrt{j-1}}{(\log \log(j-1))^p} < |X| \leq \frac{\sqrt{j}}{(\log \log j)^p}\}] j \frac{(\log j)^a (\log \log j)^{b+p'Q}}{j^{Q/2}} \\ & \leq K_1 + C \mathbb{E}[|X|^2 (\log |X|)^a (\log \log |X|)^{b+(p'-p)Q+2p}] \leq K < \infty, \end{aligned}$$

whenever $(p' - p)Q + 2p < -1$. So, (3.10) is satisfied.

Now, we turn to prove Propositions 3.1 and 3.2.

Proof of Proposition 3.1: Let $0 < \delta < \frac{1}{4}\sqrt{1+a}$. Observe that, if n is large enough,

$$\begin{aligned}
& \mathbb{P}\left\{M_n \geq \epsilon\sqrt{2B_n \log \log n}\right\} \\
= & \mathbb{P}\left\{M_n \geq \epsilon\sqrt{2B_n \log \log n}, \Delta_n \leq \frac{\sqrt{n}}{(\log \log n)^2}\right\} \\
& + \mathbb{P}\left\{M_n \geq \epsilon\sqrt{2B_n \log \log n}, \Delta_n > \frac{\sqrt{n}}{(\log \log n)^2}\right\} \\
\leq & \mathbb{P}\left\{\overline{M}'_n \geq \epsilon\sqrt{2B_n \log \log n} - \frac{\sqrt{n}}{(\log \log n)^2}\right\} \\
& + \mathbb{P}\left\{M_n \geq \frac{\sqrt{1+a}}{4}\phi(n), \Delta_n > \frac{\sqrt{n}}{(\log \log n)^2}\right\} \\
\leq & \mathbb{P}\left\{\overline{M}'_n \geq \sqrt{B_n}\left[\epsilon\sqrt{2 \log \log n} - \frac{2}{(\log \log n)^2}\right]\right\} + II_n \\
\leq & \mathbb{P}\left\{\sup_{0 \leq s \leq 1} |W(s)| \geq \epsilon\sqrt{2 \log \log n} - \frac{2}{(\log \log n)^2} - \frac{1}{(\log \log n)^{p'}}\right\} + p_n + II_n \\
\leq & \mathbb{P}\left\{\sup_{0 \leq s \leq 1} |W(s)| \geq \epsilon\sqrt{2 \log \log n} - \frac{3}{(\log \log n)^{p'}}\right\} + p_n + II_n
\end{aligned}$$

for all $\epsilon \in (\sqrt{1+a} - \delta, \sqrt{1+a} + \delta)$, where II_n is defined in Lemmas 3.4 with $\lambda = \sqrt{1+a}/4$ and p_n is defined in 3.5. Also, if n is large enough,

$$\begin{aligned}
& \mathbb{P}\left\{M_n \geq \epsilon\sqrt{2B_n \log \log n}\right\} \\
\geq & \mathbb{P}\left\{M_n \geq \epsilon\sqrt{2B_n \log \log n}, \Delta_n \leq \frac{\sqrt{n}}{(\log \log n)^2}\right\} \\
\geq & \mathbb{P}\left\{\overline{M}'_n \geq \epsilon\sqrt{2B_n \log \log n} + \frac{\sqrt{n}}{(\log \log n)^2}, \Delta_n \leq \frac{\sqrt{n}}{(\log \log n)^2}\right\} \\
\geq & \mathbb{P}\left\{\overline{M}'_n \geq \sqrt{B_n}\left[\epsilon\sqrt{2 \log \log n} + \frac{2}{(\log \log n)^2}\right]\right\} \\
& - \mathbb{P}\left\{\overline{M}'_n \geq \frac{\sqrt{1+a}}{4}\phi(n), \Delta_n > \frac{\sqrt{n}}{(\log \log n)^2}\right\} \\
\geq & \mathbb{P}\left\{\sup_{0 \leq s \leq 1} |W(s)| \geq \epsilon\sqrt{2 \log \log n} + \frac{3}{(\log \log n)^{p'}}\right\} - p_n - I_n
\end{aligned}$$

for all $\epsilon \in (\sqrt{1+a} - \delta, \sqrt{1+a} + \delta)$, where I_n is defined in Lemma 3.3 with $\lambda = \sqrt{1+a}/4$.

Similarly, if n is large enough,

$$\begin{aligned}
& \mathbb{P}\left\{|N| \geq \epsilon\sqrt{2 \log \log n} + \frac{3}{(\log \log n)^{p'}}\right\} - p_n - I_n \\
\leq & \mathbb{P}\left\{|S_n| \geq \epsilon\sqrt{2B_n \log \log n}\right\} \\
\leq & \mathbb{P}\left\{|N| \geq \epsilon\sqrt{2 \log \log n} - \frac{3}{(\log \log n)^{p'}}\right\} + p_n + II_n.
\end{aligned}$$

Letting $q_n = p_n + I_n + II_n$ completes the proof by Lemmas 3.3, 3.4 and 3.5.

Proof Proposition 3.2: Let $\{W_n(\cdot)\}$ be a sequence of standard Wiener processes being defined in the proof of Lemma 3.5, and let p_n be defined in (3.11). And set

$$q_n^* = \mathbb{P}\left(\sup_{0 \leq s \leq 1} |M_{[ns]}/\sqrt{B_n} - W_n(sB_n)/\sqrt{B_n}| \geq 3/(\log \log n)^{p'}\right).$$

Then q_n^* satisfies (3.4) and (3.5), and also

$$q_n^* \leq \mathbb{P}(\Delta_n \geq \sqrt{n}/(\log \log n)^2) + p_n.$$

By Lemma 3.5,

$$\sum_{n=1}^{\infty} \frac{(\log \log n)^b}{n \log n} p_n \leq K_1(b, p, p') < \infty.$$

Also, following the lines in the proof of (3.7) we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(\log \log n)^b}{n \log n} \mathbb{P}(\Delta_n \geq \sqrt{n}/(\log \log n)^2) \\ & \leq \sum_{n \in \mathcal{L}} \frac{(\log \log n)^b}{n \log n} \cdot n \mathbb{P}(|X| > \sqrt{n}/(\log \log n)^p) + \sum_{n \notin \mathcal{L}} \frac{(\log \log n)^{b+2}}{n^{3/2} \log n} \beta_n \\ & \leq \sum_{n=1}^{\infty} \frac{(\log \log n)^b}{\log n} \mathbb{P}(|X| > \sqrt{n}/(\log \log n)^p) \\ & \quad + \sum_{n=1}^{\infty} \frac{(\log \log n)^{b+2}}{\sqrt{n} \log n} \mathbb{E}[|X| I\{|X| > \sqrt{n}/(\log \log n)^p\}] \\ & \leq C \mathbb{E}[X^2 (\log |X|)^{-1} (\log \log |X|)^{b+2p}] + C \mathbb{E}[X^2 (\log |X|)^{-1} (\log \log |X|)^{b+2+p}] \\ & \leq C \mathbb{E} X^2 < \infty. \end{aligned}$$

So, q_n^* satisfies (3.6).

4 Proofs of the Theorems.

4.1 Proofs of the direct parts.

Without losing of generality, we assume that $\mathbb{E}X = 0$ and $\mathbb{E}X^2 = 1$.

Proof of the direct part of Theorem 1.1: Let $\delta > 0$ small enough and $\{q_n\}$ be such that (3.1), (3.2) and (3.3) hold. Then

$$\lim_{\epsilon \searrow \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} q_n = 0,$$

by (3.3). Notice that $a_n(\epsilon) \rightarrow 0$. By (3.1), we have that for n large enough,

$$\begin{aligned} & \mathbb{P}\left\{ \sup_{0 \leq s \leq 1} |W(s)| \geq \sqrt{2 \log \log n} (\epsilon + a_n(\epsilon)) + \frac{3}{(\log \log n)^{p'}} \right\} - q_n \\ & \leq \mathbb{P}\left\{ M_n \geq \sqrt{2 B_n \log \log n} (\epsilon + a_n(\epsilon)) \right\} \\ & \leq \mathbb{P}\left\{ \sup_{0 \leq s \leq 1} |W(s)| \geq \sqrt{2 \log \log n} (\epsilon + a_n(\epsilon)) - \frac{3}{(\log \log n)^{p'}} \right\} + q_n, \\ & \quad \forall \epsilon \in (\sqrt{1+a} - \delta/2, \sqrt{1+a} + \delta/2). \end{aligned}$$

On the other hand, by Proposition 2.1,

$$\begin{aligned} & \lim_{\epsilon \searrow \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \\ & \quad \cdot \mathbb{P} \left\{ \sup_{0 \leq s \leq 1} |W(s)| \geq \sqrt{2 \log \log n} (\epsilon + a_n(\epsilon)) \pm \frac{3}{(\log \log n)^{p'}} \right\} \\ & = 2 \sqrt{\frac{1}{\pi(a+1)}} \exp\{-2\tau\sqrt{1+a}\} \Gamma(b+1/2). \end{aligned}$$

It follows that

$$\begin{aligned} & \lim_{\epsilon \searrow \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \\ & \quad \cdot \mathbb{P} \left\{ M_n \geq \sqrt{2B_n \log \log n} (\epsilon + a_n(\epsilon)) \right\} \\ & = 2 \sqrt{\frac{1}{\pi(a+1)}} \exp\{-2\tau\sqrt{1+a}\} \Gamma(b+1/2). \end{aligned} \quad (4.1)$$

Similarly,

$$\begin{aligned} & \lim_{\epsilon \searrow \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \\ & \quad \cdot \mathbb{P} \left\{ |S_n| \geq \sqrt{2B_n \log \log n} (\epsilon + a_n(\epsilon)) \right\} \\ & = \sqrt{\frac{1}{\pi(a+1)}} \exp\{-2\tau\sqrt{1+a}\} \Gamma(b+1/2). \end{aligned} \quad (4.2)$$

Finally, noticing the condition (1.4), we have

$$0 \leq n - B_n \leq 2n \mathbb{E}[X^2 I\{|X| \geq \sqrt{n}/(\log \log n)^p\}] = o(n(\log \log n)^{-1}).$$

Let $a'_n(\epsilon) = \sqrt{n/B_n}(\epsilon + a_n(\epsilon)) - \epsilon$. Then

$$\mathbb{P} \left\{ M_n \geq \phi(n)(\epsilon + a_n(\epsilon)) \right\} = \mathbb{P} \left\{ M_n \geq \sqrt{2B_n \log \log n} (\epsilon + a'_n(\epsilon)) \right\},$$

$$\mathbb{P} \left\{ |S_n| \geq \phi(n)(\epsilon + a_n(\epsilon)) \right\} = \mathbb{P} \left\{ |S_n| \geq \sqrt{2B_n \log \log n} (\epsilon + a'_n(\epsilon)) \right\},$$

and,

$$a'_n(\epsilon) \log \log n = \epsilon \frac{(n - B_n) \log \log n}{\sqrt{B_n}(\sqrt{n} + \sqrt{B_n})} + \sqrt{\frac{n}{B_n}} a_n(\epsilon) \log \log n \rightarrow \tau$$

as $n \rightarrow \infty$ and $\epsilon \searrow \sqrt{1+a}$. Now, (1.5) and (1.6) follow from (4.1) and (4.2), respectively.

Proof of the direct part of Theorem 1.2: We show (1.8) only, since the proof of (1.9) is similar.

Noticing $n \geq B_n \sim n$ and Proposition 3.2, for any $0 < \delta < 1$ we have for n large enough and all

$\epsilon > 0$,

$$\begin{aligned}
& \mathbb{P}\left\{\sup_{0 \leq s \leq 1} |W(s)| \geq [\epsilon(1 + \delta) + 2|a_n| + 3/\log \log n] \sqrt{2 \log \log n}\right\} - q_n^* \\
& \leq \mathbb{P}\left\{\sup_{0 \leq s \leq 1} |W(s)| \geq (\epsilon + a_n)(1 + \delta) \sqrt{2 \log \log n} + 3/(\log \log n)^{p'}\right\} - q_n^* \\
& \leq \mathbb{P}\left\{M_n \geq (\epsilon + a_n)(1 + \delta) \sqrt{2B_n \log \log n}\right\} \\
& \leq \mathbb{P}\left\{M_n \geq (\epsilon + a_n)\phi(n)\right\} \leq \mathbb{P}\left\{M_n \geq (\epsilon + a_n) \sqrt{2B_n \log \log n}\right\} \\
& \leq \mathbb{P}\left\{\sup_{0 \leq s \leq 1} |W(s)| \geq (\epsilon + a_n) \sqrt{2 \log \log n} - 3/(\log \log n)^{p'}\right\} + q_n^* \\
& \leq \mathbb{P}\left\{\sup_{0 \leq s \leq 1} |W(s)| \geq (\epsilon + a_n - 3/\log \log n) \sqrt{2 \log \log n}\right\} + q_n^*.
\end{aligned}$$

So, by Propositions 2.2 and 3.2,

$$\begin{aligned}
& (1 + \delta)^{-2(b+1)} \frac{2}{(b+1)\sqrt{\pi}} \Gamma(b+3/2) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+2}} \\
& \leq \liminf_{\epsilon \searrow 0} \epsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^b}{n \log n} \mathbb{P}\left\{M_n \geq (\epsilon + a_n)\phi(n)\right\} \\
& \leq \limsup_{\epsilon \searrow 0} \epsilon^{2(b+1)} \sum_{n=1}^{\infty} \frac{(\log \log n)^b}{n \log n} \mathbb{P}\left\{M_n \geq (\epsilon + a_n)\phi(n)\right\} \\
& \leq \frac{2}{(b+1)\sqrt{\pi}} \Gamma(b+3/2) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+2}}.
\end{aligned}$$

Letting $\delta \rightarrow 0$ completes the proof.

4.2 Proofs of the converse parts.

Now, we turn to prove the converse parts of Theorem 1.1 and 1.2. First, we show that each of (1.5), (1.6), (1.8) and (1.9) implies

$$\mathbb{E}X^2 < \infty, \quad \mathbb{E}X = 0 \quad \text{and} \quad \mathbb{E}\left[X^2(\log |X|)^a (\log \log |X|)^{b-1}\right] < \infty, \quad (4.3)$$

where $a = -1$ in Theorem 1.2. We only give the proof that (1.6) implies (4.3), since other proofs are similar. Let $\{\tilde{X}, \tilde{X}_n; n \geq 1\}$ be the symmetrization of $\{X, X_n; n \geq 1\}$, and let $\tilde{S}_n = \sum_{k=1}^n \tilde{X}_k$. Then by (1.6),

$$\limsup_{\epsilon \searrow \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \mathbb{P}\left\{|\tilde{S}_n| \geq 2\sigma\phi(n)(\epsilon + a_n(\epsilon))\right\} \leq K < \infty.$$

For $M > 0$, define $Y = Y(M) = \tilde{X}I\{|\tilde{X}| < M\}$ and $Y_n = Y_n(M) = \tilde{X}_nI\{|\tilde{X}_n| < M\}$. Observing that $\tilde{X}I\{|\tilde{X}| < M\} - \tilde{X}I\{|\tilde{X}| \geq M\} \stackrel{D}{=} \tilde{X}$ and $\tilde{X}I\{|\tilde{X}| < M\} - \tilde{X}I\{|\tilde{X}| \geq M\} + \tilde{X} = 2Y$, we obtain

that

$$\begin{aligned}
& \limsup_{\epsilon \searrow \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \mathbb{P}\left\{ \left| \sum_{k=1}^n Y_k \right| \geq 2\sigma\phi(n)(\epsilon + a_n(\epsilon)) \right\} \\
& \leq 2 \limsup_{\epsilon \searrow \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \mathbb{P}\left\{ |\tilde{S}_n| \geq 2\sigma\phi(n)(\epsilon + a_n(\epsilon)) \right\} \\
& \leq 2K < \infty.
\end{aligned} \tag{4.4}$$

However, since Y is a bounded random variable which satisfies conditions (1.3) and (1.4), by the direct part of Theorem 1.1 we have

$$\begin{aligned}
& \lim_{\epsilon \searrow \sqrt{1+a}} (\epsilon^2 - a - 1)^{b+1/2} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \mathbb{P}\left\{ \left| \sum_{k=1}^n Y_k \right| \geq \sqrt{\mathbf{E}Y^2} \phi(n)(\epsilon + a_n(\epsilon)) \right\} \\
& = \sqrt{\frac{1}{\pi(a+1)}} \exp\{-2\tau\sqrt{1+a}\} \Gamma(b+1/2) > 0.
\end{aligned} \tag{4.5}$$

Putting (4.4) and (4.5) together yields $\sqrt{\mathbf{E}\tilde{X}^2 I\{|\tilde{X}| < M\}} = \sqrt{\mathbf{E}Y^2} \leq 2\sigma$. Then, letting $M \rightarrow \infty$ yields $\mathbf{E}X^2 < \infty$.

$\mathbf{E}X = 0$ is obvious when $\mathbf{E}X^2 < \infty$, for otherwise we have

$$\mathbb{P}\{|S_n| \geq \epsilon\sigma\phi(n)\} \rightarrow 1, \quad \forall \epsilon > 0,$$

which implies that

$$\sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \mathbb{P}\{|S_n| \geq \epsilon\sigma\phi(n)\} = \infty, \quad \forall \epsilon > 0, a \geq -1 \text{ and } b \geq -1.$$

Now, by (1.6) and the Lévy inequality we obtain that for some $\epsilon > 0$,

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \mathbb{P}\{\max_{k \leq n} |X_k| \geq 3\epsilon\sigma\phi(n)\} \\
& \leq C \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \mathbb{P}\{\max_{k \leq n} |X_k| \geq 2\epsilon\sigma\phi(n) + 2\sqrt{n\mathbf{E}X^2}\} \\
& \leq C \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \mathbb{P}\{\max_{k \leq n} |S_k| \geq \epsilon\sigma\phi(n) + \sqrt{n\mathbf{E}X^2}\} \\
& \leq C \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \mathbb{P}\{|S_n| \geq \epsilon\sigma\phi(n)\} < \infty.
\end{aligned}$$

Observe that

$$\mathbb{P}\{\max_{k \leq n} |X_k| \geq 3\epsilon\sigma\phi(n)\} \leq \frac{\mathbf{E}X^2}{18\epsilon^2 \log \log n} \rightarrow 0.$$

We conclude that

$$\begin{aligned}
& \sum_{n=1}^{\infty} (\log n)^a (\log \log n)^b \mathbb{P}\{|X| \geq 3\epsilon\sigma\phi(n)\} \\
& \leq C \sum_{n=1}^{\infty} \frac{(\log n)^a (\log \log n)^b}{n} \mathbb{P}\{\max_{k \leq n} |X_k| \geq 3\epsilon\sigma\phi(n)\} < \infty,
\end{aligned}$$

which implies

$$\mathbb{E}[X^2(\log |X|)^a(\log \log |X|)^{b-1}] < \infty.$$

(4.3) is proved.

Next, we show that $\mathbb{E}X^2 = \sigma^2$. By the direct part of Theorem 1.2, (1.8) and (1.9) shall hold with $\mathbb{E}X^2$ taking the place of σ^2 , which are obviously contradictory to (1.8) and (1.9) themselves, respectively, if $\mathbb{E}X^2 \neq \sigma^2$. Notice that (4.1) and (4.2) hold whenever (4.3) is satisfied. However, if $\mathbb{E}X^2 \neq \sigma^2$, (1.5) and (1.6) are contradictory to (4.1) and (4.2), respectively, since $B_n \sim n\mathbb{E}X^2$.

Finally, we show (1.7). Suppose that (1.7) fails. Without losing of generality, we can assume that $\sigma^{-2}\mathbb{E}[X^2I\{|X| \geq \sqrt{n}/(\log \log n)^p\}] \geq \tau_0/\log \log n$ for some $\tau_0 > 0$ and all $n \geq 1$. Then $n\sigma^2 - B_n \geq n\mathbb{E}[X^2I\{|X| \geq \sqrt{n}/(\log \log n)^p\}] \geq n\sigma^2\tau_0/\log \log n$. Let $a'_n(\epsilon) = \sqrt{1 + \tau_0/\log \log n}(\epsilon + a_n(\epsilon)) - \epsilon$. Then

$$a'_n(\epsilon) \log \log n \rightarrow \tau + \tau_0\sqrt{1 + a/2},$$

and

$$\begin{aligned} \mathbb{P}\{M_n \geq \sigma\phi(n)(\epsilon + a_n(\epsilon))\} &\leq \mathbb{P}\{M_n \geq \sqrt{2B_n \log \log n}(\epsilon + a'_n(\epsilon))\}, \\ \mathbb{P}\{|S_n| \geq \sigma\phi(n)(\epsilon + a_n(\epsilon))\} &\leq \mathbb{P}\{|S_n| \geq \sqrt{2B_n \log \log n}(\epsilon + a'_n(\epsilon))\}, \end{aligned}$$

It follows that (1.5) and (1.6) are contradictory to (4.1) and (4.2), respectively. The proof is now completed.

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