On the rates of the other law of the logarithm*

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Abstract. Let $X, X_1, X_2, \ldots$ be i.i.d. random variables, and set $S_n = X_1 + \ldots + X_n$, $M_n = \max_{k \le n} |S_k|$, $n \ge 1$. Let $a_n = o(\sqrt{n}/\log n)$. By using the strong approximation, we prove that: if $\mathsf{E}X = 0$, $\mathsf{Var}X = \sigma^2 > 0$ and $\mathsf{E}|X|^{2+\epsilon} < \infty$ for some $\epsilon > 0$, then for any $r > 1$,

$$\lim_{\epsilon \nearrow 1/\sqrt{r-1}} [\epsilon^{-2} - (r-1)] \sum_{n=1}^{\infty} n^{r-2} \mathsf{P}\Big\{ M_n \le \epsilon\sigma\sqrt{\pi^2 n/(8\log n)} + a_n \Big\} = \frac{4}{\pi}.$$

We also show that the widest a_n is $o(\sqrt{n}/\log n)$.

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1 Introduction and main results.

Let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d random variables with a common distribution function F, and set $S_n = \sum_{k=1}^n X_k$, $M_n = \max_{k\leq n} |S_k|$, $n \geq 1$. Also let $\log x = \ln(x \vee e)$, $\log\log x = \log(\log x)$ and $\phi(x) = \sqrt{\pi^2 x/(8\log x)}$. The following is the well known complete convergence first established by Hsu and Robbins (1947):

$$\sum_{n=1}^{\infty} \mathsf{P}(|S_n| \geq \epsilon n) < \infty, \quad \epsilon > 0$$

if and only if $\mathsf{E}X = 0$ and $\mathsf{E}X^2 < \infty$. Baum and Katz (1965) extended this result and proved the following theorem.

Theorem A *Let $1 \leq p < 2$ and $r \geq p$. Then*

$$\sum_{n=1}^{\infty} n^{r-2} \mathsf{P}(|S_n| \geq \epsilon n^{1/p}) < \infty, \quad \epsilon > 0$$

if and only if $\mathsf{E}X = 0$ and $\mathsf{E}|X|^{rp} < \infty$.

Many authors considered various extensions of the results of Hsu-Robbins and Baum-Katz. Some of them study the precise asymptotics of the infinite sums as $\epsilon \to 0$ (c.f. Heyde (1975), Chen (1978), Spătaru (1999) and Gut and Spătaru (2000a)). But, this kind of results do not hold for $p = 2$. However, by replacing $n^{1/p}$ by $\sqrt{n \log\log n}$, Gut and Spătaru (2000b) established an analogous result called the precise asymptotics of the law of the iterated logarithm. By replacing $n^{1/p}$ by $\sqrt{n \log n}$, Lai (1974) and Chow and Lai (1975) consider the following result on the law of the logarithm.

Theorem B *Suppose that $\mathsf{Var}X = \sigma^2$ and $r \geq 1$. Then the following are equivalent:*

$$\sum_{n=1}^{\infty} n^{r-2} \mathsf{P}(M_n \geq \epsilon\sqrt{2n\log n}) < \infty; \quad \text{for all} \quad \epsilon > \sigma\sqrt{r-1};$$

$$\sum_{n=1}^{\infty} n^{r-2} \mathsf{P}(|S_n| \geq \epsilon\sqrt{2n\log n}) < \infty, \quad \text{for all} \quad \epsilon > \sigma\sqrt{r-1};$$

$$\sum_{n=1}^{\infty} n^{r-2} \mathsf{P}(|S_n| \geq \epsilon\sqrt{2n\log n}) < \infty, \quad \text{for some} \quad \epsilon > 0;$$

$$\mathsf{E}X = 0 \quad \text{and} \quad \mathsf{E}|X|^{2r}/(\log|X|)^r < \infty.$$

Liang, et al (2003) studied the precise asymptotics of the second infinite serie in Theorem B for $1 < r < 3/2$ under the condition $\mathsf{E}|X|^{2r} < \infty$. Zhang (2003) studied all the cases of $r > 1$ and obtained the sufficient and necessary condition for such kind of results to hold.

By a small deviation theorem of Mogul'skiĭ (1974) (c.f., Lemma 3.1), it is easy to get the following results on the other law of the logarithm.

Theorem 1.1 *Suppose that* $EX = 0$*,* $\mathsf{Var}X = \sigma^2$ *and* $r > 1$*. Then*

$$\sum_{n=1}^{\infty} n^{r-2}(\log n)^a P(M_n \le \epsilon\sqrt{n/\log n}) < \infty \quad \text{for all} \quad \epsilon < \sigma\sqrt{\frac{\pi^2}{8(r-1)}}$$

and

$$\sum_{n=1}^{\infty} n^{r-2}(\log n)^a P(M_n \le \epsilon\sqrt{n/\log n}) = \infty \quad \text{for all} \quad \epsilon > \sigma\sqrt{\frac{\pi^2}{8(r-1)}}.$$

The purpose of this paper is to consider the precise asymptotics of the infinite series in Theorem 1.1 for all $r > 1$. Here is our main result.

Theorem 1.2 *Let* $r > 1$ *and* $a > -1$ *and let* $a_n(\epsilon)$ *be a function of* ϵ *such that*

$$a_n(\epsilon)\log n \to \tau \quad \text{as} \quad n \to \infty \text{ and } \epsilon \nearrow 1/\sqrt{r-1}. \tag{1.1}$$

Assume that

$$EX = 0, \quad EX^2 = \sigma^2 \ (0 < \sigma < \infty) \quad \text{and} \quad E[|X|^{2+\epsilon}] < \infty, \qquad \text{for some } 0 < \epsilon < 1. \tag{1.2}$$

Then

$$\lim_{\epsilon \nearrow 1/\sqrt{r-1}} [\epsilon^{-2} - (r-1)]^{a+1} \sum_{n=1}^{\infty} n^{r-2}(\log n)^a P\Big\{M_n \le \sigma\phi(n)(\epsilon + a_n(\epsilon))\Big\}$$
$$= \frac{4}{\pi}\exp\{2\tau(r-1)^{3/2}\}\Gamma(a+1). \tag{1.3}$$

Here, $\Gamma(\cdot)$ *is a gamma function. Conversely, if (1.3) holds for some* $r > 1$*,* $a > -1$ *and* $\epsilon > 0$*, then* $EX = 0$ *and* $\mathsf{Var}X = \sigma^2$*.*

Also, we have a refinement of Theorem 1.1 as

Theorem 1.3 *Let* $r > 1$ *and* a *be two real numbers. Suppose that the condition (1.3) is satisfied, then for any eventually non-decreasing* $\psi : [1, \infty) \to (0, \infty)$*,*

$$\sum_{n=1}^{\infty} n^{r-2}(\log n)^a P\Big\{M_n \le \sigma\sqrt{\pi^2 n/(8\psi(n))}\Big\} < \infty \text{ or } = \infty$$
$$\text{according as } \sum_{n=1}^{\infty} n^{r-2}(\log n)^a \exp\{-\psi(n)\} < \infty \text{ or } = \infty. \tag{1.4}$$

We conjecture that (1.3) is true whenever $\mathsf{E}X = 0$, $\mathsf{Var}X = \sigma^2 > 0$ and $\mathsf{E}X^2 I\{|X| \ge t\} = o\big((\log t)^{-1}\big)$ as $t \to \infty$, and (1.4) is true whenever $\mathsf{E}X = 0$, $\mathsf{Var}X = \sigma^2 > 0$ and $\mathsf{E}X^2 I\{|X| \ge t\} = O\big((\log t)^{-1}\big)$ as $t \to \infty$.

The proofs of Theorems 1.2 and 1.3 are given in Section 4. Before that, we first verify (1.3) under the assumption that F is the normal distribution in Section 2, after which, by using the strong approximation method, we then show that the probability in (1.3) can be replaced by those for normal random variables in Section 3. Throughout this paper, we let $K(\alpha, \beta, \cdots)$, $C(\alpha, \beta, \cdots)$ etc denote positive constants which depend on $\alpha, \beta, \cdots$ only, whose values can differ in different places. The notation $a_n \sim b_n$ means that $a_n/b_n \to 1$, and $a_n \approx b_n$ means that $C^{-1}b_n \le a_n \le Cb_n$ for some $c > 0$ and all n large enough.

2 Normal cases.

In this section, we prove Theorem 1.2 in the case that $\{X, X_n; n \ge 1\}$ are normal random variables. Let $\{W(t); t \ge 0\}$ be a standard Wiener process and N a standard normal variable. Our result is as follows.

Proposition 2.1 *Let $r > 1$ and $a > -1$ and let $a_n(\epsilon)$ be a function of ϵ satisfying (1.1). Then*

$$\lim_{\epsilon \nearrow 1/\sqrt{r-1}} [\epsilon^{-2} - (r-1)]^{a+1} \sum_{n=1}^{\infty} n^{r-2} (\log n)^a P\Big\{ \sup_{0 \le s \le 1} |W(s)| \le \sqrt{\pi^2/(8\log n)}(\epsilon + a_n(\epsilon)) \Big\}$$
$$= \frac{4}{\pi} \exp\{2\tau(r-1)^{3/2}\}\Gamma(a+1). \tag{2.1}$$

The following lemma will be used in the proofs.

Lemma 2.1 *Let $\{W(t); t \ge 0\}$ be a standard Wiener process. Then for all $x > 0$,*

$$P\big(\sup_{0 \le s \le 1} |W(s)| \le x \big) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\Big\{ -\frac{\pi^2(2k+1)^2}{8x^2} \Big\}.$$

In particular,

$$\frac{2}{\pi} \exp\Big\{ -\frac{\pi^2}{8x^2} \Big\} \le P\big(\sup_{0 \le s \le 1} |W(s)| \le x \big) \le \frac{4}{\pi} \exp\Big\{ -\frac{\pi^2}{8x^2} \Big\} \tag{2.2}$$

and

$$P\big(\sup_{0 \le s \le 1} |W(s)| \le x \big) \sim \frac{4}{\pi} \exp\Big\{ -\frac{\pi^2}{8x^2} \Big\} \ \text{ as } \ x \to 0.$$

Proof. It is well known. See Ciesielaki and Taylor (1962).

Lemma 2.2 *Let $\alpha_n(\epsilon) > 0$, $\beta_n(\epsilon) > 0$ and $f(\epsilon) > 0$ satisfying*

$$\alpha_n(\epsilon) \sim \beta_n(\epsilon) \quad \text{ as } \ n \to \infty \ \text{ and } \ \epsilon \to \epsilon_0,$$

$$f(\epsilon)\beta_n(\epsilon) \to 0 \ \text{ as } \epsilon \to \epsilon_0, \forall n.$$

Then

$$\limsup_{\epsilon \to \epsilon_0} (\liminf_{\epsilon \to \epsilon_0}) f(\epsilon) \sum_{n=1}^{\infty} \alpha_n(\epsilon) = \limsup_{\epsilon \to \epsilon_0} (\liminf_{\epsilon \to \epsilon_0}) f(\epsilon) \sum_{n=1}^{\infty} \beta_n(\epsilon).$$

Proof. Easy.

Now, we turn to prove the proposition 2.1. By Lemma 2.1 and the Condition (1.1) we have

$$\begin{aligned}
&\mathsf{P}\Big\{\sup_{0\le s\le 1}|W(s)|\ge\sqrt{\frac{\pi^2}{8\log n}}(\epsilon+a_n(\epsilon))\Big\}\sim\frac{4}{\pi}\Big\{-\frac{\log n}{(\epsilon+a_n(\epsilon))^2}\Big\}\\
&=\frac{4}{\pi}\Big\{-\frac{\log n}{\epsilon^2+2\epsilon a_n(\epsilon)+a_n(\epsilon)^2}\Big\}\sim\frac{4}{\pi}\Big\{-\frac{\log n}{\epsilon^2}\Big\}\exp\Big\{\frac{2}{\epsilon^3}a_n(\epsilon)\log n\Big\}\\
&\sim\frac{4}{\pi}\Big\{-\frac{\log n}{\epsilon^2}\Big\}\exp\Big\{2\tau(r-1)^{3/2}\Big\}
\end{aligned}$$

as $n\to\infty$ and $\epsilon\nearrow 1/\sqrt{r-1}$. We conclude that

$$\begin{aligned}
&\lim_{\epsilon\nearrow 1/\sqrt{r-1}}[\epsilon^{-2}-(r-1)]^{a+1}\sum_{n=1}^{\infty}n^{r-2}(\log n)^a\mathsf{P}\Big\{\sup_{0\le s\le 1}|W(s)|\ge\sqrt{\pi^2/(8\log n)}(\epsilon+a_n(\epsilon))\Big\}\\
=\;&\lim_{\epsilon\nearrow 1/\sqrt{r-1}}[\epsilon^{-2}-(r-1)]^{a+1}\sum_{n=1}^{\infty}n^{r-2}(\log n)^a\exp\Big\{-\frac{\log n}{\epsilon^2}\Big\}\frac{4}{\pi}\exp\{2\tau(r-1)^{3/2}\}\\
&\qquad(\text{ by Lemma 2.2})\\
=\;&\lim_{\epsilon\nearrow 1/\sqrt{r-1}}[\epsilon^{-2}-(r-1)]^{a+1}\sum_{n=1}^{\infty}\int_n^{n+1}x^{r-2}(\log x)^a\exp\Big\{-\frac{\log x}{\epsilon^2}\Big\}dx\frac{4}{\pi}\exp\{2\tau(r-1)^{3/2}\}\\
&\qquad(\text{ by Lemma 2.2})\\
=\;&\lim_{\epsilon\nearrow 1/\sqrt{r-1}}[\epsilon^{-2}-(r-1)]^{a+1}\int_e^{\infty}x^{r-2}(\log x)^a\exp\Big\{-\frac{\log x}{\epsilon^2}\Big\}dx\frac{4}{\pi}\exp\{2\tau(r-1)^{3/2}\}\\
=\;&\lim_{\epsilon\nearrow 1/\sqrt{r-1}}[\epsilon^{-2}-(r-1)]^{a+1}\int_1^{\infty}y^a\exp\Big\{-[\epsilon^{-2}-(r-1)]y\Big\}dy\frac{4}{\pi}\exp\{2\tau(r-1)^{3/2}\}\\
=\;&\lim_{\epsilon\nearrow 1/\sqrt{r-1}}\int_{\epsilon^{-2}-(r-1)}^{\infty}y^ae^y\frac{4}{\pi}\exp\{2\tau(r-1)^{3/2}\}=\frac{4}{\pi}\exp\{2\tau(r-1)^{3/2}\}\Gamma(a+1).
\end{aligned}$$

(2.1) is proved.

3 Approximation.

The purpose of this section is to use strong approximation and Feller's (1945) and Einmahl's (1989) truncation methods to show that the probability in (1.3) for M_n can be approximated by those for $\sqrt{n}\sup_{0\le s\le 1}|W(s)|$.

Suppose that $\mathsf{E}X=0$ and $\mathsf{E}X^2=\sigma^2<\infty$. Without losing of generality, we assume that $\sigma=1$. Let $0<p<1/2$. For each n and $1\le j\le n$, we let $X'_{nj}=X_{nj}I\{|X_j|\le\sqrt{n}/n^p\}$, $X^*_{nj}=X'_{nj}-\mathsf{E}[X'_{nj}]$, $S'_{nj}=\sum_{i=1}^j X'_{nj}$, $S^*_{nj}=\sum_{i=1}^j X^*_{nj}$, $M^*_n=\max_{k\le n}|S^*_{nk}|$ and $B_n=\sum_{k=1}^n\mathsf{Var}(X^*_{nk})$. The following proposition is the main result of this section.

Proposition 3.1 *Suppose* $\mathsf{E}[|X|^{2+\epsilon}]<\infty$ *for some* $0<\epsilon<1$. *Let* $a>-1$, $r>1$ *and* $0<p<$

$\epsilon/(4(2+\epsilon))$. *Then there exist* $\delta > 0$ *and a sequence of positive numbers* $\{q_n\}$ *such that*

$$\begin{aligned} & P\Big\{\sup_{0\le s\le 1}|W(s)| \le \epsilon\sqrt{\frac{\pi^2}{8\log n}+\frac{5}{(\log n)^2}}\Big\} - q_n \\ \le\ & P\Big\{M_n \le \epsilon\phi(n)\Big\} \\ \le\ & P\Big\{\sup_{0\le s\le 1}|W(s)| \le \epsilon\sqrt{\frac{\pi^2}{8\log n}-\frac{5}{(\log n)^2}}\Big\} + q_n, \\ & \forall\epsilon\in(\frac{1}{\sqrt{r-1}}-\delta, \frac{1}{\sqrt{r-1}}+\delta), \quad n\ge 1 \end{aligned} \tag{3.1}$$

and

$$\sum_{n=1}^{\infty} n^{r-2}(\log n)^a q_n \le K(r,a,p,\epsilon,\delta) < \infty. \tag{3.2}$$

To show this result, we need some lemmas.

Lemma 3.1 *For any* $x > 0$ *and* $0 < \delta < 1$, *there exists a positive constant* $C = C(x,\delta)$ *such that*

(a) $C^{-1}\exp\{-\frac{1+\delta}{x^2}\log n\} \le P\{M_n^* \le x\phi(n)\} \le C\exp\{-\frac{1-\delta}{x^2}\log n\}$,

(b) $C^{-1}\exp\{-\frac{1+\delta}{x^2}\log n\} \le P\{M_n \le x\phi(n)\} \le C\exp\{-\frac{1-\delta}{x^2}\log n\}$.

Proof. This lemma is so-called small deviation theorem. It follows from Theorem 2 of Shao (1995) by noting that $B_n \sim n$. (See also Shao 1991).

Lemma 3.2 *For any* $x > 0$, $A > 0$ *and* $0 < \delta < 1$, *there exists a positive constant* $C = C(x,\delta)$ *such that*

$$P\{\max_{k\le q}|S_k+z_1| \vee \max_{q<k\le n}|S_k+z_2| \le x\phi(n)\} \le C\exp\big\{-\frac{1-\delta}{x^2}\log n\big\}$$

holds uniformly in $|z_1| \le A\phi(n)$, $|z_2| \le A\phi(n)$ *and* $1 \le q \le n$.

Proof. Without losing of generality, we can assume that $0 < \delta < \frac{1}{8^8 5!}$. We follow the lines of the proof of (17) in Shao (1995). Write $x_n = x\phi(n)$. Put $M = \delta^{-2}$. For fixed n, define $m_0 = 0$,

$$m_i = \max\{j : j \le Mix_n^2\}, \quad \text{for } i \le l := \max\{i : m_i \le n-1\}$$

and $m_{l+1} = n$. It is easily seen that

$$(1-\delta/4)Mx_n^2 \le m_i - m_{i-1} \le (1+\delta/4)Mx_n^2$$

and

$$\frac{n}{Mx_n^2} - 1 \le l \le \frac{n}{Mx_n^2}$$

provided n is sufficiently large. From Lemmas 3 and 1 of Shao (1995) and the Anderson's inequality, it follows that, there exists an integer n_0 such that $\forall n \geq n_0$, $\forall 1 \leq j \leq l$, $\forall |y| \leq x_n + A\phi(n)$, $\forall |y_j| \leq A\phi(n)$,

$$
\begin{aligned}
& \mathsf{P}\Big(\max_{k \leq m_j - m_{j-1}} |S_{m_{j-1}+k} - S_{m_{j-1}} + y_j + y| \leq x_n \Big) \\
\leq\ & e^{-3M} + \mathsf{P}\Big(\sup_{0 \leq s \leq 1} |W(s) + (y_j + y)(m_j - m_{j-1})^{-1/2}| \leq x_n (m_j - m_{j-1})^{-1/2} \Big) \\
\leq\ & e^{-3M} + \mathsf{P}\Big(\sup_{0 \leq s \leq 1} |W(s)| \leq x_n (m_j - m_{j-1})^{-1/2} \Big) \\
\leq\ & e^{-3M} + 4\exp\Big(- \frac{\pi^2 (m_j - m_{j-1})}{8x_n^2} \Big) \leq e^{-3M} + 4\exp\Big(- \frac{\pi^2 M(1-\delta/4)}{8} \Big) \\
\leq\ & \frac{1}{2} e^{-2M} + \frac{1}{2}\exp\Big(- \frac{\pi^2 M(1-\delta/2)}{8} \Big) \leq \exp\Big(- \frac{\pi^2 M(1-\delta/2)}{8} \Big),
\end{aligned}
$$

where $\{W(t); t \geq 0\}$ is a standard Wiener process. Obviously, there exists an i such that $m_{i-1} < q \leq m_i$. Let $y_j = z_1$ if $j \leq i-1$, and z_2 if $j \geq i$. Then

$$
\begin{aligned}
& \mathsf{P}\big(\max_{k \leq q} |S_k + z_1| \vee \max_{q < k \leq n} |S_k + z_2| \leq x\phi(n) \big) \leq \mathsf{P}\big(\max_{j \leq l, j \neq i} \max_{m_{j-1} < k \leq m_j} |S_k + y_j| \leq x_n \big) \\
=\ & \mathsf{E}\Big\{ I\{ \max_{j \leq l-1, j \neq i} \max_{m_{j-1} < k \leq m_j} |S_k + y_j| \leq x_n \} I\{ \max_{m_{l-1} < k \leq m_l, l \neq i} |S_k + y_j| \leq x_n \} \Big\} \\
\leq\ & \mathsf{E}\Big\{ I\{ \max_{j \leq l-1, j \neq i} \max_{m_{j-1} < k \leq m_j} |S_k + y_j| \leq x_n \} \\
& \quad \times \mathsf{E}\big[I\{ \max_{m_{l-1} < k \leq m_l} |S_k + y_j| \leq x_n \} | S_k, k \leq m_{l-1} \big] \Big\} \\
=\ & \int_{-x_n - y_{l-1}}^{x_n - y_{l-1}} \mathsf{P}\Big(\max_{m_{l-1} < k \leq m_l} |S_k - S_{m_{l-1}} + y + y_l| \leq x_n \Big) \\
& \quad d\mathsf{P}\Big(\max_{j \leq l-1, j \neq i} \max_{m_{j-1} < k \leq m_j} |S_k + y_j| \leq x_n, S_{m_{l-1}} < y \Big) \\
\leq\ & \exp\Big(- \frac{\pi^2 M(1-\delta/2)}{8} \Big) \mathsf{P}\Big(\max_{j \leq l-1, j \neq i} \max_{m_{j-1} < k \leq m_j} |S_k + y_j| \leq x_n \Big) \\
\leq\ & \ldots \leq \exp\Big(- \frac{\pi^2 M(1-\delta/2)(l-1)}{8} \Big) \\
\leq\ & C\exp\Big(- \frac{(1-\delta)\pi^2 n}{8x_n^2} \Big) \leq C\exp\Big(- \frac{(1-\delta)}{x^2} \log n \Big).
\end{aligned}
$$

For $n \leq n_0$, it is obvious that

$$
\mathsf{P}\big(\max_{k \leq q} |S_k + z_1| \vee \max_{k < k \leq n} |S_k + z_2| \leq x\phi(n) \big) \leq 1 \leq C\exp\Big(- \frac{(1-\delta)}{x^2} \log n \Big).
$$

Lemma 2 is proved.

Lemma 3.3 *Define $\Delta_n = \max_{k \leq n} |S^*_{nk} - S_k|$. Suppose $\mathsf{E}[|X|^{2+\epsilon}] < \infty$ for some $0 < \epsilon < 1$. Let $a > -1$, $r > 1$ and $0 < p < \epsilon/(4(2+\epsilon))$. Then there exist constants $\delta_0 = \delta_0(r, a, p, \epsilon) > 0$ and $K = K(r, a, p, \epsilon, \delta_0)$ such that $\forall 0 < \delta < \delta_0$,*

$$
\sum_{n=1}^{\infty} n^{r-2} (\log n)^a I_n \leq K < \infty,
$$

where

$$I_n = P\Big(\Delta_n \ge \sqrt{n}/(\log n)^2, M_n^* \le \phi(n)\big(\frac{1}{\sqrt{r-1}} + \delta\big)\Big).$$

Proof. It is sufficient to show that

$$\sum_{n=1}^{\infty} n^{r-2}(\log n)^a \mathsf{P}\Big(\Delta_n \ge \sqrt{n}/(\log n)^2, M_n^* \le \frac{\phi(n)}{\sqrt{r-1-\delta}}\Big) \le C\mathsf{E}X^2, \tag{3.3}$$

whenever $\delta > 0$ is small enough. Let $\beta_n = n\mathsf{E}[|X|I\{|X| > \sqrt{n}/n^p\}]$. Then $|\mathsf{E}\sum_{i=1}^{j} X'_{ni}| \le \beta_n$, $1 \le j \le n$. Setting

$$\mathcal{L} = \{n : \beta_n \le \frac{1}{8}\sqrt{n}/(\log n)^2\},$$

we have

$$\{\Delta_n \ge \sqrt{n}/(\log n)^2\} \subset \bigcup_{j=1}^{n}\{X_j \ne X'_{nj}\}, \quad n \in \mathcal{L}.$$

So for $n \in \mathcal{L}$,

$$\begin{aligned} I'_n : &= \mathsf{P}\Big(\Delta_n \ge \sqrt{n}/(\log n)^2, M_n^* \le \frac{\phi(n)}{\sqrt{r-1-\delta}}\Big) \\ &\le \sum_{j=1}^{n} \mathsf{P}\Big(X_j \ne X'_{nj}, M_n^* \le \frac{\phi(n)}{\sqrt{r-1-\delta}}\Big). \end{aligned}$$

Observer that $X'_{nj} = 0$ whenever $X_j \ne X'_{nj}$, $j \le n$, so that by Lemma 3.1(a) we have for n large enough and all $1 \le j \le n$,

$$\begin{aligned} &\mathsf{P}\Big(X_j \ne X'_{nj}, M_n^* \le \frac{\phi(n)}{\sqrt{r-1-\delta}}\Big) \\ = &\ \mathsf{P}\Big(X_j \ne X'_{nj}, \max_{k\le j-1}|S_{nk}^*| \vee \max_{j<k\le n}|S_{nk}^* - X'_{nj}| \le \frac{\phi(n)}{\sqrt{r-1-\delta}}\Big) \\ = &\ \mathsf{P}(X_j \ne X'_{nj})\mathsf{P}\Big(\max_{k\le j-1}|S_{nk}^*| \vee \max_{j<k\le n}|S_{nk}^* - X'_{nj}| \le \frac{\phi(n)}{\sqrt{r-1-\delta}}\Big) \\ \le &\ \mathsf{P}\Big(X_j \ne X'_{nj})\mathsf{P}\Big(M_n^* \le \frac{\phi(n)}{\sqrt{r-1-\delta}} + |X'_{nj}|\Big) \\ \le &\ \mathsf{P}(|X| > \sqrt{n}/n^p)\mathsf{P}\Big(M_n^* \le \frac{\phi(n)}{\sqrt{r-1-\delta}} + \sqrt{n}/n^p\Big) \\ \le &\ C\mathsf{P}(|X| > \sqrt{n}/\log^p n)\exp\{(-(r-1)+\delta+\delta')\log n\} \\ \le &\ C\mathsf{E}[|X|^{2+\epsilon}]n^{-(r-1)+\delta+\delta'-(1/2-p)(2+\epsilon)}. \end{aligned}$$

Notice that $(1/2-p)(2+\epsilon) \ge 1+\epsilon/4$. Choose $0 < \delta, \delta' < \epsilon/16$. Then $\delta+\delta'-(1/2-p)(2+\epsilon) < -1-\epsilon/8$. So,

$$\sum_{n\in\mathcal{L}} n^{r-2}(\log n)^a I'_n \le C\mathsf{E}[|X|^{2+\epsilon}]\sum_{n=1}^{\infty} n^{\delta+\delta'-(1/2-p)(2+\epsilon)}(\log n)^a \le C\mathsf{E}[|X|^{2+\epsilon}].$$

Note that

$$\frac{\beta_n}{\sqrt{n}/(\log n)^2} \le C\mathsf{E}[|X|^{2+\epsilon}](\log n)^2 n^{1/2-(1/2-p)(1+\epsilon)} \to 0.$$

It follows that there are only finite many ns not in $\mathcal{L}$. So

$$\sum_{n\notin\mathcal{L}} n^{r-2}(\log n)^a I_n' < \infty.$$

(3.3) is proved.

Lemma 3.4 *Suppose* $\mathsf{E}[|X|^{2+\epsilon}] < \infty$ *for some* $0 < \epsilon < 1$*. Let* $a > -1$*,* $r > 1$ *and* $0 < p < \epsilon/(4(2+\epsilon))$*. Then there exist constants* $\delta_0 = \delta_0(r, a, p, \epsilon) > 0$ *and* $K = K(r, a, p, \epsilon, \delta_0)$ *such that* $\forall 0 < \delta < \delta_0$*,*

$$\sum_{n=1}^{\infty} n^{r-2}(\log n)^a II_n \le K < \infty,$$

where

$$II_n = \mathsf{P}\Big(\Delta_n \ge \sqrt{n}/(\log n)^2, M_n \le \phi(n)\big(\frac{1}{\sqrt{r-1}} + \delta\big)\Big).$$

Proof. It is enough to show that

$$\sum_{n=1}^{\infty} n^{r-2}(\log n)^a \mathsf{P}\Big(\Delta_n \ge \sqrt{n}/(\log n)^2, M_n \le \frac{\phi(n)}{\sqrt{r-1-\delta}}\Big) \le C\mathsf{E}|X|^{2+\epsilon},$$

whenever $\delta > 0$ is small enough. Let β_n and $\mathcal{L}$ be defined as in the proof of Lemma 3.3. Then for $n \in \mathcal{L}$,

$$\mathsf{P}\Big(\Delta_n \ge \sqrt{n}/(\log n)^2, M_n \le \frac{\phi(n)}{\sqrt{r-1-\delta}}\Big) \le \sum_{j=1}^{n} \mathsf{P}\Big(X_j \ne X_{nj}', M_n \le \frac{\phi(n)}{\sqrt{r-1-\delta}}\Big).$$

and for $1 \le j \le n$,

$$\begin{aligned}
& \mathsf{P}\Big(X_j \ne X_{nj}', M_n \le \frac{\phi(n)}{\sqrt{r-1-\delta}}\Big) \\
\le\ & \mathsf{P}\Big(X_j \ne X_{nj}', M_{j-1} \vee \max_{j<k\le n} |S_k - X_j + X_j| \le \frac{\phi(n)}{\sqrt{r-1-\delta}}, M_n \le \frac{\phi(n)}{\sqrt{1+a-\delta}}\Big) \\
\le\ & \mathsf{P}\Big(\frac{\sqrt{n}}{n^p} < |X_j| \le 2\frac{\phi(n)}{\sqrt{r-1-\delta}}, M_{j-1} \vee \max_{j<k\le n} |S_k - X_j + X_j| \le \frac{\phi(n)}{\sqrt{r-1-\delta}}\Big) \\
=\ & \int_{\frac{\sqrt{n}}{n^p} < |y| \le 2\frac{\phi(n)}{\sqrt{r-1-\delta}}} \mathsf{P}\Big(M_{j-1} \vee \max_{j<k\le n} |S_k - X_j + y| \le \frac{\phi(n)}{\sqrt{r-1-\delta}}\Big) d\mathsf{P}(X_j < y).
\end{aligned}$$

Note that $M_{j-1} \vee \max_{j<k\le n} |S_k - X_j + y| \stackrel{\mathcal{D}}{=} M_{j-1} \vee \max_{j\le k\le n-1} |S_k + y|$. By Lemma 3.2, we have

$$\begin{aligned}
\sup_{|y|\le 2\frac{\phi(n)}{\sqrt{r-1-\delta}}} & \mathsf{P}\Big(M_{j-1} \vee \max_{j<k\le n} |S_k - X_j + y| \le \frac{\phi(n)}{\sqrt{r-1-\delta}}\Big) \\
& \le C\exp\big\{(-(r-1) + \delta + \delta')\log n\big\}.
\end{aligned}$$

It follows that for $n \in \mathcal{L}$ and $1 \le j \le n$,

$$\begin{aligned}
\mathsf{P}\Big(X_j \ne X_{nj}', M_n \le \frac{\phi(n)}{\sqrt{r-1-\delta}}\Big) & \le Cn^{-(r-1)+\delta+\delta'}\mathsf{P}\Big(\frac{\sqrt{n}}{n^p} < |X_j| \le 2\frac{\phi(n)}{\sqrt{1+a-\delta}}\Big) \\
& \le Cn^{-(r-1)+\delta+\delta'}\mathsf{P}\Big(|X| > \sqrt{n}/n^p\Big).
\end{aligned}$$

The remained proof is similar to that of (3.3) with Lemma 3.1(b) instead of Lemma 3.1(a).

Lemma 3.5 *For any sequence of independent random variables $\{\xi_n; n \geq 1\}$ with mean zero and finite variance, there exists a sequence of independent normal variables $\{\eta_n; n \geq 1\}$ with $\mathsf{E}\eta_n = 0$ and $\mathsf{E}\eta_n^2 = \mathsf{E}\xi_n^2$ such that, for all $Q > 2$ and $y > 0$,*

$$\mathsf{P}\Big(\max_{k\leq n} |\sum_{i=1}^k \xi_i - \sum_{i=1}^k \eta_i| \geq y\Big) \leq (AQ)^Q y^{-Q} \sum_{i=1}^n \mathsf{E}|\xi_i|^Q,$$

whenever $\mathsf{E}|\xi_i|^Q < \infty$, $i = 1, \dots, n$. Here, A is a universal constant.

Proof. See Sakhaneko (1980,1984, 1985).

Lemma 3.6 *We have that*

$$\begin{aligned} & \mathsf{P}\big(\sup_{0\leq s\leq 1} |W(s)| \leq x - 1/(\log n)^2\big) - p_n \leq \mathsf{P}\big(M_n^* \leq x\sqrt{B_n}\big) \\ \leq \; & \mathsf{P}\big(\sup_{0\leq s\leq 1} |W(s)| \leq x + 1/(\log n)^2\big) + p_n, \quad \forall x > 0, \end{aligned} \tag{3.4}$$

where $p_n \geq 0$ satisfies

$$\sum_{n=1}^\infty n^{r-2} (\log n)^a p_n \leq K(r,a,p) < \infty. \tag{3.5}$$

Proof. By Lemma 3.5, there exist a universal constant $A > 0$ and a sequence of standard Wiener processes $\{W_n(\cdot)\}$ such that for all $Q > 2$,

$$\begin{aligned} & \mathsf{P}\Big(\max_{k\leq n} |S_{nk}^* - W_n(\frac{k}{n}B_n)| \geq \frac{1}{2}\sqrt{B_n}/(\log n)^2\Big) \\ \leq \; & (AQ)^Q \Big(\frac{(\log n)^2}{\sqrt{B_n}}\Big)^Q \sum_{k=1}^n \mathsf{E}\big|X_{nk}^*\big|^Q \leq Cn\Big(\frac{(\log n)^2}{\sqrt{n}}\Big)^Q \mathsf{E}\big[|X|^Q I\{|X| \leq \sqrt{n}/n^p\}\big]. \end{aligned}$$

On the other hand, by Lemma 1.1.1 of Csörgő and Révész (1981),

$$\begin{aligned} & \mathsf{P}\Big(|\max_{0\leq s\leq 1} |W_n(sB_n) - W_n(\frac{[ns]}{n}B_n)| \geq \frac{1}{2}\sqrt{B_n}/(\log n)^2\Big) \\ = \; & \mathsf{P}\Big(|\max_{0\leq s\leq 1} |W_n(s) - W_n(\frac{[ns]}{n})| \geq \frac{1}{2}\sqrt{\frac{1}{n}}\frac{\sqrt{n}}{(\log n)^2}\Big) \\ \leq \; & Cn\exp\Big\{-\frac{(\frac{1}{2}\sqrt{n}/(\log n)^2)^2}{3}\Big\} \leq Cn\exp\Big\{-\frac{1}{12}n/(\log n)^4\Big\}. \end{aligned}$$

Let

$$p_n = \mathsf{P}\Big(\big|M_n^*/\sqrt{B_n} - \sup_{0\leq s\leq 1} |W_n(sB_n)|/\sqrt{B_n}\big| \geq 1/(\log n)^2\Big). \tag{3.6}$$

Then p_n satisfies (3.4), since $\{W_n(tB_n)/\sqrt{B_n}; t \geq 0\} \stackrel{\mathcal{D}}{=} \{W(t); t \geq 0\}$ for each n. And also,

$$p_n \leq Cn\Big(\frac{(\log n)^2}{\sqrt{n}}\Big)^Q \mathsf{E}\big[|X|^Q I\{|X| \leq \sqrt{n}/n^p\}\big] + Cn\exp\Big\{-\frac{1}{12}n/(\log n)^4\Big\}.$$

It follows that

$$\begin{aligned} & \sum_{n=1}^\infty n^{r-2}(\log n)^a p_n \leq K_1 + C\sum_{n=1}^\infty n^{r-1-Q/2}(\log n)^{2Q+a}\mathsf{E}\big[|X|^Q I\{|X| \leq \sqrt{n}/n^p\}\big] \\ \leq \; & K_1 + C\sum_{n=1}^\infty n^{r-1-Q/2+(1/2-p)Q}(\log n)^{2Q+a} \leq K_1 + C\sum_{n=1}^\infty n^{r-1-pQ}(\log n)^{2Q+a} \leq K < \infty, \end{aligned}$$

whenever Q is large enough such that $pQ - r > 0$. So, (3.5) is satisfied.

Now, we turn to prove Proposition 3.1. Observe that, $0 \le 1 - B_n/n = o((\log n)^{-2})$. If n is large enough, then

$$
\begin{aligned}
& \mathsf{P}\Big\{M_n \le \epsilon\phi(n)\Big\} \\
= \ & \mathsf{P}\Big\{M_n \le \epsilon\phi(n), \Delta_n \le \frac{\sqrt{n}}{(\log n)^2}\Big\} + \mathsf{P}\Big\{M_n \le \epsilon\phi(n), \Delta_n > \frac{\sqrt{n}}{(\log n)^2}\Big\} \\
\le \ & \mathsf{P}\Big\{M_n^* \le \epsilon\sqrt{\frac{\pi^2 B_n}{8\log n}} + 2\frac{\sqrt{n}}{(\log n)^2}\Big\} + \mathsf{P}\Big\{M_n \le \phi(n)\big(\frac{1}{\sqrt{r-1}} + \delta\big), \Delta_n > \frac{\sqrt{n}}{(\log n)^2}\Big\} \\
\le \ & \mathsf{P}\Big\{\sup_{0\le s\le 1}|W(s)| \le \epsilon\sqrt{\frac{\pi^2}{8\log n} + \frac{5}{(\log n)^2}}\Big\} + p_n + II_n
\end{aligned}
$$

for all $\epsilon \in (1/\sqrt{r-1} - \delta, 1/\sqrt{r-1} + \delta)$, where II_n and p_n are defined in Lemmas 3.4 and 3.6, respectively. Similarly, if n is large enough, then

$$
\begin{aligned}
& \mathsf{P}\Big\{M_n \le \epsilon\phi(n)\Big\} \ge \mathsf{P}\Big\{M_n \le \epsilon\phi(n), \Delta_n \le \frac{\sqrt{n}}{(\log n)^2}\Big\} \\
\ge \ & \mathsf{P}\Big\{M_n^* \le \epsilon\phi(n) - \frac{\sqrt{n}}{(\log n)^2}, \Delta_n \le \frac{\sqrt{n}}{(\log n)^2}\Big\} \\
\ge \ & \mathsf{P}\Big\{M_n^* \le \sqrt{B_n}\big[\epsilon\sqrt{\frac{\pi^2}{8\log n}} - \frac{2}{(\log n)^2}\big]\Big\} \\
& -\mathsf{P}\Big\{M_n^* \le \epsilon\phi(n) - \frac{\sqrt{n}}{(\log n)^2} - \frac{2}{(\log n)^2}\big], \Delta_n > \frac{\sqrt{n}}{(\log n)^2}\Big\} \\
\ge \ & \mathsf{P}\Big\{\sup_{0\le s\le 1}|W(s)| \le \epsilon\sqrt{\frac{\pi^2}{8\log n}} - \frac{3}{(\log n)^2}\Big\} \\
& -\mathsf{P}\Big\{M_n^* \le \phi(n)\big(\frac{1}{\sqrt{r-1}} + \delta\big), \Delta_n > \frac{\sqrt{n}}{(\log n)^2}\Big\} \\
\ge \ & \mathsf{P}\Big\{\sup_{0\le s\le 1}|W(s)| \le \epsilon\sqrt{\frac{\pi^2}{8\log n}} - \frac{5}{(\log n)^2}\Big\} - p_n - I_n
\end{aligned}
$$

for all $\epsilon \in (1/\sqrt{r-1} - \delta, 1/\sqrt{r-1} + \delta)$, where I_n is defined in Lemma 3.3. Choosing $\delta > 0$ small enough and letting $q_n = p_n + I_n + II_n$ complete the proof by Lemmas 3.3, 3.4 and 3.6.

4 Proof of the Theorems.

Proof of Theorem 1.2: Suppose (1.2) hold. Without losing of generality, we assume that $\mathsf{E}X = 0$ and $\mathsf{E}X^2 = 1$. Let $\delta > 0$, $p > 0$ small enough and $\{q_n\}$ be such that (3.1) and (3.2) hold. Then

$$
\lim_{\epsilon \nearrow 1/\sqrt{r-1}} [\epsilon^{-2} - (r-1)]^{a+1} \sum_{n=1}^{\infty} n^{r-2}(\log n)^a q_n = 0,
$$

by (3.2). Notice that $a_n(\epsilon) \to 0$. By (3.1), we have that for n large enough,

$$
\begin{aligned}
& \mathsf{P}\Big\{ \sup_{0\le s\le 1} |W(s)| \le \sqrt{\pi^2/(8\log n)}(\epsilon + a_n(\epsilon)) + \frac{5}{(\log n)^2} \Big\} - q_n \\
\le\ & \mathsf{P}\Big\{ M_n \le \phi(n)(\epsilon + a_n(\epsilon)) \Big\} \\
\le\ & \mathsf{P}\Big\{ \sup_{0\le s\le 1} |W(s)| \le \sqrt{\pi^2/(8\log n)}(\epsilon + a_n(\epsilon)) - \frac{5}{(\log n)^2} \Big\} + q_n, \\
& \quad \forall \epsilon \in (1/\sqrt{r-1} - \delta/2, 1/\sqrt{r-1} + \delta/2).
\end{aligned}
$$

On the other hand, by Proposition 2.1,

$$
\begin{aligned}
& \lim_{\epsilon \searrow \sqrt{r-1}} [\epsilon^{-2} - (r-1)]^{a+1} \sum_{n=1}^{\infty} n^{r-2} (\log n)^a \mathsf{P}\Big\{ \sup_{0\le s\le 1} |W(s)| \le \sqrt{\pi^2/(8\log n)}(\epsilon + a_n(\epsilon)) \pm \frac{5}{(\log n)^2} \Big\} \\
& \quad = \frac{2}{\pi} \exp\{2\tau(r-1)^{3/2}\}\Gamma(a+1).
\end{aligned}
$$

(1.3) is now proved.

Conversely, suppose (1.3) hold. From Esseen (1968) (see also Petrov 1995, Page 74 (2.70)) it is easy to see that for all $m \ge 1$,

$$
\mathsf{P}(|S_m| \le 2\sqrt{m}) \le K\Big(\int_{-2\sqrt{m}}^{2\sqrt{m}} d\widetilde{F}(x) \Big)^{-1/2},
$$

where $\widetilde{F}(x)$ is the distribution function of the symmetrized X, and K is an absolute constant. So, if $\mathsf{E}X^2 = \infty$, then for any $M > 2$ we can choose $m_0 \ge 9$ large enough such that

$$
\mathsf{P}(|S_m| \le 2\sqrt{m}) \le e^{-2M}, \quad m \ge m_0. \tag{4.1}
$$

For $\epsilon > 0$, we let $m = [\epsilon^2 n/\log n]$, and $N = [n/m]$, then for all $n \ge m_0^2$ and $\epsilon \ge 1$,

$$
\begin{aligned}
\mathsf{P}\big(M_n \le \epsilon(n/\log n)^{1/2}\big) & \le \mathsf{P}\big(|S_{km} - S_{(k-1)m}| \le 2\sqrt{m}, k = 1, \dots, N\big) \\
& \le e^{-2MN} \le \exp\Big\{ -M\frac{\log n}{\epsilon^2} \Big\}.
\end{aligned}
$$

By this inequality, for any r, $a > -1$ and $0 < \epsilon_1 < \epsilon_2 < \infty$ there exists a constant $C = C(r, a, \epsilon_1, \epsilon_2)$ for which

$$
\sup_{\epsilon \in (\epsilon_1, \epsilon_2)} \sum_{n=1}^{\infty} n^{r-2} (\log n)^a \mathsf{P}\big\{ M_n \le \epsilon(n/\log n)^{1/2} \big\} \le C < \infty,
$$

which implies that

$$
\lim_{\epsilon \nearrow 1/\sqrt{r-1}} [\epsilon^{-2} - (r-1)]^{a+1} \sum_{n=1}^{\infty} n^{r-2} (\log n)^a \mathsf{P}\big\{ M_n \le \phi(n)(\epsilon + a_n(\epsilon)) \big\} = 0.
$$

This contradictory to (1.3). If $\mathsf{E}X^2 < \infty$ and $\mathsf{E}X = \mu \ne 0$, then (4.1) also hold since $|S_m|/\sqrt{m} \to \infty$ a.s. as $m \to \infty$. So, we conclude $\mathsf{E}X^2 < \infty$ and $\mathsf{E}X = 0$. At last, under $\mathsf{E}X^2 < \infty$ and $\mathsf{E}X = 0$, $\mathsf{Var}X = \sigma^2$ is obvious according (1.3) and Theorem 1.1.

Proof of Theorem 1.3: Let $\delta > 0$, $p > 0$ small enough and $\{q_n\}$ be such that (3.1) and (3.2) hold. By a standard argument (see Feller (1945)), we can assume that

$$(\frac{1}{\sqrt{r-1}} - \frac{\delta}{2})\phi(n) \le \sqrt{\frac{\pi^2 n}{8\psi(n)}} \le (\frac{1}{\sqrt{r-1}} + \frac{\delta}{2})\phi(n).$$

That is

$$(\frac{1}{\sqrt{r-1}} + \frac{\delta}{2})^{-1/2} \log n \le \psi(n) \le (\frac{1}{\sqrt{r-1}} - \frac{\delta}{2})^{-1/2} \log n.$$

Let $\epsilon = \sqrt{\pi^2 n/(8\psi(n)}/\phi(n)$. By (3.1), it follows that

$$\begin{aligned} & \mathsf{P}\Big\{ \sup_{0\le s\le 1} |W(s)| \le \sqrt{\pi^2/(8\psi(n))} + 5/(\log n)^2 \Big\} - q_n \\ \le\ & \mathsf{P}\Big\{ M_n \le \sqrt{\pi^2 n/(8\psi(n))} \Big\} \\ \le\ & \mathsf{P}\Big\{ \sup_{0\le s\le 1} |W(s)| \le \sqrt{\pi^2/(8\psi(n))} - 5/(\log n)^2 \Big\} + q_n. \end{aligned}$$

Notice

$$\frac{\pi^2}{8\big(\sqrt{\pi^2/(8\psi(n))} \pm 5/(\log n)^2\big)^2} = \psi(n)\big(1 + o(\frac{1}{\log n})\big) = \psi(n) + o(1).$$

According to (2.2), it follows that

$$c_1 \exp\{-\psi(n)\} - q_n \le \mathsf{P}\Big\{ M_n \le \sqrt{\pi^2 n/(8\psi(n))} \Big\} \le c_2 \exp\{-\psi(n)\} + q_n.$$

By (3.2), Theorem 1.3 is now proved.

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