

A law of the iterated logarithm for negatively associated random fields*

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Abstract

The exponential inequality of the maximum partial sums is a key to establish the law of the iterated logarithm of negatively associated random variables. In the one-indexed random sequence case, such inequalities are established by Shao (2000) by using his comparison theorem between negatively associated and independent random variables. In the multi-indexed random field case, the comparison theorem fails. The purpose of this paper is to establish the Kolmogorov exponential inequality of the maximum partial sums of a negatively associated random field via a different method. By using this inequality, the sufficient and necessary condition for the law of the iterated logarithm of a negatively associated random field to hold is obtained.

Key Words: negative association, law of the iterated logarithm, random field, Kolmogorov exponential inequality, the maximum partial sums.

Abbreviated Title: LIL for NA Fields

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1 Introduction and Results

A finite family of random variables $\{X_i; 1 \leq i \leq n\}$ is said to be negatively associated if for every pair of disjoint subsets A and B of $\{1, 2, \dots, n\}$,

$$\text{Cov}\{f(X_i; i \in A), g(X_j; j \in B)\} \leq 0 \quad (1.1)$$

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whenever f and g are coordinatewise non-decreasing and the covariance exists. An infinite family is negatively associated if every finite subfamily is negatively associated. The concept of the negative association was introduced by Alam and Saxena (1981) and Joag-Dev and Proschan (1983). As pointed out and proved by Joag-Dev and Proschan (1983), a number of well-known multivariate distributions possess the NA property. Negative association has found important and wide applications in multivariate statistical analysis and reliability theory. Many investigators also discuss applications of NA to probability, stochastic processes and statistics. We refer to Joag-Dev and Proschan (1983) for fundamental properties, Newman (1984) for the central limit theorem, Matula (1992) for the three series theorem, Su, et al. (1997) for the moment inequality, Roussas (1996) for the Hoeffding inequality, and Shao (2000) for the Rosenthal-type maximal inequality and the Kolmogorov exponential inequality. Shao and Su (1999) established the law of the iterated logarithm for negatively associated random variables with finite variances.

Theorem A *Let $\{X_i; i \geq 1\}$ be a strictly stationary negatively associated sequence with $EX_1 = 0$, $EX_1^2 < \infty$ and $\sigma^2 := EX_1^2 + 2 \sum_{i=2}^{\infty} E(X_1 X_i) > 0$. Let $S_n = \sum_{i=1}^n X_i$. Then*

$$\limsup_{n \rightarrow \infty} \frac{S_n}{(2n \log \log n)^{1/2}} = \sigma \quad a.s. \quad (1.2)$$

Here and in the sequel of this paper, $\log x = \ln(x \vee e)$

Let $\{X_{\mathbf{n}}; \mathbf{n} \in \mathbb{N}^d\}$ be a field of random variables, where $d \geq 2$ is a positive integer, \mathbb{N}^d denotes the d -dimensional lattice of positive integers. Through this paper, for $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$, $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$ and $m \in \mathbb{N}$, we denote $|\mathbf{n}| = n_1 \cdots n_d$, $\|\mathbf{n}\| = n_1 + \cdots + n_d$, $\mathbf{k}\mathbf{n} = (k_1 n_1, \dots, k_d n_d)$ and $\mathbf{k}m = (k_1 m, \dots, k_d m)$. Also, $\mathbf{k} \leq \mathbf{n}$ (resp. $\mathbf{k} \geq \mathbf{n}$) means $k_i \leq n_i$ (resp. $k_i \geq n_i$), $i = 1, 2, \dots, d$. Denote by $S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}$ and $\mathbf{1} = (1, \dots, 1) \in \mathbb{N}^d$. It is known that, if $\{X_{\mathbf{n}}\}$ is a field of i.i.d.r.v.s, then

$$\limsup_{\mathbf{n} \rightarrow \infty} \frac{|S_{\mathbf{n}}|}{(2d|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}} = (EX_{\mathbf{1}}^2)^{1/2} \quad a.s. \quad (1.3)$$

if and only if $EX_{\mathbf{1}} = 0$ and $EX_{\mathbf{1}}^2 \log^{d-1}(|X_{\mathbf{1}}|) / \log \log(|X_{\mathbf{1}}|) < \infty$, where $\mathbf{n} \rightarrow \infty$ means $n_1 \rightarrow \infty, \dots, n_d \rightarrow \infty$. When $\{X_{\mathbf{n}}; \mathbf{n} \in \mathbb{N}^d\}$ is a negatively associated field of random variables, Zhang (2000), Zhang and Wen (2001a) and Zhang and Wang (1999) established

the central limit theorem, the weak convergence, the law of large numbers and the complete convergence similar to those for fields of independent random variables. This paper is to establish the law of the iterated logarithm similar to (1.3) for a negatively associated random field.

Theorem 1.1 *Let $d \geq 2$ be a positive integer, and $\{X_{\mathbf{n}}; \mathbf{n} \in \mathbb{N}^d\}$ be a strictly stationary negatively associated field of random variables satisfying*

$$EX_{\mathbf{1}} = 0 \quad \text{and} \quad EX_{\mathbf{1}}^2 \log^{d-1}(|X_{\mathbf{1}}|) / \log \log(|X_{\mathbf{1}}|) < \infty. \quad (1.4)$$

Denote by $\Upsilon(\mathbf{j} - \mathbf{i}) = \text{Cov}(X_{\mathbf{j}}, X_{\mathbf{i}})$ and $\sigma^2 = \sum_{\mathbf{j} \in \mathbb{Z}^d} \Upsilon(\mathbf{j})$. Then

$$\limsup_{\mathbf{n} \rightarrow \infty} \frac{S_{\mathbf{n}}}{(2d|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}} = \sigma \quad \text{a.s.} \quad (1.5)$$

The following theorem tells us that the condition (1.4) is necessary for the law of the iterated logarithm to hold.

Theorem 1.2 *Let $d \geq 1$ be a positive integer, and $\{X_{\mathbf{n}}; \mathbf{n} \in \mathbb{N}^d\}$ be a negatively associated field of identically distributed random variables. If*

$$P\left(\limsup_{\mathbf{n} \rightarrow \infty} \frac{|S_{\mathbf{n}}|}{(2d|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}} < \infty\right) > 0, \quad (1.6)$$

then (1.4) holds.

In showing the law of the iterated logarithm, a main step is to establish the exponential inequalities. In the case of $d = 1$, such exponential inequalities for negatively associated random variables is established by Shao (2000) by using a comparison theorem between negatively associated and independent random variables. However, if $d \geq 2$, such comparison theorem fails for the maximum partial sums (cf., Bulinski and Suquet 2001). In section 2, we establish a Kolmogorov type exponential inequality of the maximum partial sums of a negatively associated field via a different method. Theorems 1.1 and 1.2 are proved in Section 3.

2 Moment inequalities and exponential inequalities

First, we have the following moment inequalities and exponential inequalities for the partial sums.

Lemma 2.1 Let $p \geq 2$ and let $\{Y_{\mathbf{k}}; \mathbf{k} \leq \mathbf{n}\}$ be a negatively associated field of random variables with $EY_{\mathbf{k}} = 0$ and $E|Y_{\mathbf{k}}|^p < \infty$. Then

$$E\left|\sum_{\mathbf{i} \leq \mathbf{n}} Y_{\mathbf{i}}\right|^p \leq 2(15p/\ln p)^p \left\{ \left(\sum_{\mathbf{i} \leq \mathbf{n}} EY_{\mathbf{i}}^2 \right)^{p/2} + \sum_{\mathbf{i} \leq \mathbf{n}} E|Y_{\mathbf{i}}|^p \right\}.$$

Lemma 2.2 Let $\{Y_{\mathbf{k}}; \mathbf{k} \leq \mathbf{n}\}$ be a negatively associated field of random variables with zero means and finite second moments. Let $T_{\mathbf{k}} = \sum_{\mathbf{i} \leq \mathbf{k}} Y_{\mathbf{i}}$ and $B_{\mathbf{n}}^2 = \sum_{\mathbf{k} \leq \mathbf{n}} EY_{\mathbf{k}}^2$. Then for all $x > 0$ and $a > 0$,

$$P(T_{\mathbf{n}} \geq x) \leq P\left(\max_{\mathbf{k} \leq \mathbf{n}} Y_{\mathbf{k}} > a\right) + \exp\left(-\frac{x^2}{2(ax + B_{\mathbf{n}}^2)}\right). \quad (2.1)$$

Lemma 2.3 Let $\{Y_{\mathbf{k}}; \mathbf{k} \leq \mathbf{n}\}$ be a negatively associated field of random variables with $EY_{\mathbf{k}} = 0$ and $E|Y_{\mathbf{k}}|^3 < \infty$. Denote by $T_{\mathbf{k}} = \sum_{\mathbf{i} \leq \mathbf{k}} Y_{\mathbf{i}}$ and $B_{\mathbf{n}}^2 = \sum_{\mathbf{k} \leq \mathbf{n}} EY_{\mathbf{k}}^2$. Then for all $x > 0$,

$$\begin{aligned} P(T_{\mathbf{n}} \geq xB_{\mathbf{n}}) &\geq (1 - \Phi(x+1)) - 6B_{\mathbf{n}}^{-2} \sum_{1 \leq \mathbf{i} \neq \mathbf{j} \leq \mathbf{n}} |E(Y_{\mathbf{i}}Y_{\mathbf{j}})| \\ &\quad - 12B_{\mathbf{n}}^{-3} \sum_{\mathbf{k} \leq \mathbf{n}} E|Y_{\mathbf{k}}|^3 \end{aligned} \quad (2.2)$$

where $\Phi(x)$ is the distribution of a standard normal variable.

Proofs of Lemmas 2.1- 2.3: In the case of $d = 1$, Lemma 2.1 is proved by Shao (2000), and Lemma 2.3 is proved by Shao and Su (1999). Also, (2.1) follows from the following inequality easily:

$$P(T_{\mathbf{n}} \geq x) \leq P\left(\max_{\mathbf{k} \leq \mathbf{n}} Y_{\mathbf{k}} > a\right) + \exp\left\{\frac{x}{a} - \frac{x}{a} \ln\left(\frac{xa}{B_{\mathbf{n}}^2} + 1\right)\right\}.$$

The later is proved by Su, et al.(1997). Since Lemmas 2.1-2.3 do not involve the partial order of the index set, so they are valid for $d \geq 2$ also. In fact, when $d \geq 2$, there is a one-one map $\pi : \{\mathbf{k} : \mathbf{k} \leq \mathbf{n}\} \rightarrow \{1, 2, \dots, |\mathbf{n}|\}$. By noting that $\{Y_{\pi^{-1}(i)} : i = 1, \dots, |\mathbf{n}|\}$ is a negatively associated sequence and $\sum_{i=1}^{|\mathbf{n}|}(\cdot) = \sum_{\mathbf{k} \leq \mathbf{n}}(\cdot)$, $\max_{\mathbf{k} \leq \mathbf{n}} Y_{\mathbf{k}} = \max_{i \leq |\mathbf{n}|} Y_{\pi^{-1}(i)}$, the results follow.

In a same may, one can extend (1.2) of shao (2000) to the case of $d \geq 1$.

Lemma 2.4 Let $\{Y_{\mathbf{k}}; \mathbf{k} \leq \mathbf{n}\}$ be a negatively associated field and $\{Y_{\mathbf{k}}^*; \mathbf{k} \leq \mathbf{n}\}$ be a field of independent random variables such that for each \mathbf{k} , $Y_{\mathbf{k}}$ and $Y_{\mathbf{k}}^*$ have the same distribution.

Then

$$Ef\left(\sum_{\mathbf{k} \leq \mathbf{n}} Y_{\mathbf{k}}\right) \leq Ef\left(\sum_{\mathbf{k} \leq \mathbf{n}} Y_{\mathbf{k}}^*\right)$$

for any convex function f on \mathbb{R} , whenever the expectations exist.

It shall be mentioned that it is impossible to find a one-one map $\pi : \{\mathbf{k} : \mathbf{k} \leq \mathbf{n}\} \rightarrow \{1, 2, \dots, |n|\}$ such that $\sum_{i=1}^{|\mathbf{m}|} Y_{\pi^{-1}(i)} = \sum_{\mathbf{k} \leq \mathbf{m}} Y_{\mathbf{k}}$ for all $\mathbf{m} \leq \mathbf{n}$. So, the inequalities for maximum partial sums can not be extended directly.

For maximum partial sums, Zhang and Wen (2001a) established two moment inequalities.

Proposition 2.1 (Zhang and Wen 2001a) *Let $p \geq 2$, and let $\{Y_{\mathbf{k}}; \mathbf{k} \leq \mathbf{n}\}$ be a negatively associated field of random variables with $EY_{\mathbf{k}} = 0$ and $E|Y_{\mathbf{k}}|^p < \infty$. Suppose that $\{\epsilon_{\mathbf{k}}; \mathbf{k} \leq \mathbf{n}\}$ is a field of i.i.d.r.v.s with $P(\epsilon_{\mathbf{k}} = \pm 1) = 1/2$. Also, suppose that $\{\epsilon_{\mathbf{k}}; \mathbf{k} \leq \mathbf{n}\}$ is independent of $\{Y_{\mathbf{k}}; \mathbf{k} \leq \mathbf{n}\}$. Denote by $T_{\mathbf{k}} = \sum_{\mathbf{i} \leq \mathbf{k}} Y_{\mathbf{i}}$, $M_{\mathbf{n}} = \max_{\mathbf{k} \leq \mathbf{n}} |T_{\mathbf{k}}|$, $\tilde{T}_{\mathbf{k}} = \sum_{\mathbf{i} \leq \mathbf{k}} Y_{\mathbf{i}}$, $\tilde{M}_{\mathbf{n}} = \max_{\mathbf{k} \leq \mathbf{n}} |\tilde{T}_{\mathbf{k}}|$ and $\|\mathbf{X}\|_p = (E|X|^p)^{1/p}$. Then*

$$\|M_{\mathbf{n}}\|_p \leq 5\|\tilde{M}_{\mathbf{n}}\|_p + \|M_{\mathbf{n}}\|_1, \quad (2.3)$$

and there exists a constant A_p such that

$$E|M_{\mathbf{n}}|^p \leq A_p \left\{ (E|M_{\mathbf{n}}|)^p + \left(\sum_{\mathbf{k} \leq \mathbf{n}} EY_{\mathbf{k}}^2 \right)^{p/2} + \sum_{\mathbf{k} \leq \mathbf{n}} E|Y_{\mathbf{k}}|^p \right\}. \quad (2.4)$$

Proposition 2.2 (Zhang and Wen 2001a) *Let $\{Y_{\mathbf{k}}; \mathbf{k} \leq \mathbf{n}\}$ be a strictly stationary negatively associated field of random variables with $EY_{\mathbf{1}} = 0$ and $0 < EY_{\mathbf{1}}^2 < \infty$. Denote by $T_{\mathbf{k}} = \sum_{\mathbf{i} \leq \mathbf{k}} Y_{\mathbf{i}}$. Then there exists a constant K , depending only on d , such that*

$$\limsup_{n \rightarrow \infty} |n|^{-1} E \max_{\mathbf{k} \leq \mathbf{n}} T_{\mathbf{k}}^2 \leq KEY_{\mathbf{1}}^2. \quad (2.5)$$

Now, we begin to establish the following Kolmogorov type exponential inequality for maximum partial sums.

Proposition 2.3 *Let $\{Y_{\mathbf{k}}; \mathbf{k} \leq \mathbf{n}\}$ be a negatively associated field of random variables with $EY_{\mathbf{k}} = 0$ and $|Y_{\mathbf{k}}| \leq b$ a.s for some $0 < b < \infty$. Denote by $T_{\mathbf{k}} = \sum_{\mathbf{i} \leq \mathbf{k}} Y_{\mathbf{i}}$, $M_{\mathbf{n}} = \max_{\mathbf{k} \leq \mathbf{n}} |T_{\mathbf{k}}|$ and $B_{\mathbf{n}}^2 = \sum_{\mathbf{k} \leq \mathbf{n}} EY_{\mathbf{k}}^2$. Then for all $x > 0$,*

$$P(M_{\mathbf{n}} - 2EM_{\mathbf{n}} \geq 20x) \leq 2^{d+1} \exp\left(-\frac{x^2}{2(bx + B_{\mathbf{n}}^2)}\right). \quad (2.6)$$

Proof. Let $\{\epsilon_{\mathbf{k}}; \mathbf{k} \leq \mathbf{n}\}$ is a field of i.i.d.r.v.s with $P(\epsilon_{\mathbf{k}} = \pm 1) = 1/2$. Also, assume that $\{\epsilon_{\mathbf{k}}; \mathbf{k} \leq \mathbf{n}\}$ is independent of $\{Y_{\mathbf{k}}; \mathbf{k} \leq \mathbf{n}\}$. Denote by $\tilde{T}_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} \epsilon_{\mathbf{k}} Y_{\mathbf{k}}$, $T_{\mathbf{n},1} = \sum_{\mathbf{k} \leq \mathbf{n}, \epsilon_{\mathbf{k}}=1} Y_{\mathbf{k}}$ and $T_{\mathbf{n},2} = \sum_{\mathbf{k} \leq \mathbf{n}, \epsilon_{\mathbf{k}}=-1} Y_{\mathbf{k}}$. First, we show that

$$\mathbb{E}e^{M_{\mathbf{n}}} \leq 2^d e^{2\|M_{\mathbf{n}}\|_1} \mathbb{E}e^{10|\tilde{T}_{\mathbf{n}}|} \leq 2^{d+1} e^{2\|M_{\mathbf{n}}\|_1} \prod_{\mathbf{k} \leq \mathbf{n}} \mathbb{E}e^{20\epsilon_{\mathbf{k}} Y_{\mathbf{k}}}. \quad (2.7)$$

By the Lévy inequality, we have

$$P\{\tilde{M}_{\mathbf{n}} \geq x\} = \mathbb{E}_Y P_{\epsilon}\{\tilde{M}_{\mathbf{n}} \geq x\} \leq 2^d \mathbb{E}_Y P_{\epsilon}\{|\tilde{T}_{\mathbf{n}}| \geq x\} \leq 2^d P\{|\tilde{T}_{\mathbf{n}}| \geq x\}, \quad \forall x \geq 0,$$

where $\mathbb{E}_Y(\cdot) = \mathbb{E}[\cdot | \epsilon_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d]$, and \mathbb{E}_{ϵ} , P_{ϵ} etc are defined similarly. Then

$$\mathbb{E}e^{10\tilde{M}_{\mathbf{n}}} \leq 2^d \mathbb{E}e^{10|\tilde{T}_{\mathbf{n}}|}.$$

So, from (2.3) it follows that

$$\begin{aligned} \mathbb{E}e^{M_{\mathbf{n}}} &= 1 + \mathbb{E}M_{\mathbf{n}} + \sum_{q=2}^{\infty} \frac{\mathbb{E}M_{\mathbf{n}}^q}{q!} \\ &\leq 1 + \|M_{\mathbf{n}}\|_1 + \sum_{q=2}^{\infty} \frac{(5\|\tilde{M}_{\mathbf{n}}\|_q + \|M_{\mathbf{n}}\|_1)^q}{q!} \\ &\leq 1 + \|M_{\mathbf{n}}\|_1 + \sum_{q=2}^{\infty} \frac{(2\|5\tilde{M}_{\mathbf{n}} + \|M_{\mathbf{n}}\|_1\|_q)^q}{q!} \\ &= 1 + \|M_{\mathbf{n}}\|_1 + \sum_{q=2}^{\infty} \frac{\mathbb{E}(10\tilde{M}_{\mathbf{n}} + 2\|M_{\mathbf{n}}\|_1)^q}{q!} \\ &\leq 1 + \sum_{q=1}^{\infty} \frac{\mathbb{E}(10\tilde{M}_{\mathbf{n}} + 2\|M_{\mathbf{n}}\|_1)^q}{q!} = \mathbb{E}\left\{1 + \sum_{q=1}^{\infty} \frac{(10\tilde{M}_{\mathbf{n}} + 2\|M_{\mathbf{n}}\|_1)^q}{q!}\right\} \\ &= \mathbb{E}e^{10\tilde{M}_{\mathbf{n}} + 2\|M_{\mathbf{n}}\|_1} \leq 2^d e^{2\|M_{\mathbf{n}}\|_1} \mathbb{E}e^{10|\tilde{T}_{\mathbf{n}}|}. \end{aligned}$$

Note that

$$\mathbb{E}e^{10\tilde{T}_{\mathbf{n}}} = \mathbb{E}e^{10T_{\mathbf{n},1} - 10T_{\mathbf{n},2}} \leq \frac{1}{2} \mathbb{E}(e^{20T_{\mathbf{n},1}} + e^{-20T_{\mathbf{n},2}}).$$

For fixed $\{\epsilon_{\mathbf{k}}; \mathbf{k} \leq \mathbf{n}\}$, we have by Lemma 2.4 or the definition (1.1),

$$\begin{aligned} \mathbb{E}_Y e^{20T_{\mathbf{n},1}} &\leq \prod_{\epsilon_{\mathbf{k}}=1} \mathbb{E}_Y e^{20Y_{\mathbf{k}}} = \prod_{\epsilon_{\mathbf{k}}=1} \mathbb{E}_Y e^{20\epsilon_{\mathbf{k}} Y_{\mathbf{k}}} \\ &\leq \prod_{\epsilon_{\mathbf{k}}=1} \mathbb{E}_Y e^{20\epsilon_{\mathbf{k}} Y_{\mathbf{k}}} \cdot \prod_{\epsilon_{\mathbf{k}}=-1} \mathbb{E}_Y e^{20\epsilon_{\mathbf{k}} Y_{\mathbf{k}}} = \prod_{\mathbf{k} \leq \mathbf{n}} \mathbb{E}_Y e^{20\epsilon_{\mathbf{k}} Y_{\mathbf{k}}}, \end{aligned}$$

since $\mathbb{E}_Y e^{20\epsilon_k Y_k} \geq e^{\mathbb{E}_Y(20\epsilon_k Y_k)} = 1$. It follows that

$$\begin{aligned} \mathbb{E} e^{20T_{n,1}} &= \mathbb{E}_\epsilon \mathbb{E}_Y e^{20T_{n,1}} \leq \mathbb{E}_\epsilon \left(\prod_{\mathbf{k} \leq \mathbf{n}} \mathbb{E}_Y e^{20\epsilon_k Y_k} \right) \\ &= \prod_{\mathbf{k} \leq \mathbf{n}} \mathbb{E}_\epsilon \mathbb{E}_Y e^{20\epsilon_k Y_k} = \prod_{\mathbf{k} \leq \mathbf{n}} \mathbb{E} e^{20\epsilon_k Y_k}. \end{aligned}$$

Similarly,

$$\mathbb{E} e^{-20T_{n,2}} \leq \prod_{\mathbf{k} \leq \mathbf{n}} \mathbb{E} e^{-20\epsilon_k Y_k} = \prod_{\mathbf{k} \leq \mathbf{n}} \mathbb{E} e^{20\epsilon_k Y_k}.$$

It follows that

$$\mathbb{E} e^{10\tilde{T}_n} \leq \prod_{\mathbf{k} \leq \mathbf{n}} \mathbb{E} e^{20\epsilon_k Y_k}.$$

Similarly,

$$\mathbb{E} e^{-10\tilde{T}_n} \leq \prod_{\mathbf{k} \leq \mathbf{n}} \mathbb{E} e^{20\epsilon_k Y_k}.$$

(2.7) is proved. Now, from (2.7) it follows that for any $x > 0$ and $t > 0$,

$$\mathbb{P}(M_n - 2EM_n \geq 20x) \leq e^{-20tx - 2t\|M_n\|_1} \mathbb{E} e^{tM_n} \leq 2^{d+1} e^{-20tx} \prod_{\mathbf{k} \leq \mathbf{n}} \mathbb{E} e^{20\epsilon_k Y_k}.$$

Since

$$\begin{aligned} \mathbb{E} e^{20\epsilon_k Y_k} &= \mathbb{E} \left\{ 1 + 20t\epsilon_k Y_k + \frac{e^{20t\epsilon_k Y_k} - 1 - 20t\epsilon_k Y_k}{(\epsilon_k Y_k)^2} (\epsilon_k Y_k)^2 \right\} \\ &\leq 1 + (e^{20tb} - 1 - 20tb)b^{-2} \mathbb{E} Y_k^2 \\ &\leq \exp \left\{ (e^{20tb} - 1 - 20tb)b^{-2} \mathbb{E} Y_k^2 \right\}, \end{aligned}$$

it follows that

$$\mathbb{P}(M_n - 2EM_n \geq 20x) \leq 2^{d+1} e^{-20tx} \exp \left\{ (e^{20tb} - 1 - 20tb)b^{-2} B_n^2 \right\}.$$

Letting $20t = \frac{1}{b} \log(1 + \frac{xb}{B_n^2})$ yields

$$\begin{aligned} \mathbb{P}(M_n - 2EM_n \geq 20x) &\leq 2^{d+1} \exp \left\{ \frac{x}{b} - \left(\frac{x}{b} + \frac{B_n^2}{b^2} \right) \log \left(1 + \frac{xb}{B_n^2} \right) \right\} \\ &\leq 2^{d+1} \exp \left\{ -\frac{x^2}{2(bx + B_n^2)} \right\}. \end{aligned}$$

3 Proofs of the main results

We need two more lemmas.

Lemma 3.1 *Let $d \geq 2$ and let $\{Y_{\mathbf{k}}; \mathbf{k} \in \mathbb{N}^d\}$ be a negatively associated field of identically distributed random variables with*

$$EY_{\mathbf{1}} = 0 \text{ and } EY_{\mathbf{1}}^2 \log^{d-1}(|Y_{\mathbf{1}}|)/\log \log(|Y_{\mathbf{1}}|) < \infty.$$

Denote by $T_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} Y_{\mathbf{k}}$ and $M_{\mathbf{n}} = \max_{\mathbf{k} \leq \mathbf{n}} |T_{\mathbf{k}}|$. Then

$$\begin{aligned} & \limsup_{\mathbf{n} \rightarrow \infty} \frac{M_{\mathbf{n}}}{(2|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}} \\ & \leq 20(EY_{\mathbf{1}}^2)^{1/2} + 2 \limsup_{\mathbf{n} \rightarrow \infty} \frac{EM_{\mathbf{n}}}{(2|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}}. \end{aligned} \quad (3.1)$$

Proof. Let $0 < \epsilon < 1$ be an arbitrary but fixed number. Let $b_m = \frac{\epsilon}{40}(EY_{\mathbf{1}}^2)^{1/2}(m/\log \log m)^{1/2}$, $f_{\mathbf{k}}(x) = (-b_{|\mathbf{k}|}) \vee (x \wedge b_{|\mathbf{k}|})$, $g_{\mathbf{k}}(x) = x - f_{\mathbf{k}}(x)$. Define

$$\begin{aligned} \bar{Y}_{\mathbf{k}} &= f_{\mathbf{k}}(Y_{\mathbf{k}}) - Ef_{\mathbf{k}}(Y_{\mathbf{k}}), \quad \hat{Y}_{\mathbf{k}} = g_{\mathbf{k}}(Y_{\mathbf{k}}), \\ \bar{T}_{\mathbf{n}} &= \sum_{\mathbf{k} \leq \mathbf{n}} \bar{Y}_{\mathbf{k}}, \quad \bar{M}_{\mathbf{n}} = \max_{\mathbf{k} \leq \mathbf{n}} |\bar{T}_{\mathbf{k}}|, \quad \hat{T}_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} \hat{Y}_{\mathbf{k}}. \end{aligned}$$

First, we show that

$$\frac{\hat{T}_{\mathbf{n}} - E\hat{T}_{\mathbf{n}}}{(2|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}} \rightarrow 0 \quad a.s. \quad \text{as } \mathbf{n} \rightarrow \infty. \quad (3.2)$$

Since

$$\begin{aligned} & \frac{\sum_{\mathbf{k} \leq \mathbf{n}} E|\hat{Y}_{\mathbf{k}}|}{(2|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}} \leq \frac{\sum_{\mathbf{k} \leq \mathbf{n}} E|Y_{\mathbf{k}}|I\{|Y_{\mathbf{k}}| \geq b_{|\mathbf{k}|}\}}{(2|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}} \\ & \leq C \frac{\sum_{\mathbf{k} \leq \mathbf{n}} (\frac{\log \log |\mathbf{k}|}{|\mathbf{k}|})^{1/2} EY_{\mathbf{1}}^2 I\{|Y_{\mathbf{1}}| \geq b_{|\mathbf{k}|}\}}{(2|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}} \\ & \leq C \frac{\sum_{\mathbf{k} \leq \mathbf{n}} |\mathbf{k}|^{-1/2} EY_{\mathbf{1}}^2 I\{|Y_{\mathbf{1}}| \geq b_{|\mathbf{k}|}\}}{|\mathbf{n}|^{1/2}} \\ & = o(1) \frac{\sum_{\mathbf{k} \leq \mathbf{n}} |\mathbf{k}|^{-1/2}}{|\mathbf{n}|^{1/2}} \rightarrow 0 \quad \text{as } |\mathbf{n}| \rightarrow \infty, \end{aligned} \quad (3.3)$$

it is enough to show that

$$\frac{\sum_{\mathbf{k} \leq \mathbf{n}} |\hat{Y}_{\mathbf{k}}|}{(2|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}} \rightarrow 0 \quad a.s. \quad (3.4)$$

For $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$, let $I(\mathbf{n}) = \{\mathbf{k} = (k_1, \dots, k_d) : 2^{n_i-1} \leq k_i \leq 2^{n_i} - 1, i = 1, \dots, d\}$.

(3.4) will be true if we have

$$\frac{\sum_{\mathbf{k} \in I(\mathbf{n})} \hat{Y}_{\mathbf{k}}^+}{(2\|\mathbf{n}\| \log \log 2\|\mathbf{n}\|)^{1/2}} \rightarrow 0 \quad a.s. \quad (3.5)$$

and

$$\frac{\sum_{\mathbf{k} \in I(\mathbf{n})} \widehat{Y}_{\mathbf{k}}^-}{(2^{\|\mathbf{n}\|} \log \log 2^{\|\mathbf{n}\|})^{1/2}} \rightarrow 0 \quad a.s. \quad (3.6)$$

We show (3.5) only since (3.6) can be showed similarly. Let $\alpha_m := \alpha(m) = (2m \log \log m)^{1/2}$ and $Z_{\mathbf{k}} = (\widehat{Y}_{\mathbf{k}}^+) \wedge \alpha_{|\mathbf{k}|}$. It is easily seen that $Z_{\mathbf{k}} = 0$ if $Y_{\mathbf{k}} \leq b_{|\mathbf{k}|}$, $Z_{\mathbf{k}} = Y_{\mathbf{k}} - b_{|\mathbf{k}|}$ if $b_{|\mathbf{k}|} \leq Y_{\mathbf{k}} \leq b_{|\mathbf{k}|} + \alpha_{|\mathbf{k}|}$ and $Z_{\mathbf{k}} = \alpha_{|\mathbf{k}|}$ if $Y_{\mathbf{k}} \geq b_{|\mathbf{k}|} + \alpha_{|\mathbf{k}|}$. Also, $\{Z_{\mathbf{k}}; k \in \mathbb{N}^d\}$ is a negatively associated field of random variables. Obviously,

$$\begin{aligned} \sum_{\mathbf{n}} \mathbb{P}(\widehat{Y}_{\mathbf{n}}^+ \neq Z_{\mathbf{n}}) &\leq \sum_{\mathbf{n}} \mathbb{P}(Y_{\mathbf{n}} \geq \alpha_{|\mathbf{n}|}) \leq C \sum_{m=1}^{\infty} (\log m)^{d-1} \mathbb{P}(Y_{\mathbf{1}} \geq \alpha_m) \\ &\leq C \mathbb{E} Y_{\mathbf{1}}^2 \log^{d-1}(|Y_{\mathbf{1}}|) / \log \log(|Y_{\mathbf{1}}|) < \infty. \end{aligned}$$

Also by (3.3), using the notation $2^{\mathbf{n}} = (2^{n_1}, \dots, 2^{n_d})$ for $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$, one has that,

$$\frac{|\sum_{\mathbf{k} \in I(\mathbf{n})} \mathbb{E} Z_{\mathbf{k}}|}{(2^{\|\mathbf{n}\|} \log \log 2^{\|\mathbf{n}\|})^{1/2}} \leq \frac{\sum_{\mathbf{k} \leq 2^{\mathbf{n}}} \mathbb{E} |\widehat{Y}_{\mathbf{k}}|}{(2^{\|\mathbf{n}\|} \log \log 2^{\|\mathbf{n}\|})^{1/2}} \rightarrow 0.$$

So, (3.5) is equivalent to

$$\frac{\sum_{\mathbf{k} \in I(\mathbf{n})} (Z_{\mathbf{k}} - \mathbb{E} Z_{\mathbf{k}})}{(2^{\|\mathbf{n}\|} \log \log 2^{\|\mathbf{n}\|})^{1/2}} \rightarrow 0 \quad a.s. \quad (3.7)$$

Let

$$\Lambda(\mathbf{n}) = \sum_{\mathbf{k} \in I(\mathbf{n})} \frac{\mathbb{E} Y_{\mathbf{1}}^2 I\{b_{|\mathbf{k}|} < |Y_{\mathbf{1}}| \leq 2\alpha_{|\mathbf{k}|}\}}{\alpha_{|\mathbf{k}|}^2}.$$

Note that $b_{|\mathbf{k}|} < \alpha_{|\mathbf{k}|}$ if $|\mathbf{k}|$ is large enough. From Lemma 2.1, it follows that for $\|\mathbf{n}\|$ large enough and any $\delta > 0$,

$$\begin{aligned}
& \mathbb{P}\left(\left|\sum_{\mathbf{k} \in I(\mathbf{n})} (Z_{\mathbf{k}} - \mathbb{E}Z_{\mathbf{k}})\right| \geq \delta(2^{\|\mathbf{n}\|} \log \log 2^{\|\mathbf{n}\|})^{1/2}\right) \\
& \leq C \frac{\mathbb{E}\left|\sum_{\mathbf{k} \in I(\mathbf{n})} (Z_{\mathbf{k}} - \mathbb{E}Z_{\mathbf{k}})\right|^4}{(2^{\|\mathbf{n}\|} \log \log 2^{\|\mathbf{n}\|})^2} \\
& \leq C(\alpha(2^{\|\mathbf{n}\|}))^{-4} \left\{ \left(\sum_{\mathbf{k} \in I(\mathbf{n})} \mathbb{E}|Z_{\mathbf{k}}|^2\right)^2 + \sum_{\mathbf{k} \in I(\mathbf{n})} \mathbb{E}|Z_{\mathbf{k}}|^4 \right\} \\
& \leq C(\alpha(2^{\|\mathbf{n}\|}))^{-4} \left\{ \left(\sum_{\mathbf{k} \in I(\mathbf{n})} \mathbb{E}Y_{\mathbf{1}}^2 I\{b_{|\mathbf{k}|} < |Y_{\mathbf{1}}| \leq \alpha_{|\mathbf{k}|} + b_{|\mathbf{k}|}\}\right)^2 \right. \\
& \quad \left. + \sum_{\mathbf{k} \in I(\mathbf{n})} \mathbb{E}Y_{\mathbf{1}}^4 I\{b_{|\mathbf{k}|} < |Y_{\mathbf{1}}| \leq \alpha_{|\mathbf{k}|} + b_{|\mathbf{k}|}\}\right\} \\
& \quad + C(\alpha(2^{\|\mathbf{n}\|}))^{-4} \left\{ \left(\sum_{\mathbf{k} \in I(\mathbf{n})} \alpha_{|\mathbf{k}|}^2 \mathbb{P}(|Y_{\mathbf{1}}| \geq \alpha_{|\mathbf{k}|} + b_{|\mathbf{k}|})\right)^2 \right. \\
& \quad \left. + \sum_{\mathbf{k} \in I(\mathbf{n})} \alpha_{|\mathbf{k}|}^4 \mathbb{P}(|Y_{\mathbf{1}}| \geq \alpha_{|\mathbf{k}|} + b_{|\mathbf{k}|})\right\} \\
& \leq C \left\{ \Lambda^2(\mathbf{n}) + \sum_{\mathbf{k} \in I(\mathbf{n})} \frac{\mathbb{E}Y_{\mathbf{1}}^4 I\{b_{|\mathbf{k}|} < |Y_{\mathbf{1}}| \leq 2\alpha_{|\mathbf{k}|}\}}{\alpha_{|\mathbf{k}|}^4} \right\} \\
& \quad + C \left\{ \left(\sum_{\mathbf{k} \in I(\mathbf{n})} \mathbb{P}(|Y_{\mathbf{1}}| \geq \alpha_{|\mathbf{k}|})\right)^2 + \sum_{\mathbf{k} \in I(\mathbf{n})} \mathbb{P}(|Y_{\mathbf{1}}| \geq \alpha_{|\mathbf{k}|}) \right\}.
\end{aligned}$$

Obviously,

$$\begin{aligned}
& \sum_{\mathbf{n}} \left\{ \left(\sum_{\mathbf{k} \in I(\mathbf{n})} \mathbb{P}(|Y_{\mathbf{1}}| \geq \alpha_{|\mathbf{k}|})\right)^2 + \sum_{\mathbf{k} \in I(\mathbf{n})} \mathbb{P}(|Y_{\mathbf{1}}| \geq \alpha_{|\mathbf{k}|}) \right\} \\
& \leq C \sum_{\mathbf{n}} \mathbb{P}(|Y_{\mathbf{1}}| \geq \alpha_{|\mathbf{n}|}) \leq C \mathbb{E}Y_{\mathbf{1}}^2 \log^{d-1}(|Y_{\mathbf{1}}|) / \log \log(|Y_{\mathbf{1}}|) < \infty.
\end{aligned}$$

Also,

$$\begin{aligned}
& \sum_{\mathbf{n}} \sum_{\mathbf{k} \in I(\mathbf{n})} \frac{\mathbb{E}Y_{\mathbf{1}}^4 I\{b_{|\mathbf{k}|} < |Y_{\mathbf{1}}| \leq 2\alpha_{|\mathbf{k}|}\}}{\alpha_{|\mathbf{k}|}^4} \leq \sum_{\mathbf{n}} \frac{\mathbb{E}Y_{\mathbf{1}}^4 I\{b_{|\mathbf{n}|} < |Y_{\mathbf{1}}| \leq 2\alpha_{|\mathbf{n}|}\}}{\alpha_{|\mathbf{n}|}^4} \\
& \leq \sum_{m=1}^{\infty} (\log m)^{d-1} \frac{\mathbb{E}Y_{\mathbf{1}}^4 I\{|Y_{\mathbf{1}}| \leq 2\alpha_m\}}{\alpha_m^4} \\
& \leq C \sum_{m=1}^{\infty} (\log m)^{d-1} \sum_{k=1}^m \frac{\mathbb{E}Y_{\mathbf{1}}^4 I\{2\alpha_{k-1} < |Y_{\mathbf{1}}| \leq 2\alpha_k\}}{\alpha_m^4} \\
& \leq C \sum_{k=1}^{\infty} \sum_{m=k}^{\infty} \frac{(\log m)^{d-1}}{(m \log \log m)^2} \mathbb{E}Y_{\mathbf{1}}^4 I\{2\alpha_{k-1} < |Y_{\mathbf{1}}| \leq 2\alpha_k\} \\
& \leq C \sum_{k=1}^{\infty} \frac{(\log k)^{d-1}}{k(\log \log k)^2} (k \log \log k) \mathbb{E}Y_{\mathbf{1}}^2 I\{2\alpha_{k-1} < |Y_{\mathbf{1}}| \leq 2\alpha_k\} \\
& \leq C \mathbb{E}Y_{\mathbf{1}}^2 \log^{d-1}(|Y_{\mathbf{1}}|) / \log \log(|Y_{\mathbf{1}}|) < \infty,
\end{aligned}$$

and, similarly to (3.8) of Li and Wu (1989) we have

$$\sum_{\mathbf{n}} \Lambda^2(\mathbf{n}) < \infty.$$

It follows that for arbitrary $\delta > 0$,

$$\sum_{\mathbf{n}} \mathbb{P}\left(\left|\sum_{\mathbf{k} \in I(\mathbf{n})} (Z_{\mathbf{k}} - \mathbb{E}Z_{\mathbf{k}})\right| \geq \delta(2^{\|\mathbf{n}\|} \log \log 2^{\|\mathbf{n}\|})^{1/2}\right) < \infty,$$

which implies (3.7) by the Borel-Cantelli lemma. Thus (3.2) holds.

Now, by applying (2.6) to $x = (1 + 2\epsilon)(\mathbb{E}Y_1^2)^{1/2}(2d|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}$ and $b = 2b_{|\mathbf{n}|}$, it follows that

$$\begin{aligned} & \mathbb{P}(\overline{M}_{\mathbf{n}} - 2\mathbb{E}\overline{M}_{\mathbf{n}} \geq 20(1 + 2\epsilon)(\mathbb{E}Y_1^2)^{1/2}(2d|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}) \\ & \leq 2^{d+1} \exp\{-(1 + \epsilon)d \log \log |\mathbf{n}|\} \leq 2^{d+1} (\log |\mathbf{n}|)^{-(1+\epsilon)d}. \end{aligned}$$

For $\theta > 1$ and $\mathbf{m} \in \mathbb{N}^d$, let $\mathbf{N}_{\mathbf{m}} = ([\theta^{m_1}], \dots, [\theta^{m_d}])$. It follows that

$$\begin{aligned} & \sum_{\mathbf{m}} \mathbb{P}(\overline{M}_{\mathbf{N}_{\mathbf{m}}} - 2\mathbb{E}\overline{M}_{\mathbf{N}_{\mathbf{m}}} \geq 20(1 + 2\epsilon)(\mathbb{E}Y_1^2)^{1/2}(2d|\mathbf{N}_{\mathbf{m}}| \log \log |\mathbf{N}_{\mathbf{m}}|)^{1/2}) \\ & \leq C \sum_{\mathbf{m}} \|\mathbf{m}\|^{-(1+\epsilon)d} \leq C \sum_{i=1}^{\infty} i^{d-1} i^{-(1+\epsilon)d} < \infty. \end{aligned}$$

From the Borel-Cantelli lemma, it follows that

$$\begin{aligned} & \limsup_{\mathbf{m} \rightarrow \infty} \frac{\overline{M}_{\mathbf{N}_{\mathbf{m}}}}{(2d|\mathbf{N}_{\mathbf{m}}| \log \log |\mathbf{N}_{\mathbf{m}}|)^{1/2}} \\ & \leq 20(1 + 2\epsilon)(\mathbb{E}Y_1^2)^{1/2} + 2 \limsup_{\mathbf{m} \rightarrow \infty} \frac{\mathbb{E}\overline{M}_{\mathbf{N}_{\mathbf{m}}}}{(2d|\mathbf{N}_{\mathbf{m}}| \log \log |\mathbf{N}_{\mathbf{m}}|)^{1/2}} \\ & \leq 20(1 + 2\epsilon)(\mathbb{E}Y_1^2)^{1/2} + 2 \limsup_{\mathbf{n} \rightarrow \infty} \frac{\mathbb{E}\overline{M}_{\mathbf{n}}}{(2d|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}}. \end{aligned} \quad (3.8)$$

Also, from (3.3) it follows that

$$\lim_{\mathbf{n} \rightarrow \infty} \frac{\mathbb{E}\widehat{M}_{\mathbf{n}}}{(2d|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}} = 0. \quad (3.9)$$

So, by (3.8) and (3.9) we have

$$\begin{aligned} & \limsup_{\mathbf{n} \rightarrow \infty} \frac{\overline{M}_{\mathbf{n}}}{(2d|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}} \\ & \leq \limsup_{\mathbf{n} \rightarrow \infty} \max_{\mathbf{N}_{\mathbf{m}-1} \leq \mathbf{n} \leq \mathbf{N}_{\mathbf{m}}} \frac{\overline{M}_{\mathbf{N}_{\mathbf{m}}}}{(2d|\mathbf{N}_{\mathbf{m}-1}| \log \log |\mathbf{N}_{\mathbf{m}-1}|)^{1/2}} \\ & \leq \theta^{d/2} \left(20(1 + 2\epsilon)(\mathbb{E}Y_1^2)^{1/2} + 2 \limsup_{\mathbf{n} \rightarrow \infty} \frac{\mathbb{E}\overline{M}_{\mathbf{n}}}{(2d|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}} \right) \\ & = \theta^{d/2} \left(20(1 + 2\epsilon)(\mathbb{E}Y_1^2)^{1/2} + 2 \limsup_{\mathbf{n} \rightarrow \infty} \frac{\mathbb{E}M_{\mathbf{n}}}{(2d|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}} \right) \quad a.s. \end{aligned} \quad (3.10)$$

Finally, from (3.2) and (3.10) it follows that (3.1) holds.

Lemma 3.2 *Let $\{X_{\mathbf{k}}; \mathbf{k} \in \mathbb{N}^d\}$ be a negatively associated field of bounded random variables with $EX_{\mathbf{k}} = 0$ for all $\mathbf{k} \in \mathbb{N}^d$. Denote by $S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}$. Then for any $\delta > 0$,*

$$\limsup_{\mathbf{n} \rightarrow \infty} \frac{\max_{\mathbf{k} \leq \delta \mathbf{n}} |S_{\mathbf{n}+\mathbf{k}} - S_{\mathbf{n}}|}{(2d|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}} \leq 80d(\delta(1+2\delta)^d)^{1/2} \sup_{\mathbf{k}} (EX_{\mathbf{k}}^2)^{1/2} \quad (3.11)$$

Proof. Assume that $|X_{\mathbf{k}}| \leq b$ a.s. with $0 < b < \infty$. Denote $\beta = \sup_{\mathbf{k}} (EX_{\mathbf{k}}^2)^{1/2}$. Since by Lemma 2.2, for each \mathbf{n} and \mathbf{m}

$$\mathbb{E} \left| \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}+\mathbf{m}} \right|^4 \leq A|\mathbf{n}|^2 b^4,$$

from the Theorem of Móricz (1983) it follows that for each \mathbf{n} ,

$$\mathbb{E} \max_{\mathbf{m} \leq \mathbf{n}} \left(\sum_{\mathbf{k} \leq \mathbf{m}} X_{\mathbf{k}} \right)^2 \leq \left(\mathbb{E} \max_{\mathbf{m} \leq \mathbf{n}} \left(\sum_{\mathbf{k} \leq \mathbf{m}} X_{\mathbf{k}} \right)^4 \right)^{1/2} \leq C(|\mathbf{n}|^2 b^4)^{1/2} = C|\mathbf{n}| b^2. \quad (3.12)$$

For fixed \mathbf{n} and $\mathbf{m} \leq \mathbf{n}$, let $Y_{\mathbf{k}} = 0$ for $\mathbf{k} \leq \mathbf{m}$ and $Y_{\mathbf{k}} = X_{\mathbf{k}}$ otherwise, and let $T_{\mathbf{k}} = \sum_{\mathbf{i} \leq \mathbf{k}} Y_{\mathbf{i}}$ for $\mathbf{k} \leq \mathbf{m} + \delta \mathbf{n}$, $B_{\mathbf{m}+\delta \mathbf{n}}^2 = \sum_{\mathbf{k} \leq \mathbf{m}+\delta \mathbf{n}} \mathbb{E} Y_{\mathbf{k}}^2$. Then for $\mathbf{m} \leq \mathbf{n}$,

$$B_{\mathbf{m}+\delta \mathbf{n}}^2 \leq (|\mathbf{m} + \delta \mathbf{n}| - |\mathbf{m}|) \beta^2 \leq \delta d(1+\delta)^d |\mathbf{n}| \beta^2.$$

So by Proposition 2.3,

$$\begin{aligned} & \mathbb{P} \left(\max_{\mathbf{k} \leq \delta \mathbf{n}} |S_{\mathbf{m}+\mathbf{k}} - S_{\mathbf{m}}| - 2\mathbb{E} \max_{\mathbf{k} \leq \delta \mathbf{n}} |S_{\mathbf{m}+\mathbf{k}} - S_{\mathbf{m}}| \geq 20x \right) \\ &= \mathbb{P} \left(\max_{\mathbf{k} \leq \mathbf{m}+\delta \mathbf{n}} |T_{\mathbf{k}}| - \mathbb{E} \max_{\mathbf{k} \leq \mathbf{m}+\delta \mathbf{n}} |T_{\mathbf{k}}| \geq 20x \right) \\ &\leq 2^{d+1} \exp \left\{ - \frac{x^2}{2(bx + B_{\mathbf{m}+\delta \mathbf{n}}^2)} \right\} \\ &\leq 2^{d+1} \exp \left\{ - \frac{x^2}{2(bx + \delta d(1+\delta)^d |\mathbf{n}| \beta^2)} \right\}. \end{aligned} \quad (3.13)$$

Note that by (3.12) we have

$$\begin{aligned} & \max_{\mathbf{m} \leq \mathbf{n}} \mathbb{E} \max_{\mathbf{k} \leq \delta \mathbf{n}} |S_{\mathbf{m}+\mathbf{k}} - S_{\mathbf{m}}| \leq 2\mathbb{E} \max_{\mathbf{k} \leq \mathbf{n}+\delta \mathbf{n}} |S_{\mathbf{k}}| \leq 2 \left(\mathbb{E} \max_{\mathbf{k} \leq \mathbf{n}+\delta \mathbf{n}} S_{\mathbf{k}}^2 \right)^{1/2} \\ &\leq K(|\mathbf{n} + \delta \mathbf{n}| b^2)^{1/2} \leq K(1+\delta)^{d/2} |\mathbf{n}|^{1/2} b. \end{aligned}$$

From (3.13) it follows that for $|\mathbf{n}|$ large enough

$$\begin{aligned} & \max_{\mathbf{m} \leq \mathbf{n}} \mathbb{P} \left(\max_{\mathbf{k} \leq \delta \mathbf{n}} |S_{\mathbf{m}+\mathbf{k}} - S_{\mathbf{m}}| \geq 20d(1+2\epsilon)(2\delta(1+\delta)^d d^2 \beta^2 |\mathbf{n}| \log \log |\mathbf{n}|)^{1/2} \right) \\ &\leq 2^{d+1} \exp \left\{ - (1+\epsilon)d \log \log |\mathbf{n}| \right\}. \end{aligned}$$

Now, let $I_{\mathbf{p}} = \{\mathbf{p} + \mathbf{k} : \mathbf{k} \leq [\delta \mathbf{n}]\}$. Then there are at most $[(\delta^{-1} + 1)^d] + 1$ such $I_{\mathbf{p}}$ s whose union covers $\{\mathbf{k} : \mathbf{k} \leq \mathbf{n}\}$. It follows that

$$\begin{aligned}
& \mathbb{P}\left(\max_{\mathbf{m} \leq \mathbf{n}} \max_{\mathbf{k} \leq \delta \mathbf{n}} |S_{\mathbf{m}+\mathbf{k}} - S_{\mathbf{m}}| \geq (1 + 2\epsilon)80d(\delta(1 + 2\delta))^d \beta^2 |\mathbf{n}| \log \log |\mathbf{n}|^{1/2}\right) \\
& \leq \left(\left(\frac{1}{\delta} + 1\right)^d + 1\right) \max_{\mathbf{p}} \mathbb{P}\left(\max_{\mathbf{m} \in I_{\mathbf{p}}} \max_{\mathbf{k} \leq \delta \mathbf{n}} |S_{\mathbf{m}+\mathbf{k}} - S_{\mathbf{m}}| \geq 40d(1 + 2\epsilon)(4\delta(1 + 2\delta))^d \beta^2 |\mathbf{n}| \log \log |\mathbf{n}|^{1/2}\right) \\
& \leq 4\left(\frac{1}{\delta} + 1\right)^d \max_{\mathbf{m} \leq \mathbf{n}} \mathbb{P}\left(\max_{\mathbf{k} \leq 2\delta \mathbf{n}} |S_{\mathbf{m}+\mathbf{k}} - S_{\mathbf{m}}| \geq 20d(1 + 2\epsilon)(4\delta(1 + 2\delta))^d \beta^2 |\mathbf{n}| \log \log |\mathbf{n}|^{1/2}\right) \\
& \leq 4\left(\frac{1}{\delta} + 1\right)^d 2^{d+1} \exp\left(- (1 + \epsilon)d \log \log |\mathbf{n}|\right).
\end{aligned}$$

If we choose $\mathbf{n}_{\mathbf{p}} = ([\theta^{p_1}], \dots, [\theta^{p_d}])$, then the sum of the above probability is finite. And then by the Borel-Cantelli lemma,

$$\limsup_{\mathbf{p} \rightarrow \infty} \frac{\max_{\mathbf{m} \leq \mathbf{n}_{\mathbf{p}}} \max_{\mathbf{k} \leq \delta \mathbf{n}_{\mathbf{p}}} |S_{\mathbf{m}+\mathbf{k}} - S_{\mathbf{m}}|}{(2d|\mathbf{n}_{\mathbf{p}}| \log \log |\mathbf{n}_{\mathbf{p}}|)^{1/2}} \leq 80d(\delta(1 + 2\delta))^d \beta^{1/2} \quad a.s.,$$

which implies (3.11) easily.

Now, we turn to the

Proof of Theorem 1.1: For $b > 0$, let $g_b(x) = (-b) \vee (x \wedge b)$ and $h_b(x) = x - g_b(x)$. Then $g_b(x)$ and $h_b(x)$ are both non-decreasing functions of x . Let $\bar{X}_{\mathbf{k}} = g_b(X_{\mathbf{k}}) - \mathbb{E}g_b(X_{\mathbf{k}})$, $\hat{X}_{\mathbf{k}} = h_b(X_{\mathbf{k}}) - \mathbb{E}h_b(X_{\mathbf{k}})$, $\bar{S}_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} \bar{X}_{\mathbf{k}}$ and $\bar{M}_{\mathbf{n}} = \max_{\mathbf{k} \leq \mathbf{n}} |\bar{S}_{\mathbf{k}}|$. And define $\hat{S}_{\mathbf{k}}$, $\hat{M}_{\mathbf{n}}$ similarly. Then $X_{\mathbf{k}} = \bar{X}_{\mathbf{k}} + \hat{X}_{\mathbf{k}}$ and, $\{\bar{X}_{\mathbf{k}}; \mathbf{k} \in \mathbb{N}^d\}$ and $\{\hat{X}_{\mathbf{k}}; \mathbf{k} \in \mathbb{N}^d\}$ are both strictly stationary negatively associated fields of random variables with $\mathbb{E}\bar{X}_{\mathbf{k}} = \mathbb{E}\hat{X}_{\mathbf{k}} = 0$ and $|\bar{X}_{\mathbf{k}}| \leq 2b$ a.s. Let $\bar{\Upsilon}(\mathbf{j} - \mathbf{i}) = \text{Cov}(\bar{X}_{\mathbf{j}}, \bar{X}_{\mathbf{i}})$ and $\bar{\sigma}^2 = \sum_{\mathbf{j} \in \mathbb{Z}^d} \bar{\Upsilon}(\mathbf{j})$. Then $\bar{\sigma}^2 \rightarrow \sigma^2$ as $b \rightarrow \infty$. Also by Lemma 3.1 and Proposition 2.2,

$$\begin{aligned}
& \limsup_{\mathbf{n} \rightarrow \infty} \frac{|\hat{S}_{\mathbf{n}}|}{(2d|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}} \\
& \leq 20(\mathbb{E}\hat{X}_{\mathbf{1}}^2)^{1/2} + 2 \limsup_{\mathbf{n} \rightarrow \infty} \frac{\mathbb{E}\hat{M}_{\mathbf{n}}}{(2d|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}} \\
& \leq 20(\mathbb{E}X_{\mathbf{1}}^2 I\{|X_{\mathbf{1}}| \geq b\})^{1/2} \rightarrow 0 \quad \text{as } b \rightarrow \infty \quad a.s.
\end{aligned}$$

It remains to show that

$$\limsup_{\mathbf{n} \rightarrow \infty} \frac{|\bar{S}_{\mathbf{n}}|}{(2d|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}} = \bar{\sigma}^2 \quad a.s.$$

So, without loss of generality, we can assume that $|X_{\mathbf{k}}| \leq b$ a.s. for some $0 < b < \infty$. Also, we can assume $\sigma > 0$, for otherwise, we can consider the field $\{X_{\mathbf{n}} + \epsilon Z_{\mathbf{n}}; \mathbf{n} \in \mathbb{N}^d\}$ instead, where $\{Z_{\mathbf{n}}; \mathbf{n} \in \mathbb{N}^d\}$ is a field of i.i.d. standard normal random variables and $\epsilon > 0$ is an arbitrary number. First we show that

$$\limsup_{\mathbf{n} \rightarrow \infty} \frac{|S_{\mathbf{n}}|}{(2d|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}} \leq \sigma \quad a.s. \quad (3.14)$$

Since $\mathbb{E}S_{\mathbf{1}m}^2/m^d \rightarrow \sigma^2$, we can choose m large enough such that

$$\mathbb{E}S_{\mathbf{1}m}^2 \leq (1 + \epsilon)^2 m^d \sigma^2.$$

Let $I_m = \{\mathbf{k} : 1 \leq k_i \leq m, i = 1, \dots, d\}$ and $Y_{\mathbf{k}} = \sum_{i \in I_m} X_{(\mathbf{k}-1)m+i}$. Then $\mathbb{E}Y_{\mathbf{k}}^2 = \mathbb{E}S_{\mathbf{1}m}^2$.

For $\theta > 1$, let $\mathbf{N}_{\mathbf{k}} = [\theta^{\mathbf{k}}] = ([\theta^{k_1}], \dots, [\theta^{k_d}])$. From (2.1), it follows that

$$\mathbb{P}(|S_{\mathbf{N}_{\mathbf{k}}m}| \geq x) = \mathbb{P}\left(\left|\sum_{i \leq \mathbf{N}_{\mathbf{k}}} Y_i\right| \geq x\right) \leq 2 \exp\left\{-\frac{x^2}{2(2mbx + |\mathbf{N}_{\mathbf{k}}|\mathbb{E}Y_{\mathbf{1}}^2)}\right\}.$$

Let $x = (1 + \epsilon)(\mathbb{E}Y_{\mathbf{1}}^2)^{1/2}(2d|\mathbf{N}_{\mathbf{k}}| \log \log |\mathbf{N}_{\mathbf{k}}|)^{1/2}$. It follows that

$$\begin{aligned} & \sum_{\mathbf{k}} \mathbb{P}(|S_{\mathbf{N}_{\mathbf{k}}m}| \geq (1 + \epsilon)(\mathbb{E}Y_{\mathbf{1}}^2)^{1/2}(2d|\mathbf{N}_{\mathbf{k}}| \log \log |\mathbf{N}_{\mathbf{k}}|)^{1/2}) \\ & \leq C \sum_{\mathbf{k}} \exp\{- (1 + \epsilon)d \log \log |\mathbf{N}_{\mathbf{k}}|\} \leq C \sum_{\mathbf{k}} \|\mathbf{k}\|^{-(1+\epsilon)d} \\ & \leq C \sum_{i=1}^{\infty} i^{d-1} i^{-(1+\epsilon)d} < \infty. \end{aligned}$$

From the Borel-Cantelli lemma, it follows that

$$\begin{aligned} & \limsup_{\mathbf{k} \rightarrow \infty} \frac{|S_{\mathbf{N}_{\mathbf{k}}m}|}{(2d|\mathbf{N}_{\mathbf{k}}m| \log \log |\mathbf{N}_{\mathbf{k}}m|)^{1/2}} \\ & = \frac{1}{m^{d/2}} \limsup_{\mathbf{k} \rightarrow \infty} \frac{|S_{\mathbf{N}_{\mathbf{k}}m}|}{(2d|\mathbf{N}_{\mathbf{k}}| \log \log |\mathbf{N}_{\mathbf{k}}|)^{1/2}} \\ & \leq m^{-d/2} (\mathbb{E}Y_{\mathbf{1}}^2)^{1/2} \leq (1 + \epsilon)\sigma \quad a.s. \end{aligned}$$

So, from Lemma 3.2 it follows that

$$\begin{aligned} & \limsup_{\mathbf{n} \rightarrow \infty} \frac{|S_{\mathbf{n}}|}{(2d|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}} \\ & \leq \limsup_{\mathbf{k} \rightarrow \infty} \max_{\mathbf{N}_{\mathbf{k}}m \leq \mathbf{n} \leq \mathbf{N}_{\mathbf{k}+1}m} \frac{|S_{\mathbf{n}}|}{(2d|\mathbf{N}_{\mathbf{k}}m| \log \log |\mathbf{N}_{\mathbf{k}}m|)^{1/2}} \\ & \leq \limsup_{\mathbf{k} \rightarrow \infty} \frac{|S_{\mathbf{N}_{\mathbf{k}}m}|}{(2d|\mathbf{N}_{\mathbf{k}}m| \log \log |\mathbf{N}_{\mathbf{k}}m|)^{1/2}} \\ & \quad + \limsup_{\mathbf{k} \rightarrow \infty} \max_{\mathbf{N}_{\mathbf{k}}m \leq \mathbf{n} \leq \mathbf{N}_{\mathbf{k}+1}m} \frac{|S_{\mathbf{n}} - S_{\mathbf{N}_{\mathbf{k}}m}|}{(2d|\mathbf{N}_{\mathbf{k}}m| \log \log |\mathbf{N}_{\mathbf{k}}m|)^{1/2}} \\ & \leq (1 + \epsilon)\sigma + 80d((\theta - 1)(1 + 2(\theta - 1))^d)^{1/2} (\mathbb{E}X^2)^{1/2} \quad a.s. \end{aligned}$$

Letting $\theta \rightarrow 1$ and $\epsilon \rightarrow 0$ completes the proof of (3.14).

Next we show that for any $\epsilon > 0$,

$$\limsup_{\mathbf{n} \rightarrow \infty} \frac{S_{\mathbf{n}}}{(2d|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}} \geq (1 - 9\epsilon)\sigma \quad a.s. \quad (3.15)$$

For $k \in \mathbb{N}$, let $m_k = \lfloor 2^{k^{1+\epsilon}} \rfloor$, $p_k = \lfloor k^{-2} 2^{k^{1+\epsilon}} \rfloor$, $N_k = (m_k + p_k)k^4$. For $\mathbf{k} \in \mathbb{N}^d$ and an integer q , let $\mathbf{m}_{\mathbf{k}} = (m_{k_1}, \dots, m_{k_d})$, $\mathbf{p}_{\mathbf{k}} = (p_{k_1}, \dots, p_{k_d})$, $\mathbf{N}_{\mathbf{k}} = (N_{k_1}, \dots, N_{k_d})$ and $\mathbf{k}^q = (k_1^q, \dots, k_d^q)$.

We first show the following equality

$$\sum_{\mathbf{k}} \mathbb{P}(S_{\mathbf{N}_{\mathbf{k}}} \geq (1 - 7\epsilon)\sigma(2d|\mathbf{N}_{\mathbf{k}}| \log \log |\mathbf{N}_{\mathbf{k}}|)^{1/2}) = \infty. \quad (3.16)$$

Set $I_i = I_i(\mathbf{k}) = \{(i-1)(\mathbf{m}_{\mathbf{k}} + \mathbf{p}_{\mathbf{k}}) + \mathbf{1} \leq \mathbf{m} \leq (i-1)(\mathbf{m}_{\mathbf{k}} + \mathbf{p}_{\mathbf{k}}) + \mathbf{m}_{\mathbf{k}}\}$, $A_{\mathbf{k}} = \bigcup_{1 \leq i \leq k^4} I_i$ and $B_{\mathbf{k}} = \{\mathbf{m} : \mathbf{m} \leq \mathbf{N}_{\mathbf{k}}\} \setminus A_{\mathbf{k}}$. Then $Card A_{\mathbf{k}} = |\mathbf{k}|^4 |\mathbf{m}_{\mathbf{k}}| \sim |\mathbf{N}_{\mathbf{k}}|$ and $Card B_{\mathbf{k}} = |\mathbf{N}_{\mathbf{k}}| - Card A_{\mathbf{k}} = o(|\mathbf{N}_{\mathbf{k}}|)$. Let

$$v_{i,1} = \sum_{\mathbf{j} \in I_i} X_{\mathbf{j}}$$

and

$$S_{\mathbf{k},1} = \sum_{1 \leq i \leq k^4} v_{i,1} = \sum_{\mathbf{j} \in A_{\mathbf{k}}} X_{\mathbf{j}}, \quad S_{\mathbf{k},2} = \sum_{\mathbf{j} \in B_{\mathbf{k}}} X_{\mathbf{j}}.$$

Clearly,

$$S_{\mathbf{N}_{\mathbf{k}}} = S_{\mathbf{k},1} + S_{\mathbf{k},2}$$

and

$$\sum_{\mathbf{j} \in B_{\mathbf{k}}} \mathbb{E} X_{\mathbf{j}}^2 / |\mathbf{N}_{\mathbf{k}}| \leq \mathbb{E} X_{\mathbf{1}}^2 Card B_{\mathbf{k}} / |\mathbf{N}_{\mathbf{k}}| \rightarrow 0, \quad \mathbf{k} \rightarrow \infty.$$

From (2.1), it follows that for any $\epsilon > 0$

$$\begin{aligned} & \sum_{\mathbf{k}} \mathbb{P}(|S_{\mathbf{k},2}| \geq \epsilon\sigma(2d|\mathbf{N}_{\mathbf{k}}| \log \log |\mathbf{N}_{\mathbf{k}}|)^{1/2}) \\ & \leq 2 \sum_{\mathbf{k}} \exp \left\{ - \frac{\epsilon^2 \sigma^2 2d |\mathbf{N}_{\mathbf{k}}| \log \log |\mathbf{N}_{\mathbf{k}}|}{2\{b\epsilon\sigma(2d|\mathbf{N}_{\mathbf{k}}| \log \log |\mathbf{N}_{\mathbf{k}}|)^{1/2} + \sum_{\mathbf{j} \in B_{\mathbf{k}}} \mathbb{E} X_{\mathbf{j}}^2\}} \right\} \\ & \leq C \sum_{\mathbf{k}} \exp \{ - 3d \log \log |\mathbf{N}_{\mathbf{k}}| \} < \infty. \end{aligned}$$

Thus in order to prove (3.16) is enough to show that

$$\sum_{\mathbf{k}} \mathbb{P}(S_{\mathbf{k},1} \geq (1 - 6\epsilon)\sigma(2d|\mathbf{N}_{\mathbf{k}}| \log \log |\mathbf{N}_{\mathbf{k}}|)^{1/2}) = \infty. \quad (3.17)$$

Let $B_{\mathbf{k}}^2 = \sum_{1 \leq i \leq \mathbf{k}^4} \mathbb{E} v_{i,1}^2$. It is easily seen that

$$B_{\mathbf{k}}^2 \sim |\mathbf{k}|^4 |\mathbf{m}_{\mathbf{k}}| \sigma^2 \sim |\mathbf{N}_{\mathbf{k}}| \sigma^2.$$

From Lemma 2.3, it follows that

$$\begin{aligned} & \mathbb{P}(S_{\mathbf{k},1} \geq (1 - 6\epsilon)\sigma(2d|\mathbf{N}_{\mathbf{k}}| \log \log |\mathbf{N}_{\mathbf{k}}|)^{1/2}) \\ & \geq \{1 - \Phi(1 + (1 - 5\epsilon)(2d \log \log |\mathbf{N}_{\mathbf{k}}|)^{1/2})\} - J_{\mathbf{k},1} - J_{\mathbf{k},2} \end{aligned}$$

where

$$\begin{aligned} J_{\mathbf{k},1} &= O(1)|\mathbf{N}_{\mathbf{k}}|^{-1} \sum_{1 \leq i \neq j \leq \mathbf{k}^4} |\mathbb{E}(v_{i,1} v_{j,1})|, \\ J_{\mathbf{k},2} &= O(1)|\mathbf{N}_{\mathbf{k}}|^{-3/2} \sum_{1 \leq i \leq \mathbf{k}^4} \mathbb{E}|v_{i,1}|^3. \end{aligned}$$

Obviously,

$$\begin{aligned} & \sum_{\mathbf{k}} \{1 - \Phi(1 + (1 - 5\epsilon)(2d \log \log |\mathbf{N}_{\mathbf{k}}|)^{1/2})\} \\ & \geq C \sum_{\mathbf{k}} (\log |\mathbf{N}_{\mathbf{k}}|)^{-(1-4\epsilon)d} \geq C \sum_{\mathbf{k}} \|\mathbf{k}\|^{-(1-2\epsilon)d} = \infty, \end{aligned}$$

and by Lemma 2.1,

$$\sum_{\mathbf{k}} J_{\mathbf{k},2} \leq C_p \sum_{\mathbf{k}} |\mathbf{N}_{\mathbf{k}}|^{-3/2} |\mathbf{k}|^4 (|\mathbf{m}_{\mathbf{k}}|^{3/2} (2b)^3 + |\mathbf{m}_{\mathbf{k}}| (2b)^3) \leq C \sum_{\mathbf{k}} |\mathbf{k}|^{-2} < \infty.$$

Also,

$$\begin{aligned} J_{\mathbf{k},1} &= O(1)|\mathbf{N}_{\mathbf{k}}|^{-1} \sum_{1 \leq i \neq j \leq \mathbf{N}_{\mathbf{k}}, j-i \geq p_{\mathbf{k}}} |\text{Cov}(X_i, X_j)| \\ &= O(1) \sum_{p_{\mathbf{k}} \leq j \leq \mathbf{N}_{\mathbf{k}}} |\text{Cov}(X_1, X_{j+1})| = O(1) \sum_{\mathbf{N}_{\mathbf{k}-1} < j \leq \mathbf{N}_{\mathbf{k}}} |\text{Cov}(X_1, X_{j+1})|, \end{aligned}$$

since $p_{k_i} = o(N_{k_i})$ for $i = 1, \dots, d$. It follows that

$$\sum_{\mathbf{k}} J_{\mathbf{k},1} = O(1) \sum_j |\text{Cov}(X_1, X_{j+1})| < \infty.$$

Hence (3.17) is proved.

Now, let $C_{\mathbf{k}} = \{\mathbf{m} : \mathbf{N}_{\mathbf{k}-1} < \mathbf{m} \leq \mathbf{N}_{\mathbf{k}}\}$, $D_{\mathbf{k}} = \{\mathbf{m} : \mathbf{m} \leq \mathbf{N}_{\mathbf{k}}\} \setminus C_{\mathbf{k}}$ and

$$U_{\mathbf{k}} = \sum_{\mathbf{j} \in C_{\mathbf{k}}} X_{\mathbf{j}}, \quad V_{\mathbf{k}} = \sum_{\mathbf{j} \in D_{\mathbf{k}}} X_{\mathbf{j}}.$$

Then $S_{\mathbf{N}_k} = U_k + V_k$, $CardC_k \sim |\mathbf{N}_k|$ and $CardD_k = o(|\mathbf{N}_k|)$. From (2.1), it follows that for any $\delta > 0$,

$$\sum_{\mathbf{k}} \mathbb{P}(|V_{\mathbf{k}}| \geq \delta \sigma (2d|\mathbf{N}_{\mathbf{k}}| \log \log |\mathbf{N}_{\mathbf{k}}|)^{1/2}) < \infty. \quad (3.18)$$

By the Borel-Cantelli lemma, we conclude that

$$\limsup_{\mathbf{k} \rightarrow \infty} \frac{|V_{\mathbf{k}}|}{(2d|\mathbf{N}_{\mathbf{k}}| \log \log |\mathbf{N}_{\mathbf{k}}|)^{1/2}} = 0 \quad a.s. \quad (3.19)$$

From (3.16) and (3.18), it follows that

$$\sum_{\mathbf{k}} \mathbb{P}(U_{\mathbf{k}} \geq (1 - 5\epsilon)\sigma(2d|\mathbf{N}_{\mathbf{k}}| \log \log |\mathbf{N}_{\mathbf{k}}|)^{1/2}) = \infty. \quad (3.20)$$

Note that $\{U_{\mathbf{k}}; \mathbf{k} \in \mathbb{N}^d\}$ is a negatively associated field. It follows that for any x, y and $\mathbf{i} \neq \mathbf{j}$,

$$\mathbb{P}(U_{\mathbf{i}} \geq x, U_{\mathbf{j}} \geq y) \leq \mathbb{P}(U_{\mathbf{i}} \geq x)\mathbb{P}(U_{\mathbf{j}} \geq y).$$

Hence, by the generalized Borel-Cantelli lemma, (3.20) yields

$$\limsup_{\mathbf{k} \rightarrow \infty} \frac{U_{\mathbf{k}}}{(2d|\mathbf{N}_{\mathbf{k}}| \log \log |\mathbf{N}_{\mathbf{k}}|)^{1/2}} \geq (1 - 5\epsilon)\sigma \quad a.s. \quad (3.21)$$

From (3.19) and (3.21), it follows that (3.15) holds.

Proof of Theorem 1.2: From (1.6), it follows that there exists a constant $0 < C < \infty$ such that

$$\mathbb{P}\left(\limsup_{\mathbf{n} \rightarrow \infty} \frac{|S_{\mathbf{n}}|}{(2d|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}} < C\right) > 0.$$

Then

$$\mathbb{P}\left(\limsup_{\mathbf{n} \rightarrow \infty} \frac{|X_{\mathbf{n}}|}{(2d|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}} < 2C\right) > 0. \quad (3.22)$$

Let

$$A_{\mathbf{n}} = \{|X_{\mathbf{n}}| \geq 2C(2d|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}\},$$

$$A_{\mathbf{n}}^+ = \{X_{\mathbf{n}}^+ \geq 2C(2d|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}\},$$

$$A_{\mathbf{n}}^- = \{X_{\mathbf{n}}^- \geq 2C(2d|\mathbf{n}| \log \log |\mathbf{n}|)^{1/2}\}.$$

Note that $\{I\{A_{\mathbf{n}}^+\}; \mathbf{n} \in \mathbb{N}^d\}$ and $\{I\{A_{\mathbf{n}}^-\}; \mathbf{n} \in \mathbb{N}^d\}$ are both negatively associated fields. It follows that for any \mathbf{m} and \mathbf{n} ,

$$\begin{aligned} \text{Var}\left\{\sum_{\mathbf{m} \leq \mathbf{k} \leq \mathbf{m} + \mathbf{n}} I\{A_{\mathbf{n}}^+\}\right\} &\leq \sum_{\mathbf{m} \leq \mathbf{k} \leq \mathbf{m} + \mathbf{n}} \mathbb{P}(A_{\mathbf{n}}^+), \\ \text{Var}\left\{\sum_{\mathbf{m} \leq \mathbf{k} \leq \mathbf{m} + \mathbf{n}} I\{A_{\mathbf{n}}^-\}\right\} &\leq \sum_{\mathbf{m} \leq \mathbf{k} \leq \mathbf{m} + \mathbf{n}} \mathbb{P}(A_{\mathbf{n}}^-). \end{aligned}$$

It follows that

$$\text{Var}\left\{\sum_{\mathbf{m} \leq \mathbf{k} \leq \mathbf{m} + \mathbf{n}} I\{A_{\mathbf{k}}\}\right\} \leq 2 \sum_{\mathbf{m} \leq \mathbf{k} \leq \mathbf{m} + \mathbf{n}} P(A_{\mathbf{k}}).$$

Hence from Lemma A.6 of Zhang and Wen (2001b), it follows that for any \mathbf{m}

$$(1 - P(\bigcup_{\mathbf{k} \geq \mathbf{m}} A_{\mathbf{k}}))^2 \sum_{\mathbf{k} \geq \mathbf{m}} P(A_{\mathbf{k}}) \leq 2P(\bigcup_{\mathbf{k} \geq \mathbf{m}} A_{\mathbf{k}}). \quad (3.23)$$

Since (3.22) implies $P(A_{\mathbf{n}}, i.o.) < 1$, we conclude that for some \mathbf{m} ,

$$P(\bigcup_{\mathbf{k} \geq \mathbf{m}} A_{\mathbf{k}}) := \beta < 1.$$

Then from (3.23), it follows that

$$\sum_{\mathbf{k} \geq \mathbf{m}} P(A_{\mathbf{k}}) \leq \frac{2\beta}{(1 - \beta^2)} < \infty.$$

So,

$$\sum_{\mathbf{k} \geq \mathbf{m}} P(|X_{\mathbf{1}}| \geq 2C(2d|\mathbf{k}| \log \log |\mathbf{k}|)^{1/2}) < \infty,$$

which implies

$$EX_{\mathbf{1}}^2 \log^{d-1}(|X_{\mathbf{1}}|) / \log \log(|X_{\mathbf{1}}|) < \infty. \quad (3.24)$$

Finally, from (3.24) and the law of the large numbers (c.f. Zhang and Wang 1999), it follows that

$$\lim_{\mathbf{n}} \frac{S_{\mathbf{n}} - |\mathbf{n}|EX_{\mathbf{1}}}{|\mathbf{n}|} \rightarrow 0 \quad a.s.$$

which together with (1.6) yields $EX_{\mathbf{1}} = 0$. And then (1.4) holds.

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