Precise rates in the law of the logarithm under minimal conditions *

ZHANG Li-Xin†

Department of Mathematics, Zhejiang University, Hangzhou 310028, China

ABSTRACT. Let $X, X_1, X_2, \dots$ be i.i.d. random variables, and set $S_n = X_1 + \dots + X_n$, $M_n = \max_{k\le n} |S_k|$, $n \ge 1$. Let $a_n = o(\sqrt{n/\log n})$. By using the strong approximation, we prove that, then for any $r > 1$ and $a > -1/2$,

$$\lim_{\epsilon\searrow\sqrt{r-1}} (\epsilon^2 - (r-1))^{a+1/2} \sum_{n=1}^{\infty} n^{r-2} (\log n)^a \mathsf{P}\Big\{ M_n \ge \sigma\phi(n)\epsilon + a_n \Big\}$$
$$= 2\sqrt{\frac{1}{\pi(r-1)}}\Gamma(a+1/2)$$

holds if and only if

$$\mathsf{E}X = 0, \ \ \mathsf{E}X^2 = \sigma^2 < \infty \ \text{ and } \ \mathsf{E}\big[|X|^{2r}(\log|X|)^{a-r}\big] < \infty.$$

We also show that the widest a_n is $o(\sqrt{n/\log n})$.

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†Department of Mathematics, Zhejiang University, Hangzhou 310028, China,

E-mail: lxzhang@mail.hz.zj.cn

1 Introduction and main results.

Let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d random variables with common distribution function F, and set $S_n = \sum_{k=1}^n X_k$, $M_n = \max_{k\leq n} |S_k|$, $n \geq 1$. Also let $\log x = \ln(x \vee e)$, $\log\log x = \log(\log x)$ and $\phi(x) = \sqrt{2x\log x}$. The following is the well known complete convergence first established by Hsu and Robbins (1947):

$$\sum_{n=1}^{\infty} \mathsf{P}(|S_n| \geq \epsilon n) < \infty, \quad \epsilon > 0$$

if and only if $\mathsf{E}X = 0$ and $\mathsf{E}X^2 < \infty$. Baum and Katz (1965) extended this result and proved the following theorem.

Theorem A *Let $1 \leq p < 2$ and $r \geq p$. Then*

$$\sum_{n=1}^{\infty} n^{r-2} \mathsf{P}(|S_n| \geq \epsilon n^{1/p}) < \infty, \quad \epsilon > 0$$

if and only if $\mathsf{E}X = 0$ and $\mathsf{E}|X|^{rp} < \infty$.

Many authors considered various extensions of the results of Hsu-Robbins and Baum-Katz. Some of them study the precise asymptotics of the infinite sums as $\epsilon \to 0$ (c.f. Heyde (1975), Chen (1978), Spătaru (1999) and Gut and Spătaru (2000a)). But, this kind of results do not hold for $p = 2$. However, by replacing $n^{1/p}$ by $\sqrt{n\log\log n}$, Gut and Spătaru (2000b) established an analogous result called the precise asymptotics of the law of the iterated logarithm, and Zhang (2001) gave the sufficient and necessary conditions for such kind of results to hold. By replacing $n^{1/p}$ by $\sqrt{n\log n}$, Lai (1974) and Chow and Lai (1975) consider the following result on the law of the logarithm.

Theorem B *Suppose that $\mathsf{Var}X = \sigma^2$ and $r \geq 1$. Then the following are equivalent:*

$$\sum_{n=1}^{\infty} n^{r-2} \mathsf{P}(M_n \geq \epsilon\phi(n)) < \infty; \quad \text{for all } \epsilon > \sigma\sqrt{r-1};$$

$$\sum_{n=1}^{\infty} n^{r-2} \mathsf{P}(|S_n| \geq \epsilon\phi(n)) < \infty, \quad \text{for all } \epsilon > \sigma\sqrt{r-1};$$

$$\sum_{n=1}^{\infty} n^{r-2} \mathsf{P}(|S_n| \geq \epsilon\phi(n)) < \infty, \quad \text{for some } \epsilon > 0;$$

$$\mathsf{E}X = 0 \quad \text{and} \quad \mathsf{E}|X|^{2r}/(\log|X|)^r < \infty.$$

For $r = 1$, Gut and Spătaru (2000a) gave the following precise asymptotics.

Theorem C *Suppose that $\mathsf{E}X = 0$ and $\mathsf{E}X^2 = \sigma^2 < \infty$. Then, for $0 \leq \delta \leq 1$,*

$$\lim_{\epsilon \searrow 0} \epsilon^{2\delta+2} \sum_{n=1}^{\infty} n^{-1}(\log n)^{\delta} \mathsf{P}(|S_n| \geq \epsilon\sqrt{n\log n}) = \frac{\mu^{(2\delta+2)}}{\delta+1}\sigma^{2\delta+2},$$

where $\mu^{(2\delta+2)}$ *is the* $(2\delta+2)$*-th absolute moment of the standard normal distribution.*

The purpose of this paper is to consider the precise asymptotics for all $r > 1$. We obtain the sufficient and necessary conditions for such kind of results to hold. Here are our main results.

Theorem 1.1 *Let* $r > 1$ *and* $a > -1/2$ *and let* $a_n(\epsilon)$ *be a function of* ϵ *such that*

$$a_n(\epsilon)\log n \to \tau \ \text{ as } \ n \to \infty \text{ and } \epsilon \searrow \sqrt{r-1}. \tag{1.1}$$

Suppose $\{f_n\}$ *is a sequence of non-negative numbers satisfying*

$$F_n = \sum_{k=1}^{n} f_k \sim \sum_{k=1}^{n} (\log k)^a, \quad n \to \infty. \tag{1.2}$$

Then the following are equivalent:

$$\mathsf{E}X = 0, \quad \mathsf{E}X^2 = \sigma^2 \ (0 < \sigma < \infty) \ \text{ and } \ \mathsf{E}[|X|^{2r}(\log|X|)^{a-r}] < \infty; \tag{1.3}$$

$$\begin{aligned}\lim_{\epsilon\searrow\sqrt{r-1}} [\epsilon^2-(r-1)]^{a+1/2} \sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big\{M_n \ge \sigma\phi(n)(\epsilon + a_n(\epsilon))\Big\} \\ = \frac{2}{\sqrt{\pi(r-1)}} \exp\{-2\tau\sqrt{r-1}\}\Gamma(a+1/2), \qquad \sigma > 0;\end{aligned} \tag{1.4}$$

$$\begin{aligned}\lim_{\epsilon\searrow\sqrt{r-1}} [\epsilon^2-(r-1)]^{a+1/2} \sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big\{|S_n| \ge \sigma\phi(n)(\epsilon + a_n(\epsilon))\Big\} \\ = \frac{1}{\sqrt{\pi(r-1)}} \exp\{-2\tau\sqrt{r-1}\}\Gamma(a+1/2), \qquad \sigma > 0;\end{aligned} \tag{1.5}$$

$$\sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big\{M_n \ge \epsilon\sigma\phi(n))\Big\} < \infty \ \text{ if and only if } \ \epsilon > \sqrt{r-1}; \tag{1.6}$$

$$\sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big\{|S_n| \ge \epsilon\sigma\phi(n))\Big\} < \infty \ \text{ if and only if } \ \epsilon > \sqrt{r-1}. \tag{1.7}$$

Here, $\Gamma(\cdot)$ *is a gamma function.*

Notice $\Gamma(1/2) = \sqrt{\pi}$. Letting $f_n = 1$ and $\tau = 0$ yields the following corollary.

Corollary 1.1 *Let* $r > 1$ *and* $a_n = o(\sqrt{n/\log n})$*. Then the following are equivalent:*

$$\mathsf{E}X = 0, \quad \mathsf{E}X^2 = \sigma^2 \ (0 < \sigma < \infty) \ \text{ and } \ \mathsf{E}[|X|^{2r}(\log|X|)^{-r}] < \infty; \tag{1.8}$$

$$\lim_{\epsilon\searrow\sqrt{r-1}} \sqrt{\epsilon^2-(r-1)} \sum_{n=1}^{\infty} n^{r-2} \mathsf{P}\Big\{M_n \ge \epsilon\sigma\phi(n) + a_n\Big\} = \frac{2}{\sqrt{r-1}}, \qquad \sigma > 0; \tag{1.9}$$

$$\lim_{\epsilon\searrow\sqrt{r-1}} \sqrt{\epsilon^2-(r-1)} \sum_{n=1}^{\infty} n^{r-2} \mathsf{P}\Big\{|S_n| \ge \epsilon\sigma\phi(n) + a_n\Big\} = \frac{1}{\sqrt{r-1}}, \qquad \sigma > 0. \tag{1.10}$$

Also, by Theorem 1.1, in (1.9) and (1.10) the widest a_n *is also* $o(\sqrt{n/\log n})$*.*

Remark 1.1 *Liang, Zhang and Baek (2003) established (1.10) for $1 < r < 3/2$ under the stringent conditions that $\mathsf{E}|X|^r < \infty$ and $a = \sqrt{n}/(\log n)^\gamma)$ for some $\gamma > 1/2$ by using the method of Gut and Spătaru. This method bases on the Berry-Esseen inequality, so does not work for $r \geq 3/2$ and M_n.*

The proof of Theorem 1.1 is given in Section 4. Before that, we first verify (1.4) and (1.5) under the assumption that F is the normal distribution in Section 2, after which, by using the strong approximation method, we then show that the probabilities in (1.4) and (1.5) can be replaced by those for normal random variables in Section 3. Throughout this paper, we let $K(\alpha, \beta, \cdots)$, $C(\alpha, \beta, \cdots)$ etc denote positive constants which depend on $\alpha, \beta, \cdots$ only, whose values can differ in different places. The notation $a_n \sim b_n$ means that $a_n/b_n \to 1$, and $a_n \approx b_n$ means that $C^{-1}b_n \leq a_n \leq Cb_n$ for some $c > 0$ and all n large enough.

2 Normal cases.

In this section, we prove Theorem 1.1 in the case that $\{X, X_n; n \geq 1\}$ are normal random variables. Let $\{W(t); t \geq 0\}$ be a standard Wiener process and N a standard normal variable. Our result is as follows.

Proposition 2.1 *Let $r > 1$ and $a > -1/2$ and let $a_n(\epsilon)$ be a function of ϵ satisfying (1.1). And let $\{f_n\}$ satisfy (1.2). Then*

$$\lim_{\epsilon \searrow \sqrt{r-1}} [\epsilon^2 - (r-1)]^{a+1/2} \sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big\{ \sup_{0 \leq s \leq 1} |W(s)| \geq \sqrt{2\log n}(\epsilon + a_n(\epsilon)) \Big\}$$
$$= \frac{2}{\sqrt{\pi(r-1)}} \exp\{-2\tau\sqrt{r-1}\}\Gamma(a+1/2) \tag{2.1}$$

and

$$\lim_{\epsilon \searrow \sqrt{r-1}} [\epsilon^2 - (r-1)]^{a+1/2} \sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big\{ |N| \geq \sqrt{2\log n}(\epsilon + a_n(\epsilon)) \Big\}$$
$$= \frac{1}{\sqrt{\pi(r-1)}} \exp\{-2\tau\sqrt{r-1}\}\Gamma(a+1/2). \tag{2.2}$$

The following lemma will be used in the proofs.

Lemma 2.1 *Let $\{W(t); t \geq 0\}$ be a standard Wiener process. Then for all $x > 0$,*

$$\mathsf{P}\big(\sup_{0 \leq s \leq 1} |W(s)| \geq x \big) = 1 - \sum_{k=-\infty}^{\infty} (-1)^k \mathsf{P}\big((2k-1)x \leq N \leq (2k+1)x\big)$$
$$= 4\sum_{k=0}^{\infty} (-1)^k \mathsf{P}\big(N \geq (2k+1)x\big) = 2\sum_{k=0}^{\infty} (-1)^k \mathsf{P}\big(|N| \geq (2k+1)x\big). \tag{2.3}$$

In particular,

$$\mathsf{P}\big(\sup_{0 \leq s \leq 1} |W(s)| \geq x \big) \sim 2\mathsf{P}\big(|N| \geq x\big) \sim \frac{4}{\sqrt{2\pi}x} e^{-x^2/2} \quad \text{as} \quad x \to +\infty.$$

Proof. It is well known. See Billingsley (1968).

Lemma 2.2 *Let $\alpha_n(\epsilon) > 0$, $\beta_n(\epsilon) > 0$ and $f(\epsilon) > 0$ satisfying*

$$\alpha_n(\epsilon) \sim \beta_n(\epsilon) \quad \text{as } n \to \infty \text{ and } \epsilon \to \epsilon_0$$

and

$$f(\epsilon)\beta_n(\epsilon) \to 0 \text{ as } \epsilon \to \epsilon_0, \forall n.$$

Then

$$\limsup_{\epsilon\to\epsilon_0}(\liminf_{\epsilon\to\epsilon_0}) f(\epsilon)\sum_{n=1}^{\infty}\alpha_n(\epsilon) = \limsup_{\epsilon\to\epsilon_0}(\liminf_{\epsilon\to\epsilon_0}) f(\epsilon)\sum_{n=1}^{\infty}\beta_n(\epsilon).$$

Proof. For any $\theta > 1$, there exist n_0 and a neighborhood U of ϵ_0 such that

$$\theta^{-1}\beta_n(\epsilon) \le \alpha_n(\epsilon) \le \theta\beta_n(\epsilon), \quad n \ge n_0, \epsilon \in U.$$

Then

$$\theta^{-1}\sum_{n=n_0}^{\infty}\beta_n(\epsilon) \le \sum_{n=n_0}^{\infty}\alpha_n(\epsilon) \le \theta\sum_{n=n_0}^{\infty}\beta_n(\epsilon), \quad \epsilon \in U.$$

Now, the result follows easily.

Lemma 2.3 *Let $a_n > 0$, $c_n > 0$, and $A_n = \sum_{k=1}^{n} a_k$. Suppose that the sequence $\{c_n\}$ is non-increasing and $B_n c_n \to 0$. Then*

$$\sum_{n=1}^{\infty} a_n c_n = \sum_{n=1}^{\infty} A_n(c_n - c_{n+1}).$$

Proof. By the Abel transform, it result follows immediately.

Lemma 2.4 *Let $a_n > 0$, $b_n > 0$, $c_n > 0$, $A_n = \sum_{k=1}^{n} a_k$ and $B_n = \sum_{k=1}^{n} b_k$. Suppose that the sequence $\{c_n\}$ is non-increasing, and $A_n \le B_n$, $n \ge 1$. Then*

$$\sum_{k=1}^{\infty} a_k c_k \le \sum_{k=1}^{\infty} b_k c_k \quad \text{and} \quad \sum_{k=j}^{\infty} a_k c_k \le \sum_{k=j}^{\infty} b_k c_k + B_{j-1}c_j.$$

Proof. From the Abel transform, it follows that

$$\begin{aligned}\sum_{k=1}^{n} a_k c_k &= \sum_{k=1}^{n} A_k(c_k - c_{k+1}) + A_n c_{n+1} \\ &\le \sum_{k=1}^{n} B_k(c_k - c_{k+1}) + B_n c_{n+1} = \sum_{k=1}^{n} b_k c_k\end{aligned}$$

and

$$\begin{aligned}\sum_{k=j}^{n} a_k c_k &= \sum_{k=j}^{n} A_k(c_k - c_{k+1}) + A_n c_{n+1} - A_{j-1}c_j \\ &\le \sum_{k=j}^{n} B_k(c_k - c_{k+1}) + B_n c_{n+1} = \sum_{k=j}^{n} b_k c_k + B_{j-1}c_j.\end{aligned}$$

The result follows.

Lemma 2.5 *Let $a_n > 0$, $b_n > 0$, $c_n > 0$, $A_n = \sum_{k=1}^n a_k$, $B_n = \sum_{k=1}^n b_k$. Suppose that*

$$A_n \sim B_n \text{ and } \sum_{k=1}^n b_k c_k \to \infty \quad \text{as } n \to \infty.$$

Further, suppose one of the following conditions is satisfied:

(i) *The sequence $\{c_n\}$ is eventually non-increasing;*

(ii) *The sequence $\{c_n\}$ is eventually non-decreasing, and*

$$\sum_{k=1}^n b_k c_k \approx B_n c_{n+1}; \tag{2.4}$$

Particularly, if B_n is regularly varying with a positive exponent and c_n is also regularly varying, then (2.4) is satisfied.

Then

$$\sum_{k=1}^n a_k c_k \sim \sum_{k=1}^n b_k c_k.$$

Proof. We show the result under (ii) only. For any $\theta > 1$, there exists a n_0 such that for all $n \ge n_0$, c_n is non-decreasing and $\theta^{-1} A_n \le B_n \le \theta A_n$. Then by the Abel transform, for $n > n_0$

$$\begin{aligned}
& \sum_{k=1}^n a_k c_k = \sum_{k=1}^n A_k (c_k - c_{k+1}) + A_n c_{n+1} \\
\le \;& \sum_{k=1}^{n_0} A_k (c_k - c_{k+1}) + \sum_{k=n_0+1}^{n} \theta^{-1} B_k (c_k - c_{k+1}) + \theta B_n c_{n+1} \\
= \;& \sum_{k=1}^{n_0} (A_k - \theta^{-1} B_k)(c_k - c_{k+1}) + \sum_{k=1}^{n} \theta^{-1} B_k (c_k - c_{k+1}) + \theta^{-1} B_n c_{n+1} + (\theta - \theta^{-1}) B_n c_{n+1} \\
= \;& \sum_{k=1}^{n_0} (A_k - \theta^{-1} B_k)(c_k - c_{k+1}) + \theta^{-1} \sum_{k=1}^{n} b_k c_k + (\theta - \theta^{-1}) B_n c_{n+1}.
\end{aligned}$$

It follows that

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^n a_k c_k}{\sum_{k=1}^n b_k c_k} \le \theta^{-1} + (\theta - \theta^{-1}) K.$$

Letting $\theta \to 1$ yields

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^n a_k c_k}{\sum_{k=1}^n b_k c_k} \le 1.$$

Similarly,

$$\liminf_{n\to\infty} \frac{\sum_{k=1}^n a_k c_k}{\sum_{k=1}^n b_k c_k} \ge 1.$$

The proof is completed.

Now, we turn to prove the propositions.

Proof Proposition 2.1: By Lemma 2.1 and the condition (1.1) we have

$$\begin{aligned}
&\mathsf{P}\Big\{\sup_{0\le s\le 1}|W(s)|\ge\sqrt{2\log n}(\epsilon+a_n(\epsilon))\Big\}\sim 2\mathsf{P}\Big\{|N|\ge\sqrt{2\log n}(\epsilon+a_n(\epsilon))\Big\}\\
&\quad\sim\frac{4}{\sqrt{2\pi}(\epsilon+a_n(\epsilon))\sqrt{2\log n}}\exp\Big\{-(\epsilon+a_n(\epsilon))^2\log n\Big\}\\
&\quad\sim\frac{2}{\sqrt{\pi}\epsilon}\frac{1}{\sqrt{\log n}}\exp\Big\{-\epsilon^2\log n\Big\}\exp\Big\{-2\epsilon a_n(\epsilon)\log n\Big\}\\
&\quad\sim\frac{1}{\sqrt{\log n}}\exp\Big\{-\epsilon^2\log n\Big\}\frac{2}{\sqrt{\pi(r-1)}}\exp\Big\{-2\tau\sqrt{r-1}\Big\}
\end{aligned}$$

as $n\to\infty$ and $\epsilon\searrow\sqrt{r-1}$. Also, by Lemma 2.5, we have

$$A_n=:\sum_{k=1}^{n}n^{r-2}(\log n)^{-1/2}f_n\sim B_n=:\sum_{k=1}^{n}n^{r-2}(\log n)^{a-1/2}\approx n^{r-1}(\log n)^{a-1/2}. \tag{2.5}$$

We conclude that

$$\begin{aligned}
&\lim_{\epsilon\searrow\sqrt{r-1}}[\epsilon^2-(r-1)]^{a+1/2}\sum_{n=1}^{\infty}n^{r-2}f_n\mathsf{P}\Big\{\sup_{0\le s\le 1}|W(s)|\ge\sqrt{2\log n}(\epsilon+a_n(\epsilon))\Big\}\\
=\ &\lim_{\epsilon\searrow\sqrt{r-1}}[\epsilon^2-(r-1)]^{a+1/2}\sum_{n=1}^{\infty}n^{r-2}f_n\frac{1}{\sqrt{\log n}}\exp\Big\{-\epsilon^2\log n\Big\}\frac{\exp\{-2\tau\sqrt{r-1}\}}{\sqrt{\pi(r-1)}}\\
&\text{(by Lemma 2.2)}\\
=\ &\lim_{\epsilon\searrow\sqrt{r-1}}[\epsilon^2-(r-1)]^{a+1/2}\sum_{n=1}^{\infty}A_n\Big\{\exp\{-\epsilon^2\log n\}-\exp\{-\epsilon^2\log(n+1)\}\Big\}\frac{\exp\{-2\tau\sqrt{r-1}\}}{\sqrt{\pi(r-1)}}\\
&\text{(by Lemma 2.3)}\\
=\ &\lim_{\epsilon\searrow\sqrt{r-1}}[\epsilon^2-(r-1)]^{a+1/2}\sum_{n=1}^{\infty}B_n\Big\{\exp\{-\epsilon^2\log n\}-\exp\{-\epsilon^2\log(n+1)\}\Big\}\frac{\exp\{-2\tau\sqrt{r-1}\}}{\sqrt{\pi(r-1)}}\\
&\text{(by (2.5) and Lemma 2.2)}\\
=\ &\lim_{\epsilon\searrow\sqrt{r-1}}[\epsilon^2-(r-1)]^{a+1/2}\sum_{n=1}^{\infty}n^{r-2}(\log n)^{a-1/2}\exp\Big\{-\epsilon^2\log n\Big\}\frac{\exp\{-2\tau\sqrt{r-1}\}}{\sqrt{\pi(r-1)}}\\
&\text{(by Lemma 2.3)}\\
=\ &\lim_{\epsilon\searrow\sqrt{r-1}}[\epsilon^2-(r-1)]^{a+1/2}\sum_{n=1}^{\infty}\int_n^{n+1}x^{r-2}(\log x)^a\frac{1}{\sqrt{\log x}}\exp\Big\{-\epsilon^2\log x\Big\}dx\frac{2\exp\{-2\tau\sqrt{r-1}\}}{\sqrt{\pi(r-1)}}\\
&\text{(by Lemma 2.2)}\\
=\ &\lim_{\epsilon\searrow\sqrt{r-1}}[\epsilon^2-(r-1)]^{a+1/2}\int_e^{\infty}x^{r-2}(\log x)^a\frac{1}{\sqrt{\log x}}\exp\Big\{-\epsilon^2\log x\Big\}dx\frac{2\exp\{-2\tau\sqrt{r-1}\}}{\sqrt{\pi(r-1)}}\\
=\ &\lim_{\epsilon\searrow\sqrt{r-1}}[\epsilon^2-(r-1)]^{a+1/2}\int_1^{\infty}y^{a-1/2}\exp\Big\{-[\epsilon^2-(r-1)]y\Big\}dy\frac{2\exp\{-2\tau\sqrt{r-1}\}}{\sqrt{\pi(r-1)}}\\
=\ &\lim_{\epsilon\searrow\sqrt{r-1}}\int_{\epsilon^2-(r-1)}^{\infty}y^{a-1/2}e^{-y}dy\frac{2\exp\{-2\tau\sqrt{r-1}\}}{\sqrt{\pi(r-1)}}\\
=\ &\Gamma(a+1/2)\frac{2\exp\{-2\tau\sqrt{r-1}\}}{\sqrt{\pi(r-1)}}.
\end{aligned}$$

(2.1) is proved. The proof of (2.2) is the same.

3 Approximation.

The purpose of this section is to use strong approximation and Feller's (1945) and Einmahl's (1989) truncation methods to show that the probability in (1.4) for M_n can be approximated by those for $\sqrt{n}\sup_{0\le s\le 1}|W(s)|$ and the probabilities in (1.5) for S_n can be approximated by those for $\sqrt{n}N$.

Suppose that $\mathsf{E}X=0$ and $\mathsf{E}X^2=\sigma^2<\infty$. Without losing of generality, we assume that $\sigma=1$ throughout this section. Let $p>1/2$. For each n and $1\le j\le n$, we let

$$X'_{nj}=X_jI\{|X_j|\le\sqrt{n}/(\log n)^p\},\qquad \overline{X}'_{nj}=X'_{nj}-\mathsf{E}[X'_{nj}],$$
$$S'_{nj}=\sum_{i=1}^{j}X'_{nj},\quad \overline{S}'_{nj}=\sum_{i=1}^{j}\overline{X}'_{nj},\qquad \overline{M}'_n=\max_{k\le n}|\overline{S}'_{nk}|,\quad B_n=\sum_{k=1}^{n}\mathsf{Var}(\overline{X}'_{nk})$$

and

$$X''_{nj}=X_jI\{\sqrt{n}/(\log n)^p<|X_j|\le\phi(n)\},\qquad \overline{X}''_{nj}=X''_{nj}-\mathsf{E}[X''_{nj}],$$
$$X'''_{nj}=X_jI\{|X_j|>\phi(n)\},\qquad \overline{X}'''_{nj}=X'''_{nj}-\mathsf{E}[X'''_{nj}].$$

And also define S''_{nj}, S'''_{nj}, $\overline{S}''_{nj}$, $\overline{S}'''_{nj}$, $\overline{M}''_n$ and $\overline{M}'''_n$ similarly.

The following proposition is the main result of this section.

Proposition 3.1 *Let $r>1$, $a>-1$ and $2\ge p>p'>1/2$. Suppose that the condition (1.3) is satisfied. And let $\{f_n\}$ satisfy (1.2). Then there exist $\delta>0$ and a sequence of positive numbers $\{q_n\}$ such that*

$$\begin{aligned}
&\mathsf{P}\Big\{\sup_{0\le s\le 1}|W(s)|\ge\epsilon\sqrt{2\log n}+\frac{3}{(\log n)^{p'}}\Big\}-q_n\\
\le\ &\mathsf{P}\Big\{M_n\ge\epsilon\phi(n)\Big\}\\
\le\ &\mathsf{P}\Big\{\sup_{0\le s\le 1}|W(s)|\ge\epsilon\sqrt{2\log n}-\frac{3}{(\log n)^{p'}}\Big\}+q_n,
\end{aligned}\tag{3.1}$$

$$\begin{aligned}
&\mathsf{P}\Big\{|N|\ge\epsilon\sqrt{2\log n}+\frac{3}{(\log n)^{p'}}\Big\}-q_n\\
\le\ &\mathsf{P}\Big\{|S_n|\ge\epsilon\phi(n)\Big\}\\
\le\ &\mathsf{P}\Big\{|N|\ge\epsilon\sqrt{2\log n}-\frac{3}{(\log n)^{p'}}\Big\}+q_n,\\
&\forall\epsilon\in(\sqrt{r-1}-\delta,\sqrt{r-1}+\delta),\quad n\ge 1
\end{aligned}\tag{3.2}$$

and

$$\sum_{n=1}^{\infty}n^{r-2}f_nq_n\le K(r,a,p,p',\delta)<\infty.\tag{3.3}$$

To show this proposition, we need some lemmas.

Lemma 3.1 *For any sequence of independent random variables $\{\xi_n; n \geq 1\}$ with mean zero and finite variance, there exists a sequence of independent normal variables $\{\eta_n; n \geq 1\}$ with $\mathsf{E}\eta_n = 0$ and $\mathsf{E}\eta_n^2 = \mathsf{E}\xi_n^2$ such that, for all $Q > 2$ and $y > 0$,*

$$\mathsf{P}\Big(\max_{k\leq n} |\sum_{i=1}^k \xi_i - \sum_{i=1}^k \eta_i| \geq y\Big) \leq (AQ)^Q y^{-Q} \sum_{i=1}^n \mathsf{E}|\xi_i|^Q,$$

whenever $\mathsf{E}|\xi_i|^Q < \infty$, $i = 1, \ldots, n$. Here, A is a universal constant.

Proof. See Sakhaneko (1980,1984, 1985).

Lemma 3.2 *Let $Q \geq 2$, $\xi_1, \xi_2, \ldots, \xi_n$ be independent random variables with $\mathsf{E}\xi_k = 0$ and $\mathsf{E}|\xi_k|^Q < \infty$, $k = 1, \ldots, n$. Then for all $y > 0$,*

$$\mathsf{P}\Big(\max_{k\leq n} |\sum_{i=1}^k \xi_i| \geq y\Big) \leq 2\exp\Big\{-\frac{y^2}{8\sum_{k=1}^n \mathsf{Var}(\xi_k)}\Big\} + (2AQ)^Q y^{-Q} \sum_{i=1}^n \mathsf{E}|\xi_i|^Q,$$

where A is a universal constant as in Lemma 3.1.

Proof. It follows from Lemma 3.1 easily. See also Petrov (1995, Page 78).

Lemma 3.3 *Define $\Delta_n = \max_{k\leq n} |\overline{S}'_{nk} - S_k|$. Let $r > 1$, $a > -1$ and $p > 1/2$. Suppose that the condition (1.3) is satisfied and $\mathsf{E}X^2 = 1$. Then for any $\lambda > 0$ there exist a constant $K = K(r, a, \lambda)$ such that*

$$\sum_{n=1}^\infty n^{r-2} f_n I_n \leq K < \infty, \tag{3.4}$$

where

$$I_n = \mathsf{P}\Big(\Delta_n \geq \sqrt{n}/(\log n)^2, \overline{M}'_n \geq \lambda\phi(n)\Big).$$

Proof. Let $\beta_n = n\mathsf{E}[|X|I\{|X| > \sqrt{n}/(\log n)^p\}]$. Then $|\mathsf{E}\sum_{i=1}^j X'_{ni}| \leq \beta_n$, $1 \leq j \leq n$. Setting

$$\mathcal{L} = \{n : \beta_n \leq \frac{1}{8}\sqrt{n}/(\log\log n)^2\},$$

we have

$$\{\Delta_n \geq \sqrt{n}/(\log\log n)^2\} \subset \bigcup_{j=1}^n \{X_j \neq X'_{nj}\}, \quad n \in \mathcal{L}.$$

So for $n \in \mathcal{L}$,

$$I_n \leq \sum_{j=1}^n \mathsf{P}\Big(X_j \neq X'_{nj}, \overline{M}'_n \geq \lambda\phi(n)\Big).$$

Observer that $X'_{nj} = 0$ whenever $X_j \neq X'_{nj}$, $j \leq n$, so that we have for n large enough and all $1 \leq j \leq n$,

$$\begin{aligned}
& \mathsf{P}\Big(X_j \neq X'_{nj}, \overline{M}'_n \geq \lambda\phi(n)\Big) \\
= \;& \mathsf{P}\Big(X_j \neq X'_{nj}, \max_{k\leq j-1} |\overline{S}'_{nk}| \vee \max_{j<k\leq n} |\overline{S}'_{nk} - X'_{nj}| \geq \lambda\phi(n)\Big) \\
= \;& \mathsf{P}\Big(X_j \neq X'_{nj}\Big)\mathsf{P}\Big(\max_{k\leq j-1} |\overline{S}'_{nk}| \vee \max_{j<k\leq n} |\overline{S}'_{nk} - X'_{nj}| \geq \lambda\phi(n)\Big) \\
\leq \;& \mathsf{P}\Big(X_j \neq X'_{nj}\Big)\mathsf{P}\Big(\overline{M}'_n \geq \lambda\phi(n) - |X'_{nj}|\Big) \\
\leq \;& \mathsf{P}\Big(|X| > \sqrt{n}/(\log n)^p\Big)\mathsf{P}\Big(\overline{M}'_n \geq \lambda\phi(n) - \sqrt{n}/(\log n)^p\Big) \\
\leq \;& \mathsf{P}\Big(|X| > \sqrt{n}/(\log n)^p\Big)\mathsf{P}\Big(\overline{M}'_n \geq \frac{\lambda}{2}\phi(n)\Big).
\end{aligned}$$

A straightforward application of the inequalities of Ottaviani and Bernstein yields:

$$\mathsf{P}\Big(\overline{M}'_n \geq \frac{\lambda}{2}\phi(n)\Big) \leq 2\mathsf{P}\Big(|\overline{S}'_n| \geq \frac{\lambda}{4}\phi(n)\Big) \leq n^{-\eta}$$
$$\text{for some } \eta = \eta(\lambda) > 0.$$

It follows that

$$\begin{aligned}
& \sum_{n\in\mathcal{L}} n^{r-2} f_n I_n \\
\leq \;& C\sum_{n=1}^{\infty} n^{r-1-\eta} f_n \mathsf{P}\Big(|X| > \frac{\sqrt{n}}{(\log n)^p}\Big) \\
\leq \;& \sum_{n=1}^{\infty}\sum_{j=n}^{\infty} \mathsf{P}\Big(\frac{\sqrt{j}}{(\log j)^p} < |X| \leq \frac{\sqrt{j+1}}{(\log(j+1))^p}\Big) n^{r-1-\eta} f_n \\
\leq \;& \sum_{j=1}^{\infty} \mathsf{P}\Big(\frac{\sqrt{j}}{(\log j)^p} < |X| \leq \frac{\sqrt{j+1}}{(\log(j+1))^p}\Big) \sum_{n=1}^{j} n^{r-1-\eta} f_n \\
\leq \;& C\sum_{j=1}^{\infty} \mathsf{P}\Big(\frac{\sqrt{j}}{(\log j)^p} < |X| \leq \frac{\sqrt{j+1}}{(\log(j+1))^p}\Big) \sum_{n=1}^{j} n^{r-1-\eta} (\log n)^a \\
& \qquad \text{(by Lemma 2.5)} \\
\leq \;& \sum_{j=1}^{\infty} \mathsf{P}\Big(\frac{\sqrt{j}}{(\log j)^p} < |X| \leq \frac{\sqrt{j+1}}{(\log(j+1))^p}\Big) j^{r-\eta} (\log j)^a \\
\leq \;& C\mathsf{E}\Big[|X|^{2(r-\eta)} (\log |X|)^{2p(r-\eta)+a}\Big] < \infty.
\end{aligned}$$

If $n \notin \mathcal{L}$, then we have

$$I_n \leq \mathsf{P}\Big(\overline{M}'_n \geq \lambda\phi(n)\Big) \leq n^{-\eta}.$$

It follows that

$$
\begin{aligned}
&\sum_{n\notin\mathcal{L}} n^{r-2} f_n I_n \le \sum_{n\notin\mathcal{L}} n^{r-2-\eta} f_n \\
\le\ & 8\sum_{n\notin\mathcal{L}} n^{r-5/2-\eta} f_n (\log n)^2 \beta_n \le 8\sum_{n=1}^{\infty} n^{r-3/2-\eta} f_n (\log n)^2 \mathsf{E}[|X| I\{|X| > \sqrt{n}/(\log n)^p\}] \\
\le\ & 8\sum_{n=1}^{\infty} n^{r-3/2-\eta} f_n (\log n)^2 \sum_{j=n}^{\infty} \mathsf{E}\Big[|X| I\{\frac{\sqrt{j}}{(\log j)^p} < |X| \le \frac{\sqrt{j+1}}{(\log(j+1))^p}\}\Big] \\
=\ & 8\sum_{j=1}^{\infty} \mathsf{E}\Big[|X| I\{\frac{\sqrt{j}}{(\log j)^p} < |X| \le \frac{\sqrt{j+1}}{(\log(j+1))^p}\}\Big] \sum_{n=1}^{j} n^{r-3/2-\eta} f_n (\log n)^2 \\
\le\ & C\sum_{j=1}^{\infty} \mathsf{E}\Big[|X| I\{\frac{\sqrt{j}}{(\log j)^p} < |X| \le \frac{\sqrt{j+1}}{(\log(j+1))^p}\}\Big] \sum_{n=1}^{j} n^{r-3/2-\eta} (\log n)^{a+2} \\
& \qquad (\text{ by Lemma 2.5}) \\
\le\ & C\sum_{j=1}^{\infty} \mathsf{E}\Big[|X| I\{\frac{\sqrt{j}}{(\log j)^p} < |X| \le \frac{\sqrt{j+1}}{(\log(j+1))^p}\}\Big] j^{r-1/2-\eta} (\log j)^{a+2} \\
\le\ & C\mathsf{E}\Big[|X|^{2(r-\eta)} (\log|X|)^{2p(r-1/2-\eta)+a+2}\Big] < \infty.
\end{aligned}
$$

(3.4) is proved.

Lemma 3.4 *Let $r > 1$, $a > -1$ and $p > 1/2$. Suppose the condition (1.3) is satisfied. Then for any $\lambda > 0$ there exist a constant $K = K(r, a, p, \lambda)$ such that*

$$\sum_{n=1}^{\infty} n^{r-2} f_n II_n \le K < \infty,$$

where

$$II_n = P\Big(\Delta_n \ge \sqrt{n}/(\log\log n)^2, M_n \ge \lambda\phi(n)\Big).$$

Proof. Obviously,

$$II_n \le \mathsf{P}\Big(\Delta_n \ge \sqrt{n}/(\log\log n)^2, \overline{M}_n' \ge \frac{\lambda}{3}\phi(n)\Big) + \mathsf{P}\Big(\overline{M}_n'' \ge \frac{\lambda}{3}\phi(n)\Big) + \mathsf{P}\Big(\overline{M}_n''' \ge \frac{\lambda}{3}\phi(n)\Big).$$

Observe that $\max_{k\le n} |\mathsf{E}S_{nk}'''| \le n\mathsf{E}X^2/\phi(n) = o(\sqrt{n})$. We have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big(\overline{M}_n''' \ge \frac{\lambda}{3}\phi(n)\Big) \le C\sum_{n=1}^{\infty} n^{r-2} f_n \sum_{j=1}^{n} \mathsf{P}\Big(X_j''' \ne 0\Big) \\
\le\ & \sum_{n=1}^{\infty} n^{r-1} f_n \mathsf{P}\big(|X| \ge \phi(n)\big) \\
\le\ & C\sum_{n=1}^{\infty} n^{r-1} (\log n)^a \mathsf{P}\big(|X| \ge \phi(n)\big) \quad (\text{ by Lemmas 2.5 and 2.4}) \\
\le\ & K\mathsf{E}\Big[|X|^{2r} (\log|X|)^{a-r}\Big].
\end{aligned}
$$

Also, notice that $\sum_{k=1}^{n} \mathsf{Var}(\overline{X}_{nk}'') \le n\mathsf{E}\big[X^2 I\{\sqrt{n}/(\log\log n)^p < |X| \le \phi(n)\}\big] = o(n)$. By Lemma

3.2 we have for $Q > 2r$,

$$
\begin{aligned}
& \sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big(\overline{M}''_n \geq \frac{\lambda}{3}\phi(n)\Big) \\
\leq\ & C\sum_{n=1}^{\infty} n^{r-2} f_n \exp\Big\{-\frac{\lambda^2\phi^2(n)}{3^2 8\cdot o(n)}\Big\} \\
& +C\sum_{n=1}^{\infty} n^{r-2} f_n \cdot \frac{1}{\phi^Q(n)} n\mathsf{E}\big[|X|^Q I\{|X|\leq\phi(n)\}\big] \\
\leq\ & K + C\sum_{n=1}^{\infty}\frac{n^{r-1}f_n}{\phi^Q(n)}\sum_{j=1}^{n}\mathsf{E}\big[|X|^Q I\{\phi(j-1)<|X|\leq\phi(j)\}\big] \\
\leq\ & K + C\sum_{j=1}^{\infty}\mathsf{E}\big[|X|^Q I\{\phi(j-1)<|X|\leq\phi(j)\}\big]\sum_{n=j}^{\infty}\frac{n^{r-1}}{\phi^Q(n)}f_n \\
\leq\ & K + C\sum_{j=1}^{\infty}\mathsf{E}\big[|X|^Q I\{\phi(j-1)<|X|\leq\phi(j)\}\big] \\
& \cdot\Big\{\sum_{n=j}^{\infty}\frac{n^{r-1}}{\phi^Q(n)}(\log n)^a + \sum_{n=1}^{j-1}(\log n)^a\frac{j^{r-1}}{\phi^Q(j)}\Big\} \\
& \text{(by using Lemmas 2.5 and 2.4 in the second sum)} \\
\leq\ & K + C\sum_{j=1}^{\infty}\mathsf{E}\big[|X|^Q I\{\phi(j-1)<|X|\leq\phi(j)\}\big]\frac{j^r}{\phi^Q(j)}(\log j)^a \\
\leq\ & K + C\mathsf{E}\Big[|X|^{2r}(\log|X|)^{a-r}\Big] < \infty.
\end{aligned}
$$

Finally, by noticing Lemma 3.3, we compete the proof of Lemma 3.4.

Lemma 3.5 *Let $r \geq 1$ and $a > -1$. Suppose that the conditions (1.3) and (1.2) is satisfied. Then for any $1/2 < p' < p$ we have*

$$
\begin{aligned}
& P\Big(\sup_{0\leq s\leq 1}|W(s)| \geq x + 1/(\log n)^{p'}\Big) - p_n \leq P\Big(\overline{M}'_n \geq x\sqrt{B_n}\Big) \\
\leq\ & P\Big(\sup_{0\leq s\leq 1}|W(s)| \geq x - 1/(\log n)^{p'}\Big) + p_n, \quad \forall x > 0
\end{aligned} \tag{3.5}
$$

and

$$
\begin{aligned}
& P\big(|N| \geq x + 1/(\log n)^{p'}\big) - p_n \leq P\big(|\overline{S}'_n| \geq x\sqrt{B_n}\big) \\
\leq\ & P\big(|N| \geq x - 1/(\log n)^{p'}\big) + p_n, \quad \forall x > 0,
\end{aligned} \tag{3.6}
$$

where $p_n \geq 0$ satisfies

$$\sum_{n=1}^{\infty} n^{r-2} f_n p_n \leq K(r, a, p, p') < \infty. \tag{3.7}$$

Proof. By Lemma 3.1, there exist a universal constant $A > 0$ and a sequence of standard Wiener processes $\{W_n(\cdot)\}$ such that for all $Q > 2$,

$$\begin{aligned} & \mathsf{P}\Big(\max_{k\le n}|\overline{S}'_{nk} - W_n(\frac{k}{n}B_n)| \ge \frac{1}{2}\sqrt{B_n}/(\log n)^{p'}\Big) \\ \le\ & (AQ)^Q\Big(\frac{(\log n)^{p'}}{\sqrt{B_n}}\Big)^Q\sum_{k=1}^n \mathsf{E}\big|\overline{X}'_{nk}\big|^Q \\ \le\ & Cn\Big(\frac{(\log n)^{p'}}{\sqrt{n}}\Big)^Q \mathsf{E}\big[|X|^Q I\{|X| \le \sqrt{n}/(\log n)^p\}\big]. \end{aligned}$$

On the other hand, by Lemma 1.1.1 of Csörgő and Révész (1981),

$$\begin{aligned} & \mathsf{P}\Big(|\max_{0\le s\le 1}|W_n(sB_n) - W_n(\frac{[ns]}{n}B_n)| \ge \frac{1}{2}\sqrt{B_n}/(\log n)^{p'}\Big) \\ =\ & \mathsf{P}\Big(\max_{0\le s\le 1}|W_n(s) - W_n(\frac{[ns]}{n})| \ge \frac{1}{2}\sqrt{\frac{1}{n}}\frac{\sqrt{n}}{(\log n)^{p'}}\Big) \\ \le\ & Cn\exp\Big\{-\frac{(\frac{1}{2}\sqrt{n}/(\log n)^{p'})^2}{3}\Big\} \le Cn\exp\Big\{-\frac{1}{12}n/(\log n)^{2p'}\Big\}. \end{aligned}$$

Let

$$p_n = \mathsf{P}\Big(\sup_{0\le s\le 1}\Big|\frac{\overline{S}'_{n,[ns]}}{\sqrt{B_n}} - \frac{W_n(sB_n)}{\sqrt{B_n}}\Big| \ge \frac{1}{(\log n)^{p'}}\Big). \tag{3.8}$$

Then p_n satisfies (3.5) and (3.6), since $\{W_n(tB_n)/\sqrt{B_n}; t\ge 0\} \overset{\mathcal{D}}{=} \{W(t); t \ge 0\}$ for each n. And also,

$$p_n \le Cn\Big(\frac{(\log n)^{p'}}{\sqrt{n}}\Big)^Q \mathsf{E}\big[|X|^Q I\{|X| \le \sqrt{n}/(\log n)^p\}\big] + Cn\exp\Big\{-\frac{1}{12}n/(\log n)^{2p'}\Big\}.$$

It follows that

$$\begin{aligned} & \sum_{n=1}^\infty n^{r-2}f_n p_n \le K_1 + C\sum_{n=1}^\infty n^{r-1-Q/2}f_n(\log n)^{p'Q}\mathsf{E}\big[|X|^Q I\{|X| \le \sqrt{n}/(\log n)^p\}\big] \\ \le\ & K_1 + C\sum_{n=1}^\infty n^{r-1-Q/2}f_n(\log n)^{p'Q} \\ & \quad \cdot\sum_{j=1}^n \mathsf{E}\big[|X|^Q I\{\frac{\sqrt{j-1}}{(\log(j-1))^p} < |X| \le \frac{\sqrt{j}}{(\log j)^p}\}\big] \\ \le\ & K_1 + C\sum_{j=1}^\infty \mathsf{E}\big[|X|^Q I\{\frac{\sqrt{j-1}}{(\log(j-1))^p} < |X| \le \frac{\sqrt{j}}{(\log j)^p}\}\big]\sum_{n=j}^\infty\{n^{r-1-Q/2}(\log n)^{p'Q}\}f_n \\ \le\ & K_1 + C\sum_{j=1}^\infty \mathsf{E}\big[|X|^Q I\{\frac{\sqrt{j-1}}{(\log(j-1))^p} < |X| \le \frac{\sqrt{j}}{(\log j)^p}\}\big] \\ & \qquad \cdot\Big\{\sum_{n=j}^\infty\{n^{r-1-Q/2}(\log n)^{p'Q}\}(\log n)^a + \sum_{n=1}^{j-1}(\log n)^a j^{r-1-Q/2}(\log j)^{p'Q}\Big\} \\ & \qquad \text{(by using Lemmas 2.5 and 2.4 in the second sum)} \\ \le\ & K_1 + C\sum_{j=1}^\infty \mathsf{E}\big[|X|^Q I\{\frac{\sqrt{j-1}}{(\log(j-1))^p} < |X| \le \frac{\sqrt{j}}{(\log j)^p}\}\big]j^{r-Q/2}(\log j)^{p'Q+a} \\ \le\ & K_1 + C\mathsf{E}\big[|X|^{2r}(\log|X|)^{(p'-p)Q+2rp+a}\big] \le K < \infty, \end{aligned}$$

whenever $(p'-p)Q + 2rp + a < -1$. So, (3.7) is satisfied.

Now, we turn to prove Proposition 3.1. Let $0 < \delta < \frac{1}{4}\sqrt{r-1}$. Recall $\sigma = 1$, $\phi(n) = \sqrt{2n \log n}$ and notice that

$$\begin{aligned} 0 &\le n - B_n \le n\mathsf{E}n\mathsf{E}[X^2 I\{|X| \ge \sqrt{n}/(\log n)^p] \\ &= n\mathsf{E}[|X|^{2r}(\log|X|)^{a-r}] \cdot o(n^{(r-1)/2}) = n \cdot o(n^{(r-1)/2}) = n \cdot o((\log n)^4). \end{aligned} \tag{3.9}$$

Then if n is large enough,

$$\begin{aligned} &\mathsf{P}\Big\{M_n \ge \epsilon\phi(n)\Big\} \\ = \;& \mathsf{P}\Big\{M_n \ge \epsilon\phi(n), \Delta_n \le \frac{\sqrt{n}}{(\log n)^2}\Big\} + \mathsf{P}\Big\{M_n \ge \epsilon\phi(n), \Delta_n > \frac{\sqrt{n}}{(\log n)^2}\Big\} \\ \le \;& \mathsf{P}\Big\{\overline{M}'_n \ge \epsilon\phi(n) - \frac{\sqrt{n}}{(\log n)^2}\Big\} + \mathsf{P}\Big\{M_n \ge \frac{\sqrt{r-1}}{4}\phi(n), \Delta_n > \frac{\sqrt{n}}{(\log n)^2}\Big\} \\ \le \;& \mathsf{P}\Big\{\overline{M}'_n \ge \sqrt{B_n}\big[\epsilon\sqrt{2\log n} - \frac{2}{(\log n)^2}\big]\Big\} + II_n \\ \le \;& \mathsf{P}\Big\{\sup_{0\le s\le 1}|W(s)| \ge \epsilon\sqrt{2\log n} - \frac{2}{(\log n)^2} - \frac{1}{(\log n)^{p'}}\Big\} + p_n + II_n \\ \le \;& \mathsf{P}\Big\{\sup_{0\le s\le 1}|W(s)| \ge \epsilon\sqrt{2\log n} - \frac{3}{(\log n)^{p'}}\Big\} + p_n + II_n \end{aligned}$$

for all $\epsilon \in (\sqrt{r-1} - \delta, \sqrt{r-1} + \delta)$, where II_n is defined in Lemmas 3.4 with $\lambda = \sqrt{r-1}/4$ and p_n is defined in 3.5. Also, if n is large enough,

$$\begin{aligned} &\mathsf{P}\Big\{M_n \ge \epsilon\phi(n)\Big\} \ge \mathsf{P}\Big\{M_n \ge \epsilon\phi(n), \Delta_n \le \frac{\sqrt{n}}{(\log n)^2}\Big\} \\ \ge \;& \mathsf{P}\Big\{\overline{M}'_n \ge \epsilon\phi(n) + \frac{\sqrt{n}}{(\log n)^2}, \Delta_n \le \frac{\sqrt{n}}{(\log n)^2}\Big\} \\ \ge \;& \mathsf{P}\Big\{\overline{M}'_n \ge \epsilon\phi(n) + \frac{\sqrt{n}}{(\log n)^2}\Big\} - \mathsf{P}\Big\{\overline{M}'_n \ge \frac{\sqrt{r-1}}{4}\phi(n), \Delta_n > \frac{\sqrt{n}}{(\log n)^2}\Big\} \\ \ge \;& \mathsf{P}\Big\{\overline{M}'_n \ge \sqrt{B_n}\big[\epsilon\sqrt{2\log n} + \frac{2}{(\log n)^2}\big]\Big\} - \mathsf{P}\Big\{\overline{M}'_n \ge \frac{\sqrt{r-1}}{4}\phi(n), \Delta_n > \frac{\sqrt{n}}{(\log n)^2}\Big\} \\ \ge \;& \mathsf{P}\Big\{\sup_{0\le s\le 1}|W(s)| \ge \epsilon\sqrt{2\log n} + \frac{3}{(\log n)^{p'}}\Big\} - p_n - I_n \end{aligned}$$

for all $\epsilon \in (\sqrt{1+a} - \delta, \sqrt{1+a} + \delta)$, where I_n is defined in Lemma 3.3 with $\lambda = \sqrt{1+a}/4$.

Similarly, if n is large enough,

$$\begin{aligned} &\mathsf{P}\Big\{|N| \ge \epsilon\sqrt{2\log n} + \frac{3}{(\log n)^{p'}}\Big\} - p_n - I_n \\ \le \;& \mathsf{P}\Big\{|S_n| \ge \epsilon\phi(n)\Big\} \\ \le \;& \mathsf{P}\Big\{|N| \ge \epsilon\sqrt{2\log n} - \frac{3}{(\log n)^{p'}}\Big\} + p_n + II_n \end{aligned}$$

holds for all $\epsilon \in (\sqrt{1+a} - \delta, \sqrt{1+a} + \delta)$. Letting $q_n = p_n + I_n + II_n$ completes the proof by Lemmas 3.3, 3.4 and 3.5.

4 Proofs of the Theorem.

Obviously, (1.4)$\Longrightarrow$ (1.6) and (1.5) $\Longrightarrow$ (1.7). It remains for us to show (1.3) $\Longrightarrow$ (1.4) and (1.5), and (1.6) or (1.7) $\Longrightarrow$ (1.3).

First, we show that (1.4) and (1.5) hold under (1.3). Without losing of generality, we assume that $\mathsf{E}X = 0$ and $\mathsf{E}X^2 = 1$. Let $\delta > 0$ small enough and $\{q_n\}$ be such that (3.1), (3.2) and (3.3) hold. Then

$$\lim_{\epsilon \searrow \sqrt{r-1}} [\epsilon^2 - (r-1)]^{a+1/2} \sum_{n=1}^{\infty} n^{r-2} f_n q_n = 0,$$

by (3.3). Notice that $a_n(\epsilon) \to 0$. By (3.1) and (3.2), we have that for n large enough,

$$\begin{aligned}
& \mathsf{P}\Big\{ \sup_{0\le s\le 1} |W(s)| \ge \sqrt{2\log n}(\epsilon + a_n(\epsilon)) + \frac{3}{(\log n)^{p'}} \Big\} - q_n \\
\le\ & \mathsf{P}\Big\{ M_n \ge \phi(n)(\epsilon + a_n(\epsilon)) \Big\} \\
\le\ & \mathsf{P}\Big\{ \sup_{0\le s\le 1} |W(s)| \ge \sqrt{2\log n}(\epsilon + a_n(\epsilon)) - \frac{3}{(\log n)^{p'}} \Big\} + q_n, \\
& \mathsf{P}\Big\{ |N| \ge \sqrt{2\log n}(\epsilon + a_n(\epsilon)) + \frac{3}{(\log n)^{p'}} \Big\} - q_n \\
\le\ & \mathsf{P}\Big\{ |S_n| \ge \phi(n)(\epsilon + a_n(\epsilon)) \Big\} \\
\le\ & \mathsf{P}\Big\{ |N| \ge \sqrt{2\log n}(\epsilon + a_n(\epsilon)) - \frac{3}{(\log n)^{p'}} \Big\} + q_n, \\
& \qquad \forall \epsilon \in (\sqrt{r-1} - \delta/2, \sqrt{r-1} + \delta/2).
\end{aligned}$$

On the other hand, by Proposition 2.1,

$$\begin{aligned}
& \lim_{\epsilon \searrow \sqrt{r-1}} [\epsilon^2 - (r-1)]^{a+1/2} \sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big\{ \sup_{0\le s\le 1} |W(s)| \ge \sqrt{2\log n}(\epsilon + a_n(\epsilon)) \pm \frac{3}{(\log n)^{p'}} \Big\} \\
& = \frac{2}{\sqrt{\pi(r-1)}} \exp\{-2\tau\sqrt{r-1}\} \Gamma(a+1/2)
\end{aligned}$$

and

$$\begin{aligned}
& \lim_{\epsilon \searrow \sqrt{r-1}} [\epsilon^2 - (r-1)]^{a+1/2} \sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big\{ |N| \ge \sqrt{2\log n}(\epsilon + a_n(\epsilon)) \pm \frac{3}{(\log n)^{p'}} \Big\} \\
& = \frac{1}{\sqrt{\pi(r-1)}} \exp\{-2\tau\sqrt{r-1}\} \Gamma(a+1/2).
\end{aligned}$$

(1.4) and (1.5) are now proved.

Now, we show (1.7) $\Longrightarrow$ (1.3). The proof of (1.6) $\Longrightarrow$ (1.3) is the same. First, we show that $\mathsf{E}X^2 < \infty$. Let $\{\widetilde{X}, \widetilde{X}_n; n \ge 1\}$ be the symmetrization of $\{X, X_n; n \ge 1\}$, and let $\widetilde{S}_n = \sum_{k=1}^n \widetilde{X}_k$. Then by (1.7),

$$\begin{aligned}
& \sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big\{ |\widetilde{S}_n| \ge 2\epsilon\sigma\phi(n) \Big\} \\
\le\ & 2 \sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big\{ |S_n| \ge \epsilon\sigma\phi(n) \Big\} < \infty \quad \text{for all } \epsilon > \sqrt{r-1}.
\end{aligned}$$

For $M > 0$, define $Y = Y(M) = \widetilde{X} I\{|\widetilde{X}| < M\}$ and $Y_n = Y_n(M) = \widetilde{X}_n I\{|\widetilde{X}_n| < M\}$. Observing that $\widetilde{X} I\{|\widetilde{X}| < M\} - \widetilde{X} I\{|\widetilde{X}| \geq M\} \stackrel{\mathcal{D}}{=} \widetilde{X}$ and $\widetilde{X} I\{|\widetilde{X}| < M\} - \widetilde{X} I\{|\widetilde{X}| \geq M\} + \widetilde{X} = 2Y$, we obtain that

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big\{ |\sum_{k=1}^{n} Y_k| \geq 2\epsilon\sigma\phi(n) \Big\} \\ &\leq 2 \sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big\{ |\widetilde{S}_n| \geq 2\epsilon\sigma\phi(n) \Big\} < \infty \quad \text{for all } \ \epsilon > \sqrt{r-1}. \end{aligned} \tag{4.1}$$

However, since Y is a bounded random variable which satisfies the condition (1.3), by the direct part of the theorem we have

$$\sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\Big\{ |\sum_{k=1}^{n} Y_k| \geq \epsilon \sqrt{\mathsf{E} Y^2} \phi(n) \Big\} = \infty \quad \text{for all } \ \epsilon \leq \sqrt{r-1}. \tag{4.2}$$

Putting (4.1) and (4.2) together yields $\sqrt{\mathsf{E}\widetilde{X}^2 I\{|\widetilde{X}| < M\}} = \sqrt{\mathsf{E}Y^2} \leq 2\sigma$. Then, letting $M \to \infty$ yields $\mathsf{E}X^2 < \infty$.

$\mathsf{E}X = 0$ is obvious when $\mathsf{E}X^2 < \infty$, for otherwise we have

$$\mathsf{P}\{|S_n| \geq \epsilon\sigma\phi(n)\} \to 1, \quad \forall \epsilon > 0,$$

which implies that

$$\sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\{|S_n| \geq \epsilon\sigma\phi(n)\} = \infty, \quad \forall \epsilon > 0, \ r \geq 1 \ \text{ and } \ a \geq -1.$$

Now, by (1.7) and the Lévy inequality we obtain that for some $\epsilon > 0$,

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\{\max_{k \leq n} |X_k| \geq 3\epsilon\sigma\phi(n)\} \\ &\leq C \sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\{\max_{k \leq n} |X_k| \geq 2\epsilon\sigma\phi(n) + 2\sqrt{n\mathsf{E}X^2}\} \\ &\leq C \sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\{\max_{k \leq n} |S_k| \geq \epsilon\sigma\phi(n) + \sqrt{n\mathsf{E}X^2}\} \\ &\leq C \sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\{|S_n| \geq \epsilon\sigma\phi(n)\} < \infty \quad \text{for } \epsilon > \sqrt{r-1}. \end{aligned}$$

Observe that

$$\mathsf{P}\{\max_{k \leq n} |X_k| \geq 3\epsilon\sigma\phi(n)\} \leq \frac{\mathsf{E}X^2}{18\epsilon^2 \log n} \to 0.$$

We conclude that

$$\sum_{n=1}^{\infty} n^{r-1} f_n \mathsf{P}\{|X| \geq 3\epsilon\sigma\phi(n)\} \leq C \sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\{\max_{k \leq n} |X_k| \geq 3\epsilon\sigma\phi(n)\} < \infty.$$

Notice $\sum_{k=1} k^{r-1} f_k \sim \sum_{k=1}^{n} n^{r-1} (\log k)^a$ by Lemma 2.5, and $\mathsf{P}\{|X| \geq 3\epsilon\sigma\phi(n)\}$ is non-increasing in n. From Lemma 2.4 it follows that

$$\sum_{n=1}^{\infty} n^{r-1} (\log n)^a \mathsf{P}\{|X| \geq 3\epsilon\sigma\phi(n)\} \leq C \sum_{n=1}^{\infty} n^{r-2} f_n \mathsf{P}\{\max_{k \leq n} |X_k| \geq 3\epsilon\sigma\phi(n)\} < \infty,$$

which implies

$$\mathsf{E}\big[|X|^{2r}(\log|X|)^{a-r}\big] < \infty.$$

Finally, by (1.7) itself and (1.3) $\Longrightarrow$ (1.7), $\mathsf{E}X^2 = \sigma^2$ is obvious. The proof is now completed.

5 An analog of multidimensional case

Let $\{X, X_{\mathbf{n}}; \mathbf{n} \in \mathbb{N}^d\}$ be a field of i.i.d. random variables, where $d \geq 2$ is a positive integer, $\mathbb{N}^d$ denotes the d-dimensional lattice of positive integers. For $\mathbf{n} = (n_1, \cdots, n_d) \in \mathbb{N}^d$, we denote $|\mathbf{n}| = n_1 \cdots n_d$. Also, the notation $\mathbf{k} \leq \mathbf{n}$ means that $k_i \leq n_i$, $i = 1, 2, \cdots, d$. Denote by $S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}$. We have the following asymptotic result.

Theorem 5.1 *Let $r > 1$ and $a + d > 1/2$. And let $a_{\mathbf{n}}(\epsilon)$ be a function of ϵ such that*

$$a_{|\mathbf{n}|}(\epsilon) \log|\mathbf{n}| \to \tau \ \text{ as } \ \mathbf{n} \to \infty \text{ and } \epsilon \searrow \sqrt{r-1}. \tag{5.3}$$

Then the following statements are equivalent:

$$EX = 0, \quad EX^2 = \sigma^2 \ (0 < \sigma < \infty) \ \text{ and } E[|X|^{2r}(\log|X|)^{a+d-1-r}] < \infty; \tag{5.4}$$

$$\lim_{\epsilon \searrow \sqrt{r-1}} [\epsilon^2 - (r-1)]^{a+d-1/2} \sum_{\mathbf{n}} |\mathbf{n}|^{r-2}(\log|\mathbf{n}|)^a P\Big\{|S_{\mathbf{n}}| \geq \sigma\phi(|\mathbf{n}|)(\epsilon + a_{|\mathbf{n}|}(\epsilon))\Big\}$$
$$= \frac{1}{(d-1)!\sqrt{\pi(r-1)}} \exp\{-2\tau\sqrt{r-1}\}\Gamma(a+d-1/2), \qquad \sigma > 0;$$

$$\sum_{\mathbf{n}} |\mathbf{n}|^{r-2}(\log|\mathbf{n}|)^a P\Big\{|S_{\mathbf{n}}| \geq \epsilon\sigma\phi(|\mathbf{n}|)\Big\} < \infty \ \text{ if and only if } \ \epsilon > \sqrt{r-1}.$$

Proof. Notice that

$$\sum_{\mathbf{n}} |\mathbf{n}|^{r-2}(\log|\mathbf{n}|)^a \mathsf{P}\Big\{|S_{\mathbf{n}}| \geq x\phi(|\mathbf{n}|)\Big\} = \sum_{m=1}^{\infty} f_n n^{r-2}(\log n)^a \mathsf{P}\Big\{|S_n| \geq x\phi(n)\Big\},$$

where $f_n = Card\{\mathbf{k} : |\mathbf{k}| = n\}$. For f_n we have

$$\sum_{m=1}^{n} f_m = Card\{\mathbf{k} : |\mathbf{k}| \leq n\} \sim \frac{1}{(d-1)!} n(\log n)^{d-1} \sim \frac{1}{(d-1)!} \sum_{k=1}^{n} (\log k)^{d-1}.$$

By Lemma 2.5, it follows that

$$\sum_{m=1}^{n} f_m(\log m)^a \sim \frac{1}{(d-1)!} \sum_{m=1}^{n} (\log m)^{a+d-1}.$$

The conclusion of the theorem follows from Theorem 1.1 immediately.

Taking $a = 0$ and $\tau = 0$ in Theorem 5.1 yields the following corollary.

Corollary 5.1 *Let $r > 1$ and $a_{|\mathbf{n}|}(\epsilon) = o(\sqrt{|\mathbf{n}|/\log|\mathbf{n}|})$. Then*

$$\lim_{\epsilon \searrow \sqrt{r-1}} [\epsilon^2 - (r-1)]^{d-1/2} \sum_{\mathbf{n}} |\mathbf{n}|^{r-2} \mathsf{P}\Big\{|S_{\mathbf{n}}| \geq \epsilon\sigma\phi(|\mathbf{n}|) + a_{|\mathbf{n}|}\Big\}$$
$$= \frac{(d-3/2)(d-5/2)\cdots(1/2)}{(d-1)!\sqrt{(r-1)}},$$

holds if and only if $\mathsf{E}X = 0$, $\mathsf{E}X^2 = \sigma^2$ and $\mathsf{E}[|X|^{2r}(\log|X|)^{d-1-r}] < \infty$.

For multidimensional case, since we haven't found a suitable approximation of the kind of Lemma 3.1, we haven't established an analogous result for $M_{\mathbf{n}} = \max_{\mathbf{m}\leq\mathbf{n}} |S_{\mathbf{m}}|$. But we conjecture it is true.

Conjecture *Let $r > 1$ and $a + d > 1/2$. Suppose conditions (5.3) and (5.4) are satisfied. Then*

$$\lim_{\epsilon \searrow \sqrt{r-1}} [\epsilon^2 - (r-1)]^{a+d-1/2} \sum_{\mathbf{n}} |\mathbf{n}|^{r-2} (\log|\mathbf{n}|)^a \mathsf{P}\Big\{M_{\mathbf{n}} \geq \sigma\phi(|\mathbf{n}|)(\epsilon + a_{|\mathbf{n}|}(\epsilon))\Big\}$$
$$= \frac{2^d}{(d-1)!\sqrt{\pi(r-1)}} \exp\{-2\tau\sqrt{r-1}\}\Gamma(a+d-1/2).$$

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