Precise Asymptotics in Chung's law of the iterated logarithm*

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ABSTRACT. Let X, X_1, X_2, ... be i.i.d. random variables with mean zero and positive, finite variance σ^2, and set $S_n = X_1 + \ldots + X_n$, $n \geq 1$. We prove that, if $\mathsf{E}X^2 I\{|X| \geq t\} = o((\log\log t)^{-1})$ as $t \to \infty$, then for any $a > -1$ and $b > -1$,

$$\lim_{\epsilon \nearrow 1/\sqrt{1+a}} \left(\tfrac{1}{\sqrt{1+a}} - \epsilon\right)^{b+1} \sum_{n=1}^{\infty} \tfrac{(\log n)^a (\log\log n)^b}{n} \mathsf{P}\Big\{ \max_{k\leq n} |S_k| \leq \sqrt{\tfrac{\sigma^2\pi^2 n}{8\log\log n}}(\epsilon + a_n)\Big\} = \tfrac{4}{\pi}\left(\tfrac{1}{2(1+a)^{3/2}}\right)^{b+1}\Gamma(b+1),$$

whenever $a_n = o(1/\log\log n)$.

Keywords: the law of the iterated logarithm, Chung's law of the iterated logarithm, small deviation, i.i.d. random variables.

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1 Introduction and main results.

Let $\{X, X_n; n \geq 1\}$ be a sequence of i.i.d random variables with common distribution function F, mean 0 and positive, finite variance σ^2, and set $S_n = \sum_{k=1}^n X_k$, $M_n = \max_{k\leq n} |S_k|$, $n \geq 1$. Also let $\log x = \ln(x \vee e)$, $\log\log x = \log(\log x)$ and $\phi(x) = \sqrt{\pi^2 x/(8 \log\log x)}$. Then by the so-called Chung's law of the iterated logarithm (LIL) we have

$$\liminf_{n\to\infty} \phi(n) M_n = \sigma \quad a.s.. \tag{1.1}$$

This result was first proved by Chung (1948) under $\mathsf{E}|X|^3 < \infty$, and by Jain and Pruitt (1975) under the sole assumption of a finite second moment. As pointed by Csáki (1978), the assumption of a finite second moment is also necessary for (1.1) to hold.

As for the usual LIL, Gut and Spătaru (2000) proved the following two results on its precise asymptotics.

Theorem A *Suppose that* $\mathsf{E}X = 0$, $\mathsf{E}X^2 = \sigma^2$ *and* $\mathsf{E}[X^2(\log\log|X|)^{1+\delta}] < \infty$ *for some* $\delta > 0$, *and let* $a_n = O(\sqrt{n}/(\log\log n)^\gamma)$ *for some* $\gamma > 1/2$. *Then*

$$\lim_{\epsilon\searrow 1} \sqrt{\epsilon^2 - 1} \sum_{n=1}^\infty \frac{1}{n} \mathsf{P}(|S_n| \geq \epsilon\sqrt{2\sigma^2 n \log\log n} + a_n) = 1.$$

Theorem B *Suppose that* $\mathsf{E}X = 0$ *and* $\mathsf{E}X^2 = \sigma^2 < \infty$. *Then*

$$\lim_{\epsilon\searrow 0} \epsilon^2 \sum_{n=1}^\infty \frac{1}{n \log n} \mathsf{P}(|S_n| \geq \epsilon\sqrt{n \log\log n}) = \sigma^2.$$

The main purpose of this paper is to show similar results on Chung's LIL under the *minimal* conditions by using an extended Feller's and Einmahl's truncation method.

Theorem 1.1 *Let* $a > -1$ *and* $b > -1$ *and let* $a_n(\epsilon)$ *be a function of* ϵ *such that*

$$a_n(\epsilon) \log\log n \to \tau \ \text{ as } \ n \to \infty \text{ and } \epsilon \nearrow 1/\sqrt{1+a}. \tag{1.2}$$

Suppose that $\mathsf{E}X = 0$, $\mathsf{E}X^2 = \sigma^2 < \infty$ *and*

$$\mathsf{E}X^2 I\{|X| \geq t\} = o((\log\log t)^{-1}) \ \text{ as } \ t \to \infty. \tag{1.3}$$

Then

$$\lim_{\epsilon\nearrow 1/\sqrt{1+a}}\Big(\frac{1}{\sqrt{1+a}}-\epsilon\Big)^{b+1}\sum_{n=1}^{\infty}\frac{(\log n)^a(\log\log n)^b}{n}\mathsf{P}\Big\{M_n\le\sigma\phi(n)(\epsilon+a_n(\epsilon))\Big\}$$
$$=\frac{4}{\pi}\Big(\frac{1}{2(1+a)^{3/2}}\Big)^{b+1}\Gamma(b+1)\exp\{2(1+a)^{3/2}\tau\}. \tag{1.4}$$

Here, $\Gamma(\cdot)$ *is a gamma function. Conversely, if (1.4) holds for some* $a>-1$*,* $b>-1$ *and* $0<\sigma<\infty$*, then* $\mathsf{E}X=0$*,* $\mathsf{E}X^2=\sigma^2$ *and*

$$\liminf_{t\to\infty}(\log\log t)\mathsf{E}X^2I\{|X|\ge t\}=0. \tag{1.5}$$

Theorem 1.2 *Suppose that* $\mathsf{E}X=0$ *and* $\mathsf{E}X^2=\sigma^2<\infty$*. For* $b>-1$*, we have*

$$\lim_{\epsilon\nearrow\infty}\epsilon^{-2(b+1)}\sum_{n=1}^{\infty}\frac{(\log\log n)^b}{n\log n}\mathsf{P}\Big\{M_n\le\sigma\phi(n)\epsilon\Big\}$$
$$=\frac{4}{\pi}\Gamma(b+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2b+3}}. \tag{1.6}$$

Conversely, if (1.6) holds for some $b>-1$ *and* $0<\sigma<\infty$*, then* $\mathsf{E}X=0$ *and* $\mathsf{E}X^2=\sigma^2$*.*

Theorems 1.1 and 1.2 are also related to the integral test which refines (1.1). The first one of the results on the integral test for (1.1) is due to Chung (1948) who obtained that if $\mathsf{E}X=0$, $\mathsf{E}X^2=\sigma^2$ and $\mathsf{E}|X|^3<\infty$, then for any eventually non-increasing $\psi:[1,\infty)\to(0,\infty)$,

$$\mathsf{P}\big(M_n\le\sqrt{\sigma^2\pi^2n/8}\,\psi(n)\ i.o.\big)=0 \text{ or } =1$$
$$\text{according as } J(\psi):=\sum_{n=1}^{\infty}\frac{1}{n\psi(n)^2}\exp(-1/\psi(n)^2)<\infty \text{ or } =\infty. \tag{1.7}$$

Einmahl (1993) proved (1.7) under the minimal condition that

$$\mathsf{E}X^2I\{|X|\ge t\}=O((\log\log t)^{-1}) \text{ as } t\to\infty. \tag{1.8}$$

Our next theorem gives a result on a convergence rate of (1.1) and (1.7).

Theorem 1.3 *Let* $a>-1$ *and* $b>-1$*. Suppose that* $\mathsf{E}X=0$*,* $\mathsf{E}X^2=\sigma^2$*, and that the condition (1.8) is satisfied, then for any eventually non-increasing* $\psi:[1,\infty)\to(0,\infty)$*,*

$$\sum_{n=1}^{\infty}\frac{(\log n)^a(\log\log n)^b}{n}\mathsf{P}\big(M_n\le\sqrt{\sigma^2\pi^2n/8}\,\psi(n)\big)<\infty \text{ or } =\infty$$
$$\text{according as } J_{ab}(\psi):=\sum_{n=1}^{\infty}\frac{(\log n)^a(\log\log n)^b}{n}\exp(-1/\psi(n)^2)<\infty \text{ or } =\infty. \tag{1.9}$$

Conversely, if (1.9) holds for some $a > -1$, $b > -1$, $0 < \sigma < \infty$ and any eventually non-increasing $\psi(x)$, then $\mathsf{E}X = 0$, $\mathsf{E}X^2 = \sigma^2$ and

$$\liminf_{t\to\infty}(\log\log t)\mathsf{E}X^2 I\{|X| \geq t\} < \infty. \tag{1.10}$$

By (1.9), we know that the infinite series in (1.4) converges whenever $\epsilon < 1/\sqrt{1+a}$, and diverges whenever $\epsilon > 1/\sqrt{1+a}$.

Remark 1.1 *Note that the conditions (1.3) and (1.8) is sharp. A sufficient condition for them is given by*

$$\mathsf{E}X^2 \log\log|X| < \infty.$$

However, when $a > 0$, the sufficient and necessary condition for

$$\sum_{n=1}^{\infty}\frac{(\log n)^a(\log\log n)^b}{n}\mathsf{P}\big(M_n > \epsilon\sqrt{2\sigma^2 n\log\log n}\big) < \infty, \quad \epsilon > \sqrt{1+a},$$

to hold is that

$$\mathsf{E}X^2(\log|X|)^a(\log\log|X|)^{b-1} < \infty.$$

Remark 1.2 *By using a result of Einmahl (1987) instead of our Lemma 3.5, one can extend Theorems 1.1-1.3 to multidimensional random variables.*

The proofs of Theorems 1.1-1.3 are given in Section 4. Before that, we first verify (1.4), (1.6) and (1.9) under the assumption that F is the normal distribution in Section 2, after which, by using the truncation and approximation method, we then show that the probabilities in (1.4), (1.6) and (1.9) can be replaced by those for normal random variables in Section 3. Throughout this paper, we let $K(\alpha,\beta,\cdots)$, $C(\alpha,\beta,\cdots)$ etc denote positive constants which depend on $\alpha,\beta,\cdots$ only, whose values can differ in different places. $a_n \sim b_n$ means that $a_n/b_n \to 1$.

2 Normal cases.

In this section, we prove Theorems 1.1-1.3 in the case that $\{X, X_n; n \geq 1\}$ are normal random variables. Let $\{W(t); t \geq 0\}$ be a standard Wiener process. Our results are as follows.

Proposition 2.1 *Let $a > -1$ and $b > -1$ and let $a_n(\epsilon)$ be a function of ϵ satisfying (1.2). Then*

$$\lim_{\epsilon \nearrow 1/\sqrt{1+a}} \Big(\frac{1}{\sqrt{1+a}} - \epsilon\Big)^{b+1} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} \cdot P\Big\{ \sup_{0\le s\le 1} |W(s)| \le \sqrt{\frac{\pi^2}{8\log\log n}}(\epsilon + a_n(\epsilon))\Big\}$$
$$= \frac{4}{\pi}\Big(\frac{1}{2(1+a)^{3/2}}\Big)^{b+1}\Gamma(b+1)\exp\Big\{2(1+a)^{3/2}\tau\Big\}. \tag{2.1}$$

Proposition 2.2 *For any $b > -1$, we have*

$$\lim_{\epsilon\nearrow\infty} \epsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n\log n} P\Big\{ \sup_{0\le s\le 1} |W(s)| \le \epsilon\sqrt{\frac{\pi^2}{8\log\log n}}\Big\}$$
$$= \frac{4}{\pi}\Gamma(b+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2b+3}}.$$

Proposition 2.3 *For any $a > -1$ and $b > -1$, we have that*

$$\sum_{n=1}^{\infty} \frac{(\log n)^a(\log\log n)^b}{n} P\Big(\sup_{0\le s\le 1}|W(s)| \le \sqrt{\pi^2/8}\,\psi(n)\Big) < \infty \text{ or } = \infty$$

according as $J_{ab}(\psi) < \infty$ or $= \infty$.

The following lemma will be used in the proofs.

Lemma 2.1 *Let $\{W(t); t \ge 0\}$ be a standard Wiener process. Then for all $x > 0$,*

$$P\Big(\sup_{0\le s\le 1}|W(s)| \le x\Big) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{2k+1}\exp\Big\{-\frac{\pi^2(2k+1)^2}{8x^2}\Big\} \tag{2.2}$$

and

$$P\Big(\sup_{0\le s\le 1}|W(s)| \le x\Big) \sim \frac{4}{\pi}\exp\Big\{-\frac{\pi^2}{8x^2}\Big\} \text{ as } x \to 0.$$

Proof. It is well known. See Ciesielski and Taylor (1962).

Now, we turn to prove the propositions.

Proof Proposition 2.1: First, note that the limit in (2.1) does not depend on any finite terms of the infinite series. Secondly, by Lemma 2.1 and the condition (1.2) we have

$$\mathsf{P}\Big\{\sup_{0\le s\le 1}|W(s)| \le \sqrt{\frac{\pi^2}{8\log\log n}}(\epsilon + a_n(\epsilon))\Big\} \sim \frac{4}{\pi}\exp\Big\{-\frac{\log\log n}{(\epsilon + a_n(\epsilon))^2}\Big\}$$
$$= \frac{4}{\pi}\exp\Big\{-\frac{\log\log n}{\epsilon^2 + 2\epsilon a_n(\epsilon) + a_n^2(\epsilon)}\Big\}$$
$$\sim \frac{4}{\pi}\exp\Big\{-\frac{\log\log n}{\epsilon^2}\Big\}\exp\Big\{\frac{2}{\epsilon^3}a_n(\epsilon)\log\log n\Big\}$$

as $n \to \infty$, uniformly in $\epsilon \in (1/\sqrt{1+a}-\delta, 1/\sqrt{1+a})$ for some $\delta > 0$. So, for any $0 < \theta < 1$, there exist $\delta > 0$ and n_0 such that for all $n \ge n_0$ and $\epsilon \in (1/\sqrt{1+a}-\delta, 1/\sqrt{1+a})$,

$$
\begin{aligned}
& \tfrac{4}{\pi}\exp\Big\{-\tfrac{\log\log n}{\epsilon^2}\Big\}\exp\Big\{2(1+a)^{3/2}\tau-\theta\Big\} \\
\le\ & \mathsf{P}\Big\{\sup\nolimits_{0\le s\le 1}|W(s)| \le \sqrt{\tfrac{\pi^2}{8\log\log n}}(\epsilon+a_n(\epsilon))\Big\} \\
\le\ & \tfrac{4}{\pi}\exp\Big\{-\tfrac{\log\log n}{\epsilon^2}\Big\}\exp\Big\{2(1+a)^{3/2}\tau+\theta\Big\},
\end{aligned}
$$

by the condition (1.2) again. Also,

$$
\begin{aligned}
& \lim_{\epsilon\nearrow 1/\sqrt{1+a}}\Big(\frac{1}{\sqrt{1+a}}-\epsilon\Big)^{b+1}\sum_{n=1}^{\infty}\frac{(\log n)^a(\log\log n)^b}{n}\exp\Big\{-\frac{\log\log n}{\epsilon^2}\Big\} \\
=\ & \lim_{\epsilon\nearrow 1/\sqrt{1+a}}\Big(\frac{1}{\sqrt{1+a}}-\epsilon\Big)^{b+1}\int_{e^e}^{\infty}\frac{(\log x)^a(\log\log x)^b}{x}\exp\Big\{-\frac{\log\log x}{\epsilon^2}\Big\}dx \\
=\ & \lim_{\epsilon\nearrow 1/\sqrt{1+a}}\Big(\frac{1}{\sqrt{1+a}}-\epsilon\Big)^{b+1}\int_{1}^{\infty}y^b\exp\Big\{-y\Big(\frac{1}{\epsilon^2}-1-a\Big)\Big\}dy \\
=\ & \lim_{\epsilon\nearrow 1/\sqrt{1+a}}\Big(\frac{1}{\sqrt{1+a}}-\epsilon\Big)^{b+1}\Big(\frac{1}{\epsilon^2}-1-a\Big)^{-b-1}\int_{\frac{1}{\epsilon^2}-1-a}^{\infty}y^be^{-y}dy \\
=\ & \Big(\frac{1}{2(1+a)^{3/2}}\Big)^{b+1}\int_0^{\infty}y^be^{-y}dy = \Big(\frac{1}{2(1+a)^{3/2}}\Big)^{b+1}\Gamma(b+1).
\end{aligned}
$$

(2.1) is now proved.

Proof Proposition 2.2: Notice that (2.2), and for any $m \ge 1$ and all $x > 0$,

$$
\begin{aligned}
& \frac{4}{\pi}\sum_{k=0}^{2m+1}\frac{(-1)^k}{2k+1}\exp\Big\{-\frac{\pi^2(2k+1)^2}{8x^2}\Big\} \\
\le\ & \mathsf{P}\big(\sup_{0\le s\le 1}|W(s)| \le x\big) \\
\le\ & \frac{4}{\pi}\sum_{k=0}^{2m}\frac{(-1)^k}{2k+1}\exp\Big\{-\frac{\pi^2(2k+1)^2}{8x^2}\Big\}.
\end{aligned}
$$

It is sufficient to show that for any $q > 0$,

$$
\lim_{\epsilon\nearrow\infty}\epsilon^{-2(b+1)}\sum_{n=1}^{\infty}\frac{(\log\log n)^b}{n\log n}\exp\Big\{-q\frac{\log\log n}{\epsilon^2}\Big\} = \Gamma(b+1)q^{-(b+1)}. \tag{2.3}
$$

Now, the left hand of the above equality equals

$$
\begin{aligned}
& \lim_{\epsilon\nearrow\infty}\epsilon^{-2(b+1)}\int_{e^e}^{\infty}\frac{(\log\log x)^b}{x\log x}\exp\Big\{-q\frac{\log\log x}{\epsilon^2}\Big\}dx \\
=\ & \lim_{\epsilon\nearrow\infty}\epsilon^{-2(b+1)}\int_1^{\infty}y^b\exp\Big\{-y\frac{q}{\epsilon^2}\Big\}dy \\
=\ & \lim_{\epsilon\nearrow\infty}q^{-(b+1)}\int_{q/\epsilon^2}^{\infty}y^be^{-y}dy = \Gamma(b+1)q^{-(b+1)}.
\end{aligned}
$$

The proposition is proved.

Proof Proposition 2.3: It is obvious, since

$$\frac{2}{\pi}\exp\Big\{-\frac{\pi^2}{8x^2}\Big\} \le \mathsf{P}\{\sup_{0\le s\le 1}|W(s)| \le x\} \le \frac{4}{\pi}\exp\Big\{-\frac{\pi^2}{8x^2}\Big\}, \quad \forall x>0.$$

3 Truncation and Approximation.

The purpose of this section is to extend Feller's and Einmahl's truncation methods and to show that the probabilities in (1.4), (1.6) and (1.9) for M_n can be approximated by those for $\sqrt{n}\sup_{0\le s\le 1}|W(s)|$.

Suppose that $\mathsf{E}X=0$ and $\mathsf{E}X^2=\sigma^2<\infty$. Without losing of generality, we assume that $\sigma=1$. Let $0<p<1/2$. For each n and $1\le j\le n$, we let $X'_{nj}=X_{nj}I\{|X_j|\le \sqrt{n}/\log^p n\}$, $X^*_{nj}=X'_{nj}-\mathsf{E}[X'_{nj}]$, $S'_{nj}=\sum_{i=1}^j X'_{nj}$, $S^*_{nj}=\sum_{i=1}^j X^*_{nj}$, $M^*_n=\max_{k\le n}|S^*_{nk}|$ and $B_n=\sum_{k=1}^n \mathsf{Var}(X^*_{nk})$. The following two propositions are the main results of this section.

Proposition 3.1 *Let $a>-1$, $b>-1$ and $0<p<1/2$. Then there exist $\delta>0$ and a sequence of positive numbers $\{q_n\}$ such that*

$$\begin{aligned} & P\Big\{\sup_{0\le s\le 1}|W(s)| \le \epsilon\sqrt{\frac{\pi^2}{8\log\log n}-\frac{3}{(\log\log n)^2}}\Big\} - q_n \\ \le\ & P\Big\{M_n \le \epsilon\sqrt{\frac{\pi^2 B_n}{8\log\log n}}\Big\} \\ \le\ & P\Big\{\sup_{0\le s\le 1}|W(s)| \le \epsilon\sqrt{\frac{\pi^2}{8\log\log n}+\frac{3}{(\log\log n)^2}}\Big\} + q_n, \\ & \qquad \forall \epsilon\in\Big(\frac{1}{\sqrt{1+a}}-\delta, \frac{1}{\sqrt{1+a}}+\delta\Big), \quad n\ge 1 \end{aligned} \tag{3.1}$$

and

$$\sum_{n=1}^\infty \frac{(\log n)^a(\log\log n)^b}{n} q_n \le K(a,b,p,\delta) < \infty. \tag{3.2}$$

Proposition 3.2 *If n is large enough, then*

$$\begin{aligned} & P\big(\sup_{0\le s\le 1}|W(s)| \le x - 3/(\log\log n)^2\big) - q^*_n \le P\big(M_n \le x\sqrt{B_n}\big) \\ \le\ & P\big(\sup_{0\le s\le 1}|W(s)| \le x + 3/(\log\log n)^2\big) + q^*_n, \quad \forall x>0, \end{aligned} \tag{3.3}$$

where $q_n^* \geq 0$ *satisfies*

$$\sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n\log n} q_n^* \leq K(b,p) < \infty. \tag{3.4}$$

To show this two results, we need some lemmas.

Lemma 3.1 *For any* $x > 0$ *and* $0 < \delta < 1$*, there exists a positive constant* $C = C(x,\delta)$ *such that*

(a) $C^{-1}\exp\{-\frac{1+\delta}{x^2}\log\log n\} \leq P\{M_n^* \leq x\phi(n)\} \leq C\exp\{-\frac{1-\delta}{x^2}\log\log n\}$,

(b) $C^{-1}\exp\{-\frac{1+\delta}{x^2}\log\log n\} \leq P\{M_n \leq x\phi(n)\} \leq C\exp\{-\frac{1-\delta}{x^2}\log\log n\}$.

Proof. This lemma is so-called small deviation theorem. It follows from Theorem 2 of Shao (1995) by noting that $B_n \sim n$. (See also Shao 1991).

Lemma 3.2 *For any* $x > 0$*,* $A > 0$ *and* $0 < \delta < 1$*, there exists a positive constant* $C = C(x,\delta)$ *such that*

$$P\{\max_{k\leq q}|S_k + z_1| \vee \max_{q<k\leq n}|S_k + z_2| \leq x\phi(n)\} \leq C\exp\Big\{-\frac{1-\delta}{x^2}\log\log n\Big\}$$

holds uniformly in $|z_1| \leq A\phi(n)$*,* $|z_2| \leq A\phi(n)$ *and* $1 \leq q \leq n$*.*

Proof. Without losing of generality, we can assume that $0 < \delta < \frac{1}{8^8 5!}$. We follow the lines of the proof of (17) in Shao (1995). Write $x_n = x\phi(n)$. Put $M = \delta^{-2}$. For fixed n, define $m_0 = 0$,

$$m_i = \max\{j : j \leq Mix_n^2\}, \qquad \text{for } i \leq l := \max\{i : m_i \leq n-1\}$$

and $m_{l+1} = n$. It is easily seen that

$$(1-\delta/4)Mx_n^2 \leq m_i - m_{i-1} \leq (1+\delta/4)Mx_n^2$$

and

$$\frac{n}{Mx_n^2} - 1 \leq l \leq \frac{n}{Mx_n^2}$$

provided n is sufficiently large. From Lemmas 3 and 1 of Shao (1995) and the Anderson's inequality, it follows that, there exists an integer n_0 such that $\forall n \geq n_0$, $\forall 1 \leq j \leq l$,

$\forall |y| \le x_n + A\phi(n)$, $\forall |y_j| \le A\phi(n)$,

$$\begin{aligned}
& \mathsf{P}\Big(\max_{k\le m_j-m_{j-1}} |S_{m_{j-1}+k} - S_{m_{j-1}} + y_j + y| \le x_n\Big) \\
\le\ & e^{-3M} + \mathsf{P}\Big(\sup_{0\le s\le 1} |W(s) + (y_j+y)(m_j - m_{j-1})^{-1/2}| \le x_n (m_j - m_{j-1})^{-1/2}\Big) \\
\le\ & e^{-3M} + \mathsf{P}\Big(\sup_{0\le s\le 1} |W(s)| \le x_n (m_j - m_{j-1})^{-1/2}\Big) \\
\le\ & e^{-3M} + 4\exp\Big(-\frac{\pi^2(m_j - m_{j-1})}{8x_n^2}\Big) \le e^{-3M} + 4\exp\Big(-\frac{\pi^2 M(1-\delta/4)}{8}\Big) \\
\le\ & \frac{1}{2}e^{-2M} + \frac{1}{2}\exp\Big(-\frac{\pi^2 M(1-\delta/2)}{8}\Big) \le \exp\Big(-\frac{\pi^2 M(1-\delta/2)}{8}\Big),
\end{aligned}$$

where $\{W(t); t\ge 0\}$ is a standard Wiener process. Obviously, there exists an i such that $m_{i-1} < q \le m_i$. Let $y_j = z_1$ if $j \le i-1$, and z_2 if $j \ge i$. Then

$$\begin{aligned}
& \mathsf{P}\big(\max_{k\le q} |S_k + z_1| \vee \max_{q<k\le n} |S_k + z_2| \le x\phi(n)\big) \le \mathsf{P}\big(\max_{j\le l, j\ne i}\max_{m_{j-1}<k\le m_j} |S_k + y_j| \le x_n\big) \\
=\ & \mathsf{E}\Big\{I\{\max_{j\le l-1, j\ne i}\max_{m_{j-1}<k\le m_j} |S_k + y_j| \le x_n\} I\{\max_{m_{l-1}<k\le m_l, l\ne i} |S_k + y_j| \le x_n\}\Big\} \\
\le\ & \mathsf{E}\Big\{I\{\max_{j\le l-1, j\ne i}\max_{m_{j-1}<k\le m_j} |S_k + y_j| \le x_n\} \\
& \qquad \times \mathsf{E}\big[I\{\max_{m_{l-1}<k\le m_l} |S_k + y_j| \le x_n\} | S_k, k \le m_{l-1}\big]\Big\} \\
=\ & \int_{-x_n - y_{l-1}}^{x_n - y_{l-1}} \mathsf{P}\Big(\max_{m_{l-1}<k\le m_l} |S_k - S_{m_{l-1}} + y + y_l| \le x_n\Big) \\
& \qquad d\mathsf{P}\Big(\max_{j\le l-1, j\ne i}\max_{m_{j-1}<k\le m_j} |S_k + y_j| \le x_n, S_{m_{l-1}} < y\Big) \\
\le\ & \exp\Big(-\frac{\pi^2 M(1-\delta/2)}{8}\Big) \mathsf{P}\Big(\max_{j\le l-1, j\ne i}\max_{m_{j-1}<k\le m_j} |S_k + y_j| \le x_n\Big) \\
\le\ & \ldots \le \exp\Big(-\frac{\pi^2 M(1-\delta/2)(l-1)}{8}\Big) \\
\le\ & C\exp\Big(-\frac{(1-\delta)\pi^2 n}{8x_n^2}\Big) \le C\exp\Big(-\frac{(1-\delta)}{x^2}\log\log n\Big).
\end{aligned}$$

For $n \le n_0$, it is obvious that

$$\mathsf{P}\big(\max_{k\le q} |S_k + z_1| \vee \max_{k<k\le n} |S_k + z_2| \le x\phi(n)\big) \le 1 \le C\exp\Big(-\frac{(1-\delta)}{x^2}\log\log n\Big).$$

Lemma 2 is proved.

Lemma 3.3 *Define* $\Delta_n = \max_{k\le n} |S^*_{nk} - S_k|$. *Let* $a > -1$, $b > -1$ *and* $0 < p < 1/2$. *Then there exist constants* $\delta_0 = \delta_0(a,p) > 0$ *and* $K = K(a,b,p)$ *such that* $\forall 0 < \delta < \delta_0$,

$$\sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} I_n \le K\mathsf{E}X^2 < \infty,$$

where

$$I_n = P\Big(\Delta_n \geq \sqrt{n}/(\log\log n)^2, M_n^* \leq \phi(n)\big(\frac{1}{\sqrt{1+a}}+\delta\big)\Big).$$

Proof. It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} \mathsf{P}\Big(\Delta_n \geq \sqrt{n}/(\log\log n)^2, M_n^* \leq \frac{\phi(n)}{\sqrt{1+a-\delta}}\Big) \leq C\mathsf{E}X^2, \qquad (3.5)$$

whenever $0<\delta<1+a$ and $0<\delta<1-2p$. Let $\beta_n = n\mathsf{E}[|X|I\{|X|>\sqrt{n}/\log^p n\}]$. Then $|\mathsf{E}\sum_{i=1}^j X'_{ni}| \leq \beta_n$, $1 \leq j \leq n$. Setting

$$\mathcal{L} = \{n : \beta_n \leq \frac{1}{8}\sqrt{n}/(\log\log n)^2\},$$

we have

$$\{\Delta_n \geq \sqrt{n}/(\log\log n)^2\} \subset \bigcup_{j=1}^n \{X_j \neq X'_{nj}\}, \quad n \in \mathcal{L}.$$

So for $n \in \mathcal{L}$,

$$\begin{aligned} I'_n : &= \mathsf{P}\Big(\Delta_n \geq \sqrt{n}/(\log\log n)^2, M_n^* \leq \frac{\phi(n)}{\sqrt{1+a-\delta}}\Big) \\ &\leq \sum_{j=1}^n \mathsf{P}\Big(X_j \neq X'_{nj}, M_n^* \leq \frac{\phi(n)}{\sqrt{1+a-\delta}}\Big). \end{aligned}$$

Observer that $X'_{nj}=0$ whenever $X_j \neq X'_{nj}$, $j \leq n$, so that by Lemma 3.1(a) we have for n large enough and all $1 \leq j \leq n$,

$$\begin{aligned} & \mathsf{P}\Big(X_j \neq X'_{nj}, M_n^* \leq \frac{\phi(n)}{\sqrt{1+a-\delta}}\Big) \\ = & \mathsf{P}\Big(X_j \neq X'_{nj}, \max_{k \leq j-1} |S_{nk}^*| \vee \max_{j<k\leq n} |S_{nk}^* - X'_{nj}| \leq \frac{\phi(n)}{\sqrt{1+a-\delta}}\Big) \\ = & \mathsf{P}(X_j \neq X'_{nj})\mathsf{P}\Big(\max_{k \leq j-1} |S_{nk}^*| \vee \max_{j<k\leq n} |S_{nk}^* - X'_{nj}| \leq \frac{\phi(n)}{\sqrt{1+a-\delta}}\Big) \\ \leq & \mathsf{P}\Big(X_j \neq X'_{nj})\mathsf{P}\Big(M_n^* \leq \frac{\phi(n)}{\sqrt{1+a-\delta}} + |X'_{nj}|\Big) \\ \leq & \mathsf{P}(|X| > \sqrt{n}/\log^p n)\mathsf{P}\Big(M_n^* \leq \frac{\phi(n)}{\sqrt{1+a-\delta}} + \sqrt{n}/\log^p n\Big) \\ \leq & C\mathsf{P}(|X| > \sqrt{n}/\log^p n)\exp\{(-1-a+\delta+\delta')\log\log n\}, \end{aligned}$$

where $0 < \delta' < 1 - 2p - \delta$. So,

$$
\begin{aligned}
& \sum_{n \in \mathcal{L}} \frac{(\log n)^a (\log\log n)^b}{n} I_n' \le C \sum_{n=1}^{\infty} \mathsf{P}(|X| > \sqrt{n}/\log^p n)(\log n)^{\delta+\delta'-1}(\log\log n)^b \\
\le\ & \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} \mathsf{P}\big(\sqrt{j}/\log^p j < |X| \le \sqrt{j+1}/\log^p(j+1)\big)(\log n)^{\delta+\delta'-1}(\log\log n)^b \\
\le\ & \sum_{j=1}^{\infty} \mathsf{P}\big(\sqrt{j}/\log^p j < |X| \le \sqrt{j+1}/\log^p(j+1)\big) \sum_{n=1}^{j} (\log n)^{\delta+\delta'-1}(\log\log n)^b \\
\le\ & \sum_{j=1}^{\infty} \mathsf{P}\big(\sqrt{j}/\log^p j < |X| \le \sqrt{j+1}/\log^p(j+1)\big) j (\log j)^{\delta+\delta'-1}(\log\log j)^b \\
\le\ & C\mathsf{E}\Big[X^2 (\log|X|)^{\delta+\delta'+2p-1}(\log\log|X|)^b\Big] \le C\mathsf{E}X^2.
\end{aligned}
$$

If $n \notin \mathcal{L}$, then by Lemma 3.1(a) we have

$$
\begin{aligned}
I_n' \ &\le\ \mathsf{P}\Big(M_n^* \le \frac{\phi(n)}{\sqrt{1+a-\delta}}\Big) \le C \exp\{(-1-a+\delta+\delta')\log\log n\} \\
&=\ C(\log n)^{-1-a+\delta+\delta'}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \sum_{n \notin \mathcal{L}} \frac{(\log n)^a (\log\log n)^b}{n} I_n' \le C \sum_{n \notin \mathcal{L}} \frac{(\log n)^{\delta+\delta'-1}(\log\log n)^b}{n} \\
\le\ & 8C \sum_{n \notin \mathcal{L}} \frac{(\log n)^{\delta+\delta'-1}(\log\log n)^{b+2}}{n^{3/2}} \beta_n \\
\le\ & 8C \sum_{n=1}^{\infty} \frac{(\log n)^{\delta+\delta'-1}(\log\log n)^{b+2}}{n^{1/2}} \\
& \quad \cdot \sum_{j=n}^{\infty} \mathsf{E}\Big[|X| I\{\sqrt{j}/\log^p j < |X| \le \sqrt{j+1}/\log^p(j+1)\}\Big] \\
=\ & 8C \sum_{j=1}^{\infty} \mathsf{E}\Big[|X| I\{\sqrt{j}/\log^p j < |X| \le \sqrt{j+1}/\log^p(j+1)\}\Big] \\
& \quad \cdot \sum_{n=1}^{j} \frac{(\log n)^{\delta+\delta'-1}(\log\log n)^{b+2}}{n^{1/2}} \\
\le\ & C \sum_{j=1}^{\infty} \mathsf{E}\Big[|X| I\{\sqrt{j}/\log^p j < |X| \le \sqrt{j+1}/\log^p(j+1)\}\Big] \\
& \quad \cdot \sqrt{j}(\log j)^{\delta+\delta'-1}(\log\log j)^{b+2} \\
\le\ & C\mathsf{E}\Big[X^2 (\log|X|)^{\delta+\delta'-1+p}(\log\log|X|)^{b+2}\Big] \le C\mathsf{E}X^2.
\end{aligned}
$$

(3.5) is proved.

Lemma 3.4 *Let $a > -1$, $b > -1$ and $0 < p < 1/2$. Then there exist constants $\delta_0 = \delta_0(a,p) > 0$ and $K = K(a,b,p)$ such that $\forall 0 < \delta < \delta_0$,*

$$\sum_{n=1}^{\infty} \frac{\log^q n(\log\log n)^b}{n} II_n \le K\mathsf{E}X^2 < \infty,$$

where

$$II_n = \mathsf{P}\Big(\Delta_n \ge \sqrt{n}/(\log\log n)^2, M_n \le \phi(n)\big(\frac{1}{\sqrt{1+a}} + \delta\big)\Big).$$

Proof. It is enough to show that

$$\sum_{n=1}^{\infty} \frac{\log^q n(\log\log n)^b}{n} \mathsf{P}\Big(\Delta_n \ge \sqrt{n}/(\log\log n)^2, M_n \le \frac{\phi(n)}{\sqrt{1+a-\delta}}\Big) \le C\mathsf{E}X^2,$$

whenever $0 < \delta < 1 + a$ and $0 < \delta < 1 - 2p$. Let β_n and $\mathcal{L}$ be defined as in the proof of Lemma 3.3. Then for $n \in \mathcal{L}$,

$$\mathsf{P}\Big(\Delta_n \ge \sqrt{n}/(\log\log n)^2, M_n \le \frac{\phi(n)}{\sqrt{1+a-\delta}}\Big) \le \sum_{j=1}^{n} \mathsf{P}\Big(X_j \ne X'_{nj}, M_n \le \frac{\phi(n)}{\sqrt{1+a-\delta}}\Big).$$

and for $1 \le j \le n$,

$$\begin{aligned}
& \mathsf{P}\Big(X_j \ne X'_{nj}, M_n \le \frac{\phi(n)}{\sqrt{1+a-\delta}}\Big) \\
\le\ & \mathsf{P}\Big(X_j \ne X'_{nj}, M_{j-1} \vee \max_{j<k\le n} |S_k - X_j + X_j| \le \frac{\phi(n)}{\sqrt{1+a-\delta}}, M_n \le \frac{\phi(n)}{\sqrt{1+a-\delta}}\Big) \\
\le\ & \mathsf{P}\Big(\frac{\sqrt{n}}{\log^p n} < |X_j| \le 2\frac{\phi(n)}{\sqrt{1+a-\delta}}, M_{j-1} \vee \max_{j<k\le n} |S_k - X_j + X_j| \le \frac{\phi(n)}{\sqrt{1+a-\delta}}\Big) \\
=\ & \int_{\frac{\sqrt{n}}{\log^p n} < |y| \le 2\frac{\phi(n)}{\sqrt{1+a-\delta}}} \mathsf{P}\Big(M_{j-1} \vee \max_{j<k\le n} |S_k - X_j + y| \le \frac{\phi(n)}{\sqrt{1+a-\delta}}\Big) d\mathsf{P}(X_j < y).
\end{aligned}$$

Note that $M_{j-1} \vee \max_{j<k\le n} |S_k - X_j + y| \overset{\mathcal{D}}{=} M_{j-1} \vee \max_{j\le k\le n-1} |S_k + y|$. By Lemma 3.2, we have

$$\begin{aligned}
\sup_{|y| \le 2\frac{\phi(n)}{\sqrt{1+a-\delta}}} & \mathsf{P}\ \Big(M_{j-1} \vee \max_{j<k\le n} |S_k - X_j + y| \le \frac{\phi(n)}{\sqrt{1+a-\delta}}\Big) \\
& \le\ C\exp\big\{(-1-a+\delta+\delta')\log\log n\big\}.
\end{aligned}$$

It follows that for $n \in \mathcal{L}$ and $1 \le j \le n$,

$$\begin{aligned}
& \mathsf{P}\Big(X_j \ne X'_{nj}, M_n \le \frac{\phi(n)}{\sqrt{1+a-\delta}}\Big) \\
\le\ & C(\log n)^{-1-a+\delta+\delta'} \mathsf{P}\Big(\frac{\sqrt{n}}{\log^p n} < |X_j| \le 2\frac{\phi(n)}{\sqrt{1+a-\delta}}\Big) \\
\le\ & C(\log n)^{-1-a+\delta+\delta'} \mathsf{P}\Big(|X| > \sqrt{n}/\log^p n\Big).
\end{aligned}$$

The remained proof is similar to that of (3.5) with Lemma 3.1(b) instead of Lemma 3.1(a).

Lemma 3.5 *For any sequence of independent random variables $\{\xi_n; n \ge 1\}$ with mean zero and finite variance, there exists a sequence of independent normal variables $\{\eta_n; n \ge 1\}$ with $\mathsf{E}\eta_n = 0$ and $\mathsf{E}\eta_n^2 = \mathsf{E}\xi_n^2$ such that, for all $Q > 2$ and $y > 0$,*

$$\mathsf{P}\Big(\max_{k\le n} |\sum_{i=1}^k \xi_i - \sum_{i=1}^k \eta_i| \ge y\Big) \le (AQ)^Q y^{-Q} \sum_{i=1}^n \mathsf{E}|\xi_i|^Q,$$

whenever $\mathsf{E}|\xi_i|^Q < \infty$, $i = 1, \dots, n$. Here, A is a universal constant.

Proof. See Sakhaneko (1980,1984, 1985).

Lemma 3.6 *We have that*

$$\begin{aligned} &\mathsf{P}\big(\sup_{0\le s\le 1} |W(s)| \le x - 1/(\log\log n)^2\big) - p_n \le \mathsf{P}\big(M_n^* \le x\sqrt{B_n}\big) \\ \le\ &\mathsf{P}\big(\sup_{0\le s\le 1} |W(s)| \le x + 1/(\log\log n)^2\big) + p_n, \quad \forall x > 0, \end{aligned} \tag{3.6}$$

where $p_n \ge 0$ satisfies

$$\sum_{n=1}^\infty \frac{(\log n)^a (\log\log n)^b}{n} p_n \le K(a,b,p) < \infty. \tag{3.7}$$

Proof. By Lemma 3.5, there exist a universal constant $A > 0$ and a sequence of standard Wiener processes $\{W_n(\cdot)\}$ such that for all $Q > 2$,

$$\begin{aligned} &\mathsf{P}\Big(\max_{k\le n} |S_{nk}^* - W_n(\frac{k}{n}B_n)| \ge \frac{1}{2}\sqrt{B_n}/(\log\log n)^2\Big) \\ \le\ &(AQ)^Q \Big(\frac{(\log\log n)^2}{\sqrt{B_n}}\Big)^Q \sum_{k=1}^n \mathsf{E}\big|X_{nk}^*\big|^Q \\ \le\ &Cn\Big(\frac{(\log\log n)^2}{\sqrt{n}}\Big)^Q \mathsf{E}\big[|X|^Q I\{|X| \le \sqrt{n}/\log^p n\}\big]. \end{aligned}$$

On the other hand, by Lemma 1.1.1 of Csörgő and Révész (1981),

$$\begin{aligned} &\mathsf{P}\Big(|\max_{0\le s\le 1} |W_n(sB_n) - W_n(\frac{[ns]}{n}B_n)| \ge \frac{1}{2}\sqrt{B_n}/(\log\log n)^2\Big) \\ =\ &\mathsf{P}\Big(|\max_{0\le s\le 1} |W_n(s) - W_n(\frac{[ns]}{n})| \ge \frac{1}{2}\sqrt{\frac{1}{n}}\frac{\sqrt{n}}{(\log\log n)^2}\Big) \\ \le\ &Cn\exp\Big\{-\frac{(\frac{1}{2}\sqrt{n}/(\log\log n)^2)^2}{3}\Big\} \le Cn\exp\Big\{-\frac{1}{12}n/(\log\log n)^4\Big\}. \end{aligned}$$

Let

$$p_n = \mathsf{P}\Big(\big|M_n^*/\sqrt{B_n} - \sup_{0\le s\le 1} |W_n(sB_n)|/\sqrt{B_n}\big| \ge 1/(\log\log n)^2\Big). \tag{3.8}$$

Then p_n satisfies (3.6), since $\{W_n(tB_n)/\sqrt{B_n}; t\ge 0\} \overset{\mathcal{D}}{=} \{W(t); t\ge 0\}$ for each n. And also,

$$p_n \le Cn\Big(\frac{(\log\log n)^2}{\sqrt{n}}\Big)^Q \mathsf{E}\big[|X|^Q I\{|X|\le \sqrt{n}/\log^p n\}\big] + Cn\exp\Big\{-\frac{1}{12}n/(\log\log n)^4\Big\}.$$

It follows that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(\log n)^a(\log\log n)^b}{n}p_n\\
\le\ & K_1 + C\sum_{n=1}^{\infty}\frac{(\log n)^a(\log\log n)^{b+2Q}}{n^{Q/2}}\mathsf{E}\big[|X|^Q I\{|X|\le \sqrt{n}/\log^p n\}\big]\\
\le\ & K_1 + C\sum_{n=1}^{\infty}\frac{(\log\log n)^{b+2Q}}{n(\log n)^{p(Q-2)-a}}\mathsf{E}X^2 \le K<\infty,
\end{aligned}$$

whenever $p(Q-2)-a>1$. So, (3.7) is satisfied.

Now, we turn to prove Propositions 3.1 and 3.2.

Proof of Proposition 3.1: Let $0<\delta<\frac{1}{2\sqrt{1+a}}$. Observe that, if n is large enough,

$$\begin{aligned}
&\mathsf{P}\Big\{M_n\le \epsilon\sqrt{\frac{\pi^2 B_n}{8\log\log n}}\Big\}\\
=\ & \mathsf{P}\Big\{M_n\le \epsilon\sqrt{\frac{\pi^2 B_n}{8\log\log n}}, \Delta_n \le \frac{\sqrt{n}}{(\log\log n)^2}\Big\}\\
&\quad +\mathsf{P}\Big\{M_n\le \epsilon\sqrt{\frac{\pi^2 B_n}{8\log\log n}}, \Delta_n > \frac{\sqrt{n}}{(\log\log n)^2}\Big\}\\
\le\ & \mathsf{P}\Big\{M_n^*\le \epsilon\sqrt{\frac{\pi^2 B_n}{8\log\log n}} + \frac{\sqrt{n}}{(\log\log n)^2}\Big\}\\
&\quad +\mathsf{P}\Big\{M_n\le \phi(n)\big(\frac{1}{\sqrt{1+a}}+\delta\big), \Delta_n > \frac{\sqrt{n}}{(\log\log n)^2}\Big\}\\
\le\ & \mathsf{P}\Big\{\sup_{0\le s\le 1}|W(s)|\le \epsilon\sqrt{\frac{\pi^2}{8\log\log n}} + \frac{3}{(\log\log n)^2}\Big\} + p_n + II_n
\end{aligned}$$

for all $\epsilon\in(1/\sqrt{1+a}-\delta, 1/\sqrt{1+a}+\delta)$, where II_n and p_n are defined in Lemmas 3.4 and

3.6, respectively. Similarly, if n is large enough,

$$\begin{aligned}
& \mathsf{P}\Big\{M_n \le \epsilon\sqrt{\frac{\pi^2 B_n}{8\log\log n}}\Big\} \\
\ge\ & \mathsf{P}\Big\{M_n \le \epsilon\sqrt{\frac{\pi^2 B_n}{8\log\log n}}, \Delta_n \le \frac{\sqrt{n}}{(\log\log n)^2}\Big\} \\
\ge\ & \mathsf{P}\Big\{M_n^* \le \epsilon\sqrt{\frac{\pi^2 B_n}{8\log\log n}} - \frac{\sqrt{n}}{(\log\log n)^2}, \Delta_n \le \frac{\sqrt{n}}{(\log\log n)^2}\Big\} \\
\ge\ & \mathsf{P}\Big\{M_n^* \le \sqrt{B_n}\big[\epsilon\sqrt{\frac{\pi^2}{8\log\log n}} - \frac{2}{(\log\log n)^2}\big]\Big\} \\
& \quad -\mathsf{P}\Big\{M_n^* \le \sqrt{B_n}\big[\epsilon\sqrt{\frac{\pi^2}{8\log\log n}} - \frac{2}{(\log\log n)^2}\big], \Delta_n > \frac{\sqrt{n}}{(\log\log n)^2}\Big\} \\
\ge\ & \mathsf{P}\Big\{\sup_{0\le s\le 1}|W(s)| \le \epsilon\sqrt{\frac{\pi^2}{8\log\log n}} - \frac{3}{(\log\log n)^2}\Big\} \\
& \quad -\mathsf{P}\Big\{M_n^* \le \phi(n)\big(\frac{1}{\sqrt{1+a}} + \delta\big), \Delta_n > \frac{\sqrt{n}}{(\log\log n)^2}\Big\} \\
\ge\ & \mathsf{P}\Big\{\sup_{0\le s\le 1}|W(s)| \le \epsilon\sqrt{\frac{\pi^2}{8\log\log n}} - \frac{3}{(\log\log n)^2}\Big\} - p_n - I_n
\end{aligned}$$

for all $\epsilon \in (1/\sqrt{1+a} - \delta, 1/\sqrt{1+a} + \delta)$, where I_n is defined in Lemma 3.3. Choosing $\delta > 0$ small enough and letting $q_n = p_n + I_n + II_n$ complete the proof by Lemmas 3.3, 3.4 and 3.6.

Proof of Proposition 3.2: Let $\{W_n(\cdot)\}$ be a sequence of standard Wiener processes being defined in the proof of Lemma 3.6, and let p_n be defined in (3.8). And set

$$q_n^* = \mathsf{P}\Big(\big|M_n/\sqrt{B_n} - \sup_{0\le s\le 1}|W_n(sB_n)|/\sqrt{B_n}\big| \ge 3/(\log\log n)^2\Big).$$

Then q_n^* satisfies (3.3), and

$$q_n^* \le \mathsf{P}\big(\Delta_n \ge \sqrt{n}/(\log\log n)^2\big) + p_n.$$

By Lemma 3.6,

$$\sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n\log n} p_n \le K_1(b,p) < \infty.$$

Also, following the lines in the proof of (3.5) we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n\log n} \mathsf{P}\big(\Delta_n \ge \sqrt{n}/(\log\log n)^2\big) \\
\le\ & \sum_{n\in\mathcal{L}} \frac{(\log\log n)^b}{n\log n} \cdot n\mathsf{P}\big(|X| > \sqrt{n}/(\log n)^p\big) + \sum_{n\notin\mathcal{L}} \frac{(\log\log n)^{b+2}}{n^{3/2}\log n}\beta_n \\
\le\ & \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{\log n} \mathsf{P}\big(|X| > \sqrt{n}/(\log n)^p\big) \\
& \quad + \sum_{n=1}^{\infty} \frac{(\log\log n)^{b+2}}{\sqrt{n}\log n} \mathsf{E}\big[|X| I\{|X| > \sqrt{n}/(\log n)^p\}\big] \\
\le\ & C\mathsf{E}\big[X^2(\log|X|)^{2p-1}(\log\log|X|)^b\big] + C\mathsf{E}\big[X^2(\log|X|)^{p-1}(\log\log|X|)^{b+2}\big] \\
\le\ & C\mathsf{E}X^2 < \infty.
\end{aligned}$$

So, q_n^* satisfies (3.4).

4 Proofs of the Theorems.

4.1 Proofs of the direct parts.

Without losing of generality, we assume that $\mathsf{E}X = 0$ and $\mathsf{E}X^2 = 1$.

Proof of the direct part of Theorem 1.1: Let $\delta > 0$ small enough and $\{q_n\}$ be such that (3.1) and (3.2) hold. Then

$$\lim_{\epsilon \nearrow 1/\sqrt{1+a}} \Big(\frac{1}{\sqrt{1+a}} - \epsilon\Big)^{b+1} \sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} q_n = 0,$$

by (3.2). Notice that $a_n(\epsilon) \to 0$. By (3.1), we have that for n large enough,

$$\begin{aligned}
& \mathsf{P}\Big\{\sup_{0\le s\le 1} |W(s)| \le \sqrt{\frac{\pi^2}{8\log\log n}}(\epsilon + a_n(\epsilon)) - \frac{3}{(\log\log n)^2}\Big\} - q_n \\
\le\ & \mathsf{P}\Big\{M_n \le \sqrt{\frac{\pi^2 B_n}{8\log\log n}}(\epsilon + a_n(\epsilon))\Big\} \\
\le\ & \mathsf{P}\Big\{\sup_{0\le s\le 1} |W(s)| \le \sqrt{\frac{\pi^2}{8\log\log n}}(\epsilon + a_n(\epsilon)) + \frac{3}{(\log\log n)^2}\Big\} + q_n, \\
& \quad \forall \epsilon \in \Big(\frac{1}{\sqrt{1+a}} - \delta/2, \frac{1}{\sqrt{1+a}} + \delta/2\Big).
\end{aligned}$$

On the other hand, by Proposition 2.1,

$$\lim_{\epsilon\nearrow 1/\sqrt{1+a}}\Big(\frac{1}{\sqrt{1+a}}-\epsilon\Big)^{b+1}\sum_{n=1}^{\infty}\frac{(\log n)^a(\log\log n)^b}{n}$$
$$\cdot\mathsf{P}\Big\{\sup_{0\le s\le 1}|W(s)|\le\sqrt{\frac{\pi^2}{8\log\log n}}(\epsilon+a_n(\epsilon))\pm\frac{3}{(\log\log n)^2}\Big\}$$
$$=\frac{4}{\pi}\Big(\frac{1}{2(1+a)^{3/2}}\Big)^{b+1}\Gamma(b+1)\exp\Big\{2(1+a)^{3/2}\tau\Big\}.$$

It follows that

$$\lim_{\epsilon\nearrow 1/\sqrt{1+a}}\Big(\frac{1}{\sqrt{1+a}}-\epsilon\Big)^{b+1}\sum_{n=1}^{\infty}\frac{(\log n)^a(\log\log n)^b}{n}\mathsf{P}\Big\{M_n\le\sqrt{\frac{\pi^2B_n}{8\log\log n}}(\epsilon+a_n(\epsilon))\Big\}$$
$$=\frac{4}{\pi}\Big(\frac{1}{2(1+a)^{3/2}}\Big)^{b+1}\Gamma(b+1)\exp\Big\{2(1+a)^{3/2}\tau\Big\}. \tag{4.1}$$

Finally, noticing the condition (1.3), we have

$$0\le n-B_n\le 2n\mathsf{E}[X^2I\{|X|\ge\sqrt{n}/\log^p n\}]=o(n(\log\log n)^{-1}).$$

Let $a_n'(\epsilon)=\sqrt{n/B_n}(\epsilon+a_n(\epsilon))-\epsilon$. Then

$$\mathsf{P}\Big\{M_n\le\phi(n)(\epsilon+a_n(\epsilon))\Big\}=\mathsf{P}\Big\{M_n\le\sqrt{\frac{\pi^2B_n}{8\log\log n}}(\epsilon+a_n'(\epsilon))\Big\},$$

and,

$$a_n'(\epsilon)\log\log n=\epsilon\frac{(n-B_n)\log\log n}{\sqrt{B_n}(\sqrt{n}+\sqrt{B_n})}+\sqrt{\frac{n}{B_n}}a_n(\epsilon)\log\log n\to\tau$$

as $n\to\infty$ and $\epsilon\nearrow 1/\sqrt{1+a}$. Now, (1.4) follows from (4.1).

Proof of the direct part of Theorem 1.2: Noticing $B_n\sim n$ and Proposition 3.2, for any $0<\delta<1$ we have for n large enough and all $\epsilon>10^3$,

$$\begin{aligned}
&\mathsf{P}\Big\{\sup_{0\le s\le 1}|W(s)|\le(1-\delta)\epsilon\sqrt{\frac{\pi^2}{8\log\log n}}\Big\}-q_n^*\\
\le\ &\mathsf{P}\Big\{\sup_{0\le s\le 1}|W(s)|\le\epsilon\sqrt{\frac{\pi^2}{8\log\log n}}-3/(\log\log n)^2\Big\}-q_n^*\\
\le\ &\mathsf{P}\Big\{M_n\le\epsilon\sqrt{\frac{\pi^2B_n}{8\log\log n}}\Big\}\\
\le\ &\mathsf{P}\Big\{M_n\le\phi(n)\epsilon\Big\}\le\mathsf{P}\Big\{M_n\le(1+\frac{\delta}{2})\epsilon\sqrt{\frac{\pi^2B_n}{8\log\log n}}\Big\}\\
\le\ &\mathsf{P}\Big\{\sup_{0\le s\le 1}|W(s)|\le(1+\frac{\delta}{2})\epsilon\sqrt{\frac{\pi^2}{8\log\log n}}+3/(\log\log n)^2\Big\}+q_n^*\\
\le\ &\mathsf{P}\Big\{\sup_{0\le s\le 1}|W(s)|\le(1+\delta)\epsilon\sqrt{\frac{\pi^2}{8\log\log n}}\Big\}+q_n^*.
\end{aligned}$$

So, by Propositions 2.2 and 3.2,

$$
\begin{aligned}
& (1-\delta)^{2(b+1)} \frac{4}{\pi} \Gamma(b+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+3}} \\
= \;& \lim_{\epsilon \nearrow \infty} \epsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n \log n} \mathsf{P}\Big\{ \sup_{0 \le s \le 1} |W(s)| \le (1-\delta)\epsilon \sqrt{\frac{\pi^2}{8 \log\log n}} \Big\} \\
\le \;& \liminf_{\epsilon \nearrow \infty} \epsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n \log n} \mathsf{P}\Big\{ M_n \le \phi(n)\epsilon \Big\} \\
\le \;& \limsup_{\epsilon \nearrow \infty} \epsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n \log n} \mathsf{P}\Big\{ M_n \le \phi(n)\epsilon \Big\} \\
\le \;& \lim_{\epsilon \nearrow \infty} \epsilon^{-2(b+1)} \sum_{n=1}^{\infty} \frac{(\log\log n)^b}{n \log n} \mathsf{P}\Big\{ \sup_{0 \le s \le 1} |W(s)| \le (1+\delta)\epsilon \sqrt{\frac{\pi^2}{8 \log\log n}} \Big\} \\
= \;& (1+\delta)^{2(b+1)} \frac{4}{\pi} \Gamma(b+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2b+3}}.
\end{aligned}
$$

Letting $\delta \to 0$ completes the proof.

Proof of the direct part of Theorem 1.3: Choose $\delta > 0$ small enough and $\{q_n\}$ for (3.1) and (3.2) to hold. By a standard argument (see Feller 1945), we can assume that

$$
\frac{1}{\sqrt{\log\log n}} \Big(\frac{1}{\sqrt{1+a}} - \frac{\delta}{2} \Big) \le \psi(n) \le \frac{1}{\sqrt{\log\log n}} \Big(\frac{1}{\sqrt{1+a}} + \frac{\delta}{2} \Big).
$$

Then for n large enough,

$$
\begin{aligned}
& \mathsf{P}\Big\{ \sup_{0 \le s \le 1} |W(s)| \le \sqrt{\pi^2/8}\big[\psi(n) - 5/(\log\log n)^2\big] \Big\} - q_n \\
\le \;& \mathsf{P}\Big\{ \sup_{0 \le s \le 1} |W(s)| \le \sqrt{\pi^2/8}\psi(n) - 3/(\log\log n)^2 \Big\} - q_n \\
\le \;& \mathsf{P}\Big\{ M_n \le \sqrt{\pi^2 B_n/8}\psi(n) \Big\} \\
\le \;& \mathsf{P}\Big\{ \sup_{0 \le s \le 1} |W(s)| \le \sqrt{\pi^2/8}\psi(n) + 3/(\log\log n)^2 \Big\} + q_n \\
\le \;& \mathsf{P}\Big\{ \sup_{0 \le s \le 1} |W(s)| \le \sqrt{\pi^2/8}\big[\psi(n) + 5/(\log\log n)^2\big] \Big\} + q_n.
\end{aligned}
$$

Notice that $J_{ab}\big(\psi(n) \pm 5/(\log\log n)^2\big) < \infty$ if and only if $J_{ab}(\psi(n)) < \infty$. So, by Propositions 2.3 and 3.1 we have

$$
\sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} \mathsf{P}\big\{ M_n \le \sqrt{\pi^2 B_n/8}\psi(n) \big\} < \infty \text{ or } = \infty
$$
$$
\text{according as } J_{ab}(\psi) < \infty \text{ or } = \infty. \tag{4.2}
$$

Finally, if condition (1.8) is satisfied, then

$$
0 \le n - B_n = O(n/\log\log n),
$$

which implies that

$$\sqrt{n}\psi(n) = \sqrt{B_n}\psi(n)\sqrt{1+O(1/\log\log n)}.$$

Notice that $J_{ab}\big(\psi(n)\sqrt{1\pm O(1/\log\log n)}\big)<\infty$ if and only if $J_{ab}(\psi(n))<\infty$. (1.9) follows from (4.2).

4.2 Proofs of the converse parts.

Now, we turn to prove the converse parts of Theorem 1.1-1.3. The proof is organized as follows. First, we show that $\mathsf{E}X^2=\infty$ is impossible. Secondly, we show that $\mathsf{E}X\neq 0$ is impossible if $\mathsf{E}X^2<\infty$. Thirdly, we show that $\mathsf{E}X^2\neq\sigma^2$ is impossible if $\mathsf{E}X=0$ and $\mathsf{E}X^2<\infty$. At last, we show (1.5) and (1.10).

From Esseen (1968) (see also Petrov 1995) it is easy to see that for all $m\geq 1$,

$$\mathsf{P}(|S_m|\leq 2\sqrt{m})\leq K\Big(\int_{-2\sqrt{m}}^{2\sqrt{m}} d\widetilde{F}(x)\Big)^{-1/2},$$

where $\widetilde{F}(x)$ is the distribution function of the symmetrized X, and K is an absolute constant. So, if $\mathsf{E}X^2=\infty$, then for any $M>2$ we can choose $m_0\geq 9$ large enough such that

$$\mathsf{P}(|S_m|\leq 2\sqrt{m})\leq e^{-2M},\quad m\geq m_0.$$

For $\epsilon>0$, we let $m=[\epsilon^2 n/\log\log n]$, and $N=[n/m]$, then for all $n\geq m_0^2$ and $\epsilon\geq 1$,

$$\begin{aligned}\mathsf{P}\big(M_n\leq\epsilon(n/\log\log n)^{1/2}\big) &\leq \mathsf{P}\big(|S_{km}-S_{(k-1)m}|\leq 2\sqrt{m}, k=1,\ldots,N\big)\\ &\leq e^{-2MN}\leq\exp\Big\{-M\frac{\log\log n}{\epsilon^2}\Big\}.\end{aligned}\tag{4.3}$$

By this inequality, for any a, b and $0<\epsilon_1<\epsilon_2<\infty$ there exists a constant $C=C(a,b,\epsilon_1,\epsilon_2)$ for which

$$\sup_{\epsilon\in(\epsilon_1,\epsilon_2)}\sum_{n=1}^{\infty}\frac{(\log n)^a(\log\log n)^b}{n}\mathsf{P}\big(M_n\leq\epsilon(n/\log\log n)^{1/2}\big)\leq C<\infty,$$

which implies that (1.4) and (1.9) can not hold. Also, by (4.3) and (2.3), for any $b>-1$ we have

$$\begin{aligned}&\limsup_{\epsilon\nearrow\infty}\epsilon^{-2(b+1)}\sum_{n=1}^{\infty}\frac{(\log\log n)^b}{n\log n}\mathsf{P}\big(M_n\leq\epsilon(n/\log\log n)^{1/2}\big)\\ &\leq\lim_{\epsilon\nearrow\infty}\epsilon^{-2(b+1)}\sum_{n=1}^{\infty}\frac{(\log\log n)^b}{n\log n}\exp\Big\{-M\frac{\log\log n}{\epsilon^2}\Big\}\\ &=M^{-(b+1)}\Gamma(b+1)\to 0\quad\text{as } M\to\infty,\end{aligned}$$

which implies that (1.6) can not hold.

If $\mathsf{E}X^2 < \infty$ and $\mathsf{E}X = \mu \neq 0$, then

$$\mathsf{P}(M_n \le \frac{n|\mu|}{2}) \le \mathsf{P}(|S_n| \le \frac{n|\mu|}{2}) \le \mathsf{P}(|S_n - n\mu| \ge \frac{n|\mu|}{2}) \le \frac{4\mathsf{E}X^2}{\mu^2 n^2}.$$

It follows that

$$\sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} \mathsf{P}(M_n \le \frac{n|\mu|}{2}) \le \frac{4\mathsf{E}X^2}{\mu^2} \sum_{n=1}^{\infty} \frac{1}{n^3} \le \frac{8\mathsf{E}X^2}{\mu^2}.$$

It follows that for any $\epsilon > 0$,

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{(\log n)^a (\log\log n)^b}{n} \mathsf{P}(M_n \le \epsilon(n/\log\log n)^{1/2}) \\
\le & \sum_{n<(2\epsilon/|\mu|)^2} \frac{(\log n)^a (\log\log n)^b}{n} \mathsf{P}(M_n \le \epsilon(n/\log\log n)^{1/2}) \\
& + \sum_{n\ge(2\epsilon/|\mu|)^2} \frac{(\log n)^a (\log\log n)^b}{n} \mathsf{P}(M_n \le \epsilon(n/\log\log n)^{1/2}) \\
\le & \sum_{n\ge(2\epsilon/|\mu|)^2} \frac{(\log n)^a (\log\log n)^b}{n} \mathsf{P}(M_n \le \frac{n|\mu|}{2}) + \sum_{n<(2\epsilon/|\mu|)^2} \frac{(\log n)^{|a|+|b|}}{n} \\
\le & \frac{8\mathsf{E}X^2}{\mu^2} + 4\Big(\log(\frac{2\epsilon}{|\mu|})^4\Big)^{|a|+|b|+1},
\end{aligned}$$

which implies none of (1.4), (1.6) and (1.9) can hold.

Suppose that $\mathsf{E}X = 0$ and $\mathsf{E}X^2 < \infty$, and that (1.4), (1.6) and (1.9) hold for some positive constant σ. By the direct part of Theorem 1.2, (1.6) shall hold with $\mathsf{E}X^2$ taking the place of σ^2. This is obvious a contradiction if $\mathsf{E}X^2 \neq \sigma^2$. Notice that (4.1) and (4.3) hold whenever $\mathsf{E}X = 0$ and $\mathsf{E}X^2 < \infty$. However, if $\mathsf{E}X^2 \neq \sigma^2$, then (4.1) and (4.2) are contradictory to (1.4) and (1.9) respectively, since $B_n \sim n\mathsf{E}X^2$.

Finally, we show (1.5) and (1.10). Suppose that (1.5) fails. Without losing of generality, we can assume that $\sigma^{-2}\mathsf{E}[X^2 I\{|X| \ge \sqrt{n}/(\log n)^p\}] \ge \tau_0/\log\log n$ for some $\tau_0 > 0$ and all $n \ge 1$. Then $n\sigma^2 - B_n \ge n\mathsf{E}[X^2 I\{|X| \ge \sqrt{n}/(\log n)^p\}] \ge n\sigma^2\tau_0/\log\log n$. Let $a_n'(\epsilon) = \sqrt{1+\tau_0/\log\log n}\big(\epsilon + a_n(\epsilon)\big) - \epsilon$. Then

$$a_n'(\epsilon)\log\log n \to \tau + \tau_0/(2\sqrt{1+a}),$$

and

$$\mathsf{P}\Big\{M_n \le \sigma\phi(n)\big(\epsilon + a_n(\epsilon)\big)\Big\} \ge \mathsf{P}\Big\{M_n \le \sqrt{\frac{\pi^2 B_n}{8\log\log n}}\big(\epsilon + a_n'(\epsilon)\big)\Big\}.$$

It follows that (1.4) is contradictory to (4.1).

Now, suppose that (1.5) fails. Let $d_n = (\log\log n)\mathsf{E}[X^2 I\{|X| \ge \sqrt{n}/(\log n)^p\}]/\sigma^2$. Then $d_n/\log\log n$ is non-increasing in n and $d_n \to \infty$. So, one can find a non-decreasing sequence $\{b_n\}$ of positive numbers for which $e \le b_n \to \infty$,

$$\sum_{n=1}^{\infty} \frac{1}{n(\log n)(\log\log n)b_n} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{d_n}{n(\log n)(\log\log n)b_n} = \infty.$$

Define

$$\psi(n) = 1/\sqrt{(1+a)\log\log n + (1+b)\log\log\log n + \log b_n}$$

and

$$\widetilde{\psi}(n) = \psi(n)/\sqrt{1 - d_n/(2\log\log n)}.$$

Then

$$J_{ab}(\psi) = \sum_{n=1}^{\infty} \frac{1}{n(\log n)(\log\log n)b_n} < \infty$$

and

$$\begin{aligned} J_{ab}(\widetilde{\psi}) &= \sum_{n=1}^{\infty} \frac{(\log n)^a(\log\log n)^b}{n} \exp\Big\{-\frac{1-d_n/(2\log\log n)}{\psi^2(n)}\Big\} \\ &\ge \sum_{n=1}^{\infty} \frac{(\log n)^a(\log\log n)^b}{n} \exp\Big\{-\frac{1}{\psi^2(n)}\Big\} \exp\Big\{\frac{1+a}{2} d_n\Big\} \\ &\ge c\sum_{n=1}^{\infty} \frac{d_n}{n(\log n)(\log\log n)b_n} = \infty. \end{aligned}$$

However,

$$\mathsf{P}\Big\{M_n \le \sqrt{\frac{\pi^2\sigma^2 n}{8}}\psi(n)\Big\} = \mathsf{P}\Big\{M_n \le \sqrt{\frac{\pi^2 B_n}{8}}\psi(n)\sqrt{\frac{n\sigma^2}{B_n}}\Big\} \ge \mathsf{P}\Big\{M_n \le \sqrt{\frac{\pi^2 B_n}{8}}\widetilde{\psi}(n)\Big\},$$

since $n\sigma^2 - B_n \ge n\sigma^2 d_n$. It follows that (1.9) is contradictory to (4.2). The proof is now completed.

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