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On the fractal nature of increments of ℓ^p-valued Gaussian processes[1]

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Abstract

We prove that the set of points where exceptional oscillation of ℓ^p-valued Gaussian processes occur infinitely often is a random fractal, and evaluate its Hausdorff dimension. Applications to fractional Brownian motions and Ornstein–Uhlenbeck processes are also discussed. © 1997 Elsevier Science B.V.

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1. Introduction and results

Let $\{W(t);\ t \geqslant 0\}$ be a standard Wiener process. The following result about the Lévy's moduli of continuity is well known.

Theorem A. *We have*

$$\lim_{h\to 0} \sup_{0\leqslant t\leqslant 1-h} \frac{|W(t+h)-W(t)|}{(2h\log h^{-1})^{1/2}} = 1 \quad a.s., \tag{1.1}$$

and

$$\lim_{h\to 0} \sup_{0\leqslant t\leqslant 1-h} \sup_{0\leqslant s\leqslant h} \frac{|W(t+s)-W(t)|}{(2h\log h^{-1})^{1/2}} = 1 \quad a.s. \tag{1.2}$$

Orey and Taylor (1974) studied the fractal nature of the set

$$B(\alpha) =: \left\{ t\in[0,1], \limsup_{h\to 0} \frac{|W(t+h)-W(t)|}{(2h\log h^{-1})^{1/2}} \geqslant \alpha \right\} \quad (0\leqslant\alpha\leqslant 1).$$

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They showed that for each $0\leqslant\alpha\leqslant 1$, $B(\alpha)$ is a random fractal. Their main result, stated in Theorem B below, provides the Hausdorff dimension of this set. Recall (see, e.g., Falconer, 1985; Taylor, 1986) that the Hausdorff dimension $\dim B$ of a subset B of [0,1] may be defined by setting

$$\dim B = \inf\{c>0;\ s^c\text{-mes}\, B=0\}, \tag{1.3}$$

where for any continuous increasing function $\phi:[0,1]\to[0,\infty]$ with $\phi(0)=0$, the ϕ-measure of B is defined by

$$\phi\text{-mes}\, B = \lim_{h\to 0}\left(\inf\left\{\sum_i \phi(|I_i|);\ B\subset\bigcup_i I_i,\ |I_i|\leqslant h\right\}\right). \tag{1.4}$$

Here, the I_i constitute an h-cover of B (i.e., a collection of intervals with lengths not exceeding h, whose union includes B), we set $|I|$ for the Lebesgue measure of I and infimum in (1.4) is taken over all h-covers of B.

Theorem B. *For any* $\alpha\in[0,1]$ *we have almost surely*

$$\dim B(\alpha) = 1-\alpha^2. \tag{1.5}$$

Recently, it has been shown that a lot of Gaussian processes have moduli of continuity similar to (1.1) and (1.2). For example, Csörgő and Shao (1993) and Csáki et al. (1992, 1995) studied the increments of ℓ^p-valued Gaussian processes, one of their results, stated in Theorem C below, provides the moduli of continuity for ℓ^p-valued Gaussian processes. But up to now, to the best of our knowledge, little is known about the fractal nature of the type (1.5) for any Gaussian processes except the Wiener process. Recently, Deheuvels and Mason (1994, 1995) studied the fractal nature for empirical increments and processes with independent increments. But their methods are not effective for studying Gaussian processes with dependent increments.

Let $\{Y(t);\ -\infty<t<\infty\}=\{X_k(t);\ -\infty<t<\infty\}_{k=1}^{\infty}$ be a sequence of independent Gaussian processes with $\boldsymbol{E}X_k(t)=0$ and stationary increments $\sigma_k^2(h)=\boldsymbol{E}(X_k(t+h)-X_k(t))^2$, where, and throughout this paper, $\sigma_k(h)$ is assumed to be a non-decreasing continuous function for each $k\geqslant 1$. Put

$$\sigma(p,h)=\left(\sum_{k=1}^{\infty}\sigma_k^p(h)\right)^{1/p}, \tag{1.6}$$

$$\sigma^*(h)=\max_{k\geqslant 1}\sigma_k(h), \tag{1.7}$$

$$\tilde\sigma(p,h)=\begin{cases}\sigma(\frac{2p}{(2-p)},h) & \text{if } 1\leqslant p<2,\\ \sigma^*(h) & \text{if } p\geqslant 2,\end{cases} \tag{1.8}$$

$$\delta_p^p=\boldsymbol{E}|\mathrm{N}(0,1)|^p=\frac{2^{p/2}}{\sqrt{\pi}}\int_0^{\infty}x^{(p-1)/2}\mathrm{e}^{-x}\,\mathrm{d}x,\quad p\geqslant 1. \tag{1.9}$$

A function $f(x)$ on (a,b) is called quasi-increasing on (a,b) if there exists a positive c such that

$$f(x) \leqslant f(y) \quad \text{for all } a<x<y<b.$$

Theorem C. (I). *Assume that $\tilde{\sigma}(p,h)/h^{\Lambda}$ is quasi-increasing on $(0,\Lambda)$ for some $\Lambda>0$. Moreover, suppose that*

$$\sigma(p,h) = o(\tilde{\sigma}(p,h)(\log h^{-1})^{1/2}) \quad as\ h \to 0, \tag{1.10}$$

$$\limsup_{h\to 0} \max_{h^{-\varepsilon}\leqslant j\leqslant h^{-1}} \max_{k\geqslant 1} \frac{E(X_k(h)-X_k(0))(X_k((j+1)h)-X_k(jh))}{\sigma_k^2(h)} \leqslant 0 \tag{1.11}$$

for each $\varepsilon>0$. Then we have

$$\lim_{h\to 0} \sup_{0\leqslant t\leqslant 1} \sup_{0\leqslant s\leqslant h} \frac{\|Y(t+s)-Y(t)\|_{\ell^p}}{\tilde{\sigma}(p,h)(2\log h^{-1})^{1/2}} = 1 \quad a.s. \tag{1.12}$$

(II). *Assume that $\sigma(p,h)/h^{\Lambda}$ is quasi-increasing on $(0,\Lambda)$ for some $\Lambda>0$. Moreover, suppose that*

$$\tilde{\sigma}(p,h)(\log h^{-1})^{1/2} = o(\sigma(p,h)) \quad as\ h \to 0. \tag{1.10'}$$

Then we have

$$\lim_{h\to 0} \sup_{0\leqslant t\leqslant 1} \sup_{0\leqslant s\leqslant h} \frac{\|Y(t+s)-Y(t)\|_{\ell^p}}{\delta_p \sigma(p,h)} = 1 \quad a.s. \tag{1.12'}$$

If the conditions in (II) of Theorem C are satisfied, then we have

$$\begin{aligned}
&\limsup_{h\to 0} \inf_{0\leqslant t\leqslant 1} \frac{\|Y(t+h)-Y(t)\|_{\ell^p}}{\delta_p\sigma(p,h)} \\
&\quad \geqslant \limsup_{h\to 0} \min_{0\leqslant n\leqslant h^{-2}} \frac{\|Y(nh^2+h)-Y(nh^2)\|_{\ell^p}}{\delta_p\sigma(p,h)} \\
&\qquad -2\limsup_{h\to 0} \sup_{0\leqslant t\leqslant 2} \sup_{0\leqslant s\leqslant h^2} \frac{\|Y(t+s)-Y(t)\|_{\ell^p}}{\delta_p\sigma(p,h^2)} \cdot \frac{\sigma(p,h^2)}{\sigma(p,h)} \\
&\quad = \limsup_{h\to 0} \min_{0\leqslant n\leqslant h^{-2}} \frac{\|Y(nh^2+h)-Y(nh^2)\|_{\ell^p}}{\delta_p\sigma(p,h)}.
\end{aligned} \tag{1.13}$$

Similar to the proof of (3.27) of Csörgő and Shao (1993), for any $1<\theta<2$ we have that for h sufficiently small,

$$\begin{aligned}
&P\left\{ \min_{0\leqslant n\leqslant h^{-2}} \frac{\|Y(nh^2+h)-Y(nh^2)\|_{\ell^p}}{\delta_p\sigma(p,h)} \leqslant 2-\theta \right\} \\
&\leqslant 2(h^{-2}+1)\exp\left(-\frac{(\theta-1)^2\delta_p^2\sigma^2(p,h)}{8\,\tilde{\sigma}^2(p,h)} \right) \\
&\leqslant 4h^{-2}\exp\left(-4\log\frac{1}{h} \right) = 4h^2 \to 0 \quad \text{as } h\to 0,
\end{aligned}$$

which together with (1.13) implies that

$$\limsup_{h\to 0} \inf_{0\leqslant t\leqslant 1} \frac{\|Y(t+s)-Y(t)\|_{\ell^p}}{\delta_p\sigma(p,h)} \geqslant 1 \quad \text{a.s.}$$

Hence, if we define a random set similar to $B(\alpha)$, by

$$E(\alpha)=\left\{t\in[0,1];\ \limsup_{h\to 0} \frac{\|Y(t+h)-Y(t)\|_{\ell^p}}{\delta_p\sigma(p,h)} \geqslant \alpha\right\}, \quad 0\leqslant\alpha\leqslant 1,$$

then $E(\alpha)=[0,1]$ a.s. for any $0\leqslant\alpha\leqslant 1$. So, in this case there is nothing for us to consider on the fractal nature of $E(\alpha)$.

Now, we suppose the conditions in (I) of Theorem C are satisfied and define a random set similar to $B(\alpha)$, by

$$E(\alpha)=\left\{t\in[0,1];\ \limsup_{h\to 0} \frac{\|Y(t+h)-Y(t)\|_{\ell^p}}{\tilde\sigma(p,h)(2\log h^{-1})^{1/2}} \geqslant \alpha\right\}, \quad 0\leqslant\alpha\leqslant 1. \tag{1.14}$$

The purpose of this paper is to establish the following theorem, in the spirit of Theorem B, where we prove that $E(\alpha)$ is a random fractal, and evaluate its Hausdorff dimension. Applications to fractional Brownians and Ornstein–Uhlenbeck processes are discussed in Section 3.

Theorem 1.1. *Assume that $\tilde\sigma(p,h)/h^{\Lambda}$ is quasi-increasing on $(0,\Lambda)$ for some $\Lambda>0$. Moreover, suppose that* (1.10) *holds and*

$$\limsup_{h\to 0} \max_{h^{-\varepsilon}\leqslant j\leqslant h^{-1}} \max_{k\geqslant 1} \frac{\mathbf{E}(X_k(h)-X_k(0))(X_k((j+1)h)-X_k(jh))}{(\log h^{-1})^{-2}\sigma_k^2(h)} \leqslant 0 \tag{1.15}$$

for each $\varepsilon>0$. Then for any $\alpha\in[0,1]$, we have almost surely

$$\dim E(\alpha)=1-\alpha^2. \tag{1.16}$$

Also, for the random set

$$E^*(\alpha)=\left\{t\in[0,1];\ \limsup_{h\to 0} \frac{\|Y(t+h)-Y(t)\|_{\ell^p}}{\tilde\sigma(p,h)(2\log h^{-1})^{1/2}} = \alpha\right\} \quad 0\leqslant\alpha\leqslant 1 \tag{1.17}$$

we have the following results.

Theorem 1.2. *Assume that $\tilde\sigma(p,h)/h^{\Lambda}$ is quasi-increasing on $(0,\Lambda)$ for some $\Lambda>0$. Moreover, suppose that* (1.10) *holds and*

$$\limsup_{h\to 0} \max_{(\log h^{-1})^P\leqslant j\leqslant h^{-1}} \max_{k\geqslant 1} \frac{\mathbf{E}(X_k(h)-X_k(0))(X_k((j+1)h)-X_k(jh))}{(\log h^{-1})^{-2}\sigma_k^2(h)} \leqslant 0 \tag{1.18}$$

for some $P>0$. Then for any $\alpha\in[0,1]$, we have almost surely

$$\dim E^*(\alpha)=1-\alpha^2. \tag{1.19}$$

2. Proof of the results

First, we state three lemmas required in the proof. The first one of these is a version of Lemma 2.2 of Orey and Taylor (1974).

Lemma 2.1. *Suppose* $\phi:[0,1]\to[0,\infty)$ *is a continuous function with* $\phi(0)=0$. *Let* $K\subset[0,1]$ *be such that* $K=\bigcap_{m=1}^{\infty}E_m$, *where* $E_1\supset\cdots\supset E_m\cdots$ *for* $m=1,2,\ldots$, *and* $E_m=\bigcup_{k=1}^{M_m}I_{m,k}$ *with* $\{I_{m,k}: 1\leqslant k\leqslant M_m\}$ *being, for each* $m\geqslant 1$, *a collection of disjoint closed subintervals of* $[0,1]$. *Then, if there exist two constants* $\Delta>0$ *and* $d>0$ *such that, for every interval* $I\subset[0,1]$ *with* $|I|\leqslant\Delta$ *there is a constant* $m(I)$ *such that for all* $m\geqslant m(I)$,

$$M_m(I)=:\#\{I_{m,k}\subset I;\ 1\leqslant k\leqslant M_m\}\leqslant d\phi(|I|)M_m, \tag{2.1}$$

we have ϕ-mes$(K)>0$.

Lemma 2.2. *Assume that* $\tilde{\sigma}(p,h)/h^{\Lambda}$ *is quasi-increasing on* $(0,\Lambda)$ *for some* $\Lambda>0$ *and* (1.10) *holds. Then, for any* $0<\alpha_2<\alpha_1$, *there exists* $\delta=\delta(\alpha_1,\alpha_2)$ *such that*

$$P\left(\sup_{a\leqslant s<t\leqslant a+h}\|Y(t+s)-Y(s)\|_{\ell^p}>\alpha_1\tilde{\sigma}(p,h)(2\log h^{-1})^{1/2}\right)<h^{\alpha_2}$$

for all $a\geqslant 0$ *and* $0<h<\delta$.

Proof. For any fixed $\varepsilon>0$, put $\sigma_*(p,h)=\varepsilon\sup_{0\leqslant s\leqslant h}\tilde{\sigma}(p,h)(\log s^{-1})^{1/2}$, $0<h\leqslant 1$. By Lemma 2.2 of Csáki et al. (1992), Remark 3.1 and (3.30) of Csörgő and Shao (1993), there exist $C=C(\varepsilon,\Lambda)$, $h_0=h_0(\varepsilon,\Lambda)$ and a constant c_0 independent of ε such that

$$\begin{aligned}
&P\left(\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant h}\|Y(t+s)-Y(t)\|_{\ell^p}\geqslant x\tilde{\sigma}(p,h)+(1+\varepsilon)\varepsilon c_0\tilde{\sigma}(p,h)(\log h^{-1})^{1/2}\right)\\
&\leqslant P\left(\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant h}\|Y(t+s)-Y(t)\|_{\ell^p}\geqslant x\tilde{\sigma}(p,h)+(1+\varepsilon)\sigma_*(p,h)\right)\\
&\leqslant C\left(\frac{T}{h}+1\right)\exp\left(-\frac{x^2}{2(1+\varepsilon)}\right)
\end{aligned}$$

for every $x\geqslant 1$, $T\geqslant 0$ and $0<h\leqslant h_0$. And then the proof is completed immediately. □

Lemma 2.3. *Suppose* $\{\xi_i;\ i=1,\ldots,n\}$ *are mean zero Gaussian variables with* $\boldsymbol{E}\xi_i^2=1$ *and* $\boldsymbol{E}\xi_i\xi_j\leqslant\beta^2$ $(j\neq i)$, *where* $0\leqslant\beta<1$. *Then for any* $\varepsilon>0$, $\lambda>0$, $0<\theta\leqslant\frac{1}{2}$ *and all* t, *we have*

$$\begin{aligned}
&P\left(\sum_{i=1}^{n}I\{\xi_i>t\}-np_0\geqslant\lambda np_0\right)\\
&\leqslant\exp\{-\theta n((\lambda+1)p_0-(1+\theta)p_1)\}+2nP\left(\mathrm{N}(0,1)>\frac{\varepsilon}{\beta}\right).
\end{aligned} \tag{2.2}$$

$$P\left(np_0-\sum_{i=1}^n I\{\xi_i>t\}\geqslant\lambda np_0\right)$$
$$\leqslant\exp\{-\theta n((1-\theta)p_2-(1-\lambda)p_0)\}+2nP\left(\mathrm{N}(0,1)>\frac{\varepsilon}{\beta}\right), \tag{2.3}$$

where $p_0=P(\mathrm{N}(0,1)>t)$, $p_1=P(\mathrm{N}(0,1)>(t-\varepsilon)/(1-\beta^2)^{1/2})$ *and* $p_2=P(\mathrm{N}(0,1)>(t+\varepsilon)/(1-\beta^2)^{1/2})$.

Proof. Let $\{\tau,\eta_i;\ i=1,\dots,n\}$ are independent mean zero Gaussian variables with $\boldsymbol{E}\tau^2=\beta^2$ and $\boldsymbol{E}\eta_i^2=1-\beta^2$. Then $\boldsymbol{E}(\tau+\eta_i)^2=\boldsymbol{E}\xi_i^2=1$ and $\boldsymbol{E}\xi_i\xi_j\leqslant\boldsymbol{E}(\tau+\eta_i)(\tau+\eta_j)=\beta^2$ $(i\neq j)$.

Define

$$f(x)=\begin{cases}\mathrm{e}^x & \text{for } 0\leqslant x\leqslant m,\\ \mathrm{e}^m(x-m+1) & \text{for } x\geqslant m.\end{cases}$$

It is easy to see that $f(x)\leqslant\mathrm{e}^x$ for $x\geqslant 0$, $f(x)\leqslant n\mathrm{e}^m$ for $0\leqslant x\leqslant n/2$ and $f(x)$ $(x\geqslant 0)$ is an increasing convex function. It follows that

$$g(x_1,\dots,x_n)=:f\left(\theta\sum_{i=1}^n I\{x_i>t\}\right)\leqslant\mathrm{e}^{\theta\sum_{i=1}^n I\{x_i>t\}},\quad g(x_1,\dots,x_n)\leqslant n\mathrm{e}^m$$

and g is a function on $\mathbb{R}^n$ such that its second derivatives in the sense of distribution satisfy

$$D_{ij}g=\theta^2\frac{\mathrm{d}^2f}{\mathrm{d}x^2}\frac{\mathrm{d}}{\mathrm{d}x_i}I\{x_i>t\}\frac{\mathrm{d}}{\mathrm{d}x_j}I\{x_j>t\}\geqslant 0\quad(i\neq j).$$

By the well-known comparison property (cf. Theorem 3.11 of Ledoux and Talagrand, 1991, p. 74), we have

$$\boldsymbol{E}g(\xi_1,\dots,\xi_n)\leqslant\boldsymbol{E}g(\tau+\eta_1,\dots,\tau+\eta_n).$$

Choose $m=\theta(\lambda np_0+np_0)$, we conclude that

$$\begin{aligned}P\left(\sum_{i=1}^n I\{\xi_i>t\}-np_0\geqslant\lambda np_0\right)&=P\left(f\left(\theta\sum_{i=1}^n I\{\xi_i>t\}\right)\geqslant f(m)\right)\\&=P(g(\xi_1,\dots,\xi_m)\geqslant\mathrm{e}^m)\\&=P(g(\xi_1,\dots,\xi_m)\geqslant\mathrm{e}^{\theta(\lambda+1)np_0})\\&\leqslant\mathrm{e}^{-\theta(\lambda+1)np_0}\boldsymbol{E}g(\xi_1,\dots,\xi_m)\\&\leqslant\mathrm{e}^{-\theta(\lambda+1)np_0}\boldsymbol{E}g(\tau+\eta_1,\dots,\tau+\eta_m)\\&\leqslant\mathrm{e}^{-\theta(\lambda+1)np_0}\{\boldsymbol{E}\mathrm{e}^{\theta\sum_{i=1}^n I\{\tau+\eta_i>t\}}I\{|\tau|\leqslant\varepsilon\}\\&\quad+n\mathrm{e}^{\theta(\lambda+1)np_0}P(|\tau|>\varepsilon)\}\\&\leqslant\mathrm{e}^{-\theta(\lambda+1)np_0}\boldsymbol{E}\mathrm{e}^{\theta\sum_{i=1}^n I\{\eta_i>t-\varepsilon\}}+2nP(\tau>\varepsilon).\end{aligned}$$

By the fact that $\{\eta_i;\ i=1,\dots,n\}$ are independent, it is easy to see that

$$\begin{aligned}&E\mathrm{e}^{\theta\sum_{i=1}^n I\{\eta_i>t-\varepsilon\}}\\ &\quad=\mathrm{e}^{\theta np_1}(E\mathrm{e}^{\theta(I\{\eta_1>t-\varepsilon\}-p_1)})^n\\ &\quad\leqslant \mathrm{e}^{\theta np_1}(1+p_1(1-p_1)\theta^2)^n\leqslant \mathrm{e}^{\theta np_1+\theta^2 np_1(1-p_1)}.\end{aligned}$$

Then, we have

$$\begin{aligned}&P\left(\sum_{i=1}^n I\{\xi_i>t\}-np_0\geqslant\lambda np_0\right)\\ &\quad\leqslant \mathrm{e}^{-\theta(\lambda+1)np_0}\mathrm{e}^{\theta^2 np_1(1-p_1)+\theta np_1}+2nP(\tau>\varepsilon)\\ &\quad\leqslant \mathrm{e}^{-\theta n((\lambda+1)p_0-(1+\theta)p_1)}+2nP(\tau>\varepsilon),\end{aligned}$$

which implies (2.2) immediately. Note that

$$P\left(np_0-\sum_{i=1}^n I\{\xi_i>t\}\geqslant\lambda np_0\right)=P\left(\sum_{i=1}^n I\{\xi_i\leqslant t\}-n(1-p_0)\geqslant\lambda np_0\right).$$

If we choose $m=\theta(\lambda np_0+n(1-p_0))$ and define g by

$$g(x_1,\dots,x_n)=f\left(\theta\sum_{i=1}^n I\{x_i\leqslant t\}\right),$$

we can obtain (2.3) similarly. □

Proof of Theorem 1.1. Using Lemma 2.2 and following the same lines in the proof of Theorem 2 of Orey and Taylor (1974, p. 180), we can show that $\dim E(\alpha_0)\leqslant 1-\alpha_0^2$ a.s. easily.

Now, we turn to the proof of the opposite inequality. It is sufficient to show that for any $0<\alpha_0<1$, we have almost surely

$$\dim E(\alpha_0)\geqslant 1-\alpha_0^2. \tag{2.4}$$

For each fixed $1>\alpha>\alpha_0$ and $\varepsilon>0$, we will apply Lemma 2.1 with K chosen as a suitable subset of $E(\alpha_0)$ and $\phi(s)=s^{\beta-2\varepsilon}$, where $\beta=1-\alpha^2$, $0<\varepsilon<\frac{1}{2}\beta<1$. This will enable us to establish (2.4). The remainder of the proof is devoted to the construction of K and was inspired by, and accurately is a generalized version of, the arguments in Section 4 of Orey and Taylor (1974).

Let $\mathscr{T}$ denote the collection of intervals $[u,v]\subset[0,1]$ such that

$$\|Y(v)-Y(u)\|_{\ell^p}\geqslant\alpha_0\tilde\sigma(p,v-u)(2\log(v-u)^{-1})^{1/2}. \tag{2.5}$$

Theorem C tells us that

$$\|Y(t)-Y(s)\|_{\ell^p}\leqslant 2\tilde\sigma(p,|t-s|)(2\log|t-s|^{-1})^{1/2}$$

for all $s,t\in[0,1]$ with $|t-s|$ sufficiently small. Hence, there exists $b>0$ depending only on α_0 and α such that, for every sufficiently small $I=[u,v]\subset[0,1]$

$$\|Y(v)-Y(u)\|_{\ell^p}\geqslant\alpha\tilde\sigma(p,v-u)(2\log(v-u)^{-1})^{1/2} \tag{2.6}$$

implies that $[t,v]\in\mathscr{T}$ for all $t\in I(b)=[u,u+b(v-u)]$. For convenience, we assume that b is the reciprocal of an integer.

Suppose that ρ_m is the reciprocal of an integer, $\rho_{m+1}<b\rho_m$ and $b\rho_m/\rho_{m+1}$ is an integer for $m=1,2,\ldots$. Let δ be a positive number such that $\delta<\frac{1}{16}\varepsilon$. For each $m\geqslant 1$, define $A_m=[\rho_m^{-\delta}]$, $l_m=[(\rho_m^{-1}-1)/A_m]+1$ and

$$t_m(i)=iA_m\rho_m,\quad i=0,1,\ldots,l_m-1, \tag{2.7}$$

$$\mathscr{F}_m=\{[t_m(i),t_m(i)+\rho_m];\ i=0,1,\ldots,l_m-1\}. \tag{2.8}$$

We proceed with the proof by considering the cases of $1\leqslant p<2$ and $2\leqslant p<\infty$, separately.

Case I: $1\leqslant p<2$. In this case, for each $m\geqslant 1$ and any $I=[t_m(i),t_m(i)+\rho_m]\in\mathscr{F}_m$ we have

$$\begin{aligned}
&\frac{\|Y(I)\|_{\ell^p}}{\tilde\sigma(p,\rho_m)(2\log\rho_m^{-1})^{1/2}}\\
&\quad=\frac{\|Y(t_m(i)+\rho_m)-Y(t_m(i))\|_{\ell^p}}{\tilde\sigma(p,\rho_m)(2\log\rho_m^{-1})^{1/2}}\\
&\quad=\frac{\sup_{\|a\|_{l^q}\leqslant 1}\sum_{j=1}^\infty a_j(X_j(t_m(i)+\rho_m)-X_j(t_m(i)))}{\tilde\sigma(p,\rho_m)(2\log\rho_m^{-1})^{1/2}}\\
&\quad\geqslant\frac{\sum_{j=1}^\infty\sigma_j(\rho_m)^{2(p-1)/(2-p)}(X_j(t_m(i)+\rho_m)-X_j(t_m(i)))}{\tilde\sigma(p,\rho_m)(\sum_{j=1}^\infty\sigma_j(\rho_m)^{2p/(2-p)})^{(p-1)/p}(2\log\rho_m^{-1})^{1/2}}\\
&\quad=\frac{\sum_{j=1}^\infty\sigma_j(\rho_m)^{2(p-1)/(2-p)}(X_j(t_m(i)+\rho_m)-X_j(t_m(i)))}{\overline\sigma(p,\rho_m)(2\log\rho_m^{-1})^{1/2}}=:Y_{m,I},
\end{aligned} \tag{2.9}$$

where $\overline\sigma(p,\rho_m)=(\sum_{j=1}^\infty\sigma_j(\rho_m)^{2p/(2-p)})^{1/2}$, $q=p/(p-1)$ and

$$Y_{m,I}=\overline\sigma(p,\rho_m)^{-1}\sum_{j=1}^\infty\sigma_j(\rho_m)^{2(p-1)/(2-p)}(X_j(t_m(i)+\rho_m)-X_j(t_m(i))). \tag{2.10}$$

We define

$$\begin{aligned}
&\mathscr{F}_m^+=\{I\in\mathscr{F}_m;Y_{m,I}>\alpha(2\log\rho_m^{-1})^{1/2}\},\\
&\mathscr{F}_m^+(b)=\{I(b)=[u,u+b(v-u)];\ I=[u,v]\in\mathscr{F}_m^+\},\\
&N_m(J)=\#\{I\in\mathscr{F}_m^+,I\subset J\},\quad N_m=N_m([0,1]),\\
&l_m(J)=\#\{I\in\mathscr{F}_m,I\subset J\},\quad l_m=l_m([0,1]),\\
&\rho_m^{1-\beta(m)}=P(\mathrm{N}(0,1)>\alpha(2\log\rho_m^{-1})^{1/2}),
\end{aligned} \tag{2.11}$$

where $0<\beta(m)\to\beta=1-\alpha^2$ as $m\to\infty$.

From (2.9), we deduce that for m large enough, $I=[u,v]\in\mathscr{F}_m^+$ implies (2.6), and then $[t,v]\in\mathscr{T}$ for any $t\in I(b)\in\mathscr{F}_m^+(b)$.

The following lemma tells us that $N_m(J)$ has similar probability estimates as a binomial distribution with parameters $p=\rho_m^{1-\beta(m)}$ and $n=l_m(J)$.

Lemma 2.4. *For any* $0<\theta\leqslant\frac{1}{2}$, *there exists an integer* m_θ *such that*

$$P(|N_m(J)-EN_m(J)|>\lambda EN_m(J))$$
$$\leqslant 2\exp\{-\theta(\lambda-2\theta)EN_m(J)\}+\rho_m^4 \tag{2.12}$$

for all $J\subset[0,1]$, $m\geqslant m_\theta$ *and* $\lambda>0$.

Proof. By condition (1.15), we can assume that

$$\max_{A_m\leqslant j\leqslant\rho_m^{-1}}\max_{k\geqslant 1}\frac{E(X_k(\rho_m)-X_k(0))(X_k((j+1)\rho_m)-X_k(j\rho_m))}{\sigma_k^2(h)}\leqslant b_m^2$$

for all $m\geqslant 1$, where $b_m>0$ satisfies $b_m\log\rho_m\to 0$ as $m\to\infty$. It follows that

$$\max_{\substack{I,J\in\mathscr{F}_m\\ I\neq J}} EY_{m,I}Y_{m,J}\leqslant b_m^2. \tag{2.13}$$

For m large enough, let

$$p_0^{(m)}=P\left(\mathrm{N}(0,1)>\alpha(2\log\rho_m^{-1})^{1/2}\right),$$
$$p_1^{(m)}=P\left(\mathrm{N}(0,1)>\frac{(\alpha-3b_m)(2\log\rho_m^{-1})^{1/2}}{(1-b_m^2)^{1/2}}\right), \tag{2.14}$$
$$p_2^{(m)}=P\left(\mathrm{N}(0,1)>\frac{(\alpha+3b_m)(2\log\rho_m^{-1})^{1/2}}{(1-b_m^2)^{1/2}}\right).$$

Applying Lemma 2.3 to $\{Y_{m,I};\ I\subset J\}$ with $\beta=b_m$, $t=\alpha(2\log\rho_m^{-1})^{1/2}$ and $\varepsilon=3b_m(2\log\rho_m^{-1})^{1/2}$, we conclude that for any $0<\theta\leqslant\frac{1}{2}$ and $\lambda>0$.

$$P(|N_m(J)-EN_m(J)|>\lambda EN_m(J))$$
$$\leqslant\exp\{-\theta l_m(J)((\lambda+1)p_0^{(m)}-(1+\theta)p_1^{(m)})\}$$
$$+\exp\{-\theta l_m(J)((1-\theta)p_2^{(m)}-(1-\lambda)p_0^{(m)})\}$$
$$+2l_m(J)P(\mathrm{N}(0,1)>3(2\log\rho_m^{-1})^{1/2}). \tag{2.15}$$

Note that

$$\log\frac{p_1^{(m)}}{p_0^{(m)}}\sim 2\left(\alpha^2-\frac{(\alpha-3b_m)^2}{1-b_m^2}\right)\log\rho_m^{-1}\sim 6\alpha b_m\log\rho_m^{-1}=\mathrm{o}(1)\quad\text{as } m\to\infty$$

which implies $p_1^{(m)}\sim p_0^{(m)}$ as $m\to\infty$. Similarly, we have $p_2^{(m)}\sim p_0^{(m)}$ as $m\to\infty$. It follows that there exists an integer m_θ such that for $m\geqslant m_\theta$,

$$(1+\theta)\frac{p_1^{(m)}}{p_0^{(m)}}\leqslant(1+2\theta),\quad (1-\theta)\frac{p_2^{(m)}}{p_0^{(m)}}\geqslant(1-2\theta).$$

Hence, we deduce from (2.15) that

$$P(|N_m(J) - \boldsymbol{E}N_m(J)| > \lambda \boldsymbol{E}N_m(J))$$
$$\leqslant 2\exp\{-\theta(\lambda - 2\theta)\boldsymbol{E}N_m(J)\} + \rho_m^{-1}\rho_m^5 = 2\exp\{-\theta(\lambda-2\theta)\boldsymbol{E}N_m(J)\} + \rho_m^4$$

for $m \geqslant m_\theta$. We have proved Lemma 2.4. □

By using Lemma 2.4, with similar proofs to that of Lemmas 4.1 of Orey and Taylor (1974) we have the following lemma.

Lemma 2.5. *Given* $\varepsilon > 0$ *and* $\tau > 0$, *with probability* 1 *there exists an integer* $m_0 = m_0(\varepsilon, \tau)$ *such that*

$$|N_m(J) - \boldsymbol{E}N_m(J)| < \varepsilon \boldsymbol{E}N_m(J) \tag{2.16}$$

for all $J \subset [0,1]$ *such that* $|J| \geqslant \tau$, *and all* $m \geqslant m_0(\varepsilon, \tau)$.

Lemma 2.6. *Given* $\beta' < \beta = 1 - \alpha^2$, *if* $\delta < \frac{1}{2}(\beta - \beta')$, *then there exists an absolute constant* c *such that with probability* 1, *there exists* $m_1 = m_1(\beta')$ *such that*

$$N_m(J) \leqslant c|J|^{\beta'} N_m([0,1]) \tag{2.17}$$

for all $J \subset [0,1]$ *and* $m \geqslant m_1$.

Proof. By Lemma 2.5, it is sufficient to show that

$$N_m(J) \leqslant c|J|^{\beta'} \boldsymbol{E}N_m([0,1]) = c|J|^{\beta'} l_m \rho_m^{1-\beta(m)}$$

for $m \geqslant m_1$. Note that $|J| < \rho_m$ implies $N_m(J) = 0$, $\rho_m \leqslant |J| < A_m\rho_m$ implies $N_m(J) \leqslant 1$ and $|J|^{\beta'} l_m \rho_m^{1-\beta(m)} \geqslant c\rho_m^{\delta+\beta'-\beta(m)} \to \infty$, we need only to consider the case of $|J| \geqslant A_m\rho_m$. It is clearly sufficient to consider only the class $\mathscr{D}_m$ of intervals $[iA_m\rho_m, jA_m\rho_m]$, where i, j are integers and $0 \leqslant i < j \leqslant (A_m\rho_m)^{-1}$. We deduce from (2.12) that for m large enough and all $r \geqslant 4$

$$P(N_m(J) \geqslant r\boldsymbol{E}N_m(J)) \leqslant \exp(-\tfrac{1}{8}r\boldsymbol{E}N_m(J)) + \rho_m^4.$$

Note that $l_m \sim \rho_m^{-1}A_m^{-1} \sim \rho_m^{\delta-1}$ and $l_m(J) \approx |J| l_m$, we have that

$$P(|N_m(J) \geqslant c|J|^{\beta'} l_m \rho_m^{1-\beta(m)},\ J \in \mathscr{D}_m)$$
$$\leqslant \rho_m^{-2}\exp\left(-\frac{c}{8}|\rho_m|^{\beta'} l_m \rho_m^{1-\beta(m)}\right) + \rho_m^2$$
$$\leqslant \rho_m^{-2}\exp(-c_1\rho_m^{\delta+\beta'-\beta(m)}) + \rho_m^2.$$

Since, $\delta + \beta' - \beta(m) \to \delta + \beta' - \beta < 0$, it follows that

$$\sum_{m=1}^{\infty} P(N_m(J) \geqslant c|J|^{\beta'} l_m \rho_m^{1-\beta(m)},\quad J \in \mathscr{D}_m) < \infty,$$

which implies almost surely there exists $m_1 = m_1(\beta')$ such that

$$N_m(J) \leqslant c|J|^{\beta'} l_m \rho_m^{1-\beta(m)}$$

for all $J \in \mathscr{D}_m$ $(m \geqslant m_1)$. This completes the proof of the lemma immediately. □

We shall now show that there exists a sequence of sets $E_1 \supset E_2 \supset \cdots$ fulfilling the assumptions of Lemma 2.1 and such that $K = \bigcap_{m=1}^{\infty} E_m \subset E(\alpha_0)$. Since only a countable number steps of the construction are needed and each step can be carried out with probability 1, we can assume that all the steps are carried out in the same probability 1 set. Choose $\beta' = \beta - \frac{1}{4}\varepsilon$ and define $m_1 = m_1(\beta')$ such that (2.17) is valid for $m \geqslant m_1$. Suppose (ε_k) is a sequence of positive numbers with $\sum \varepsilon_k < \infty$. In the first step, we apply Lemma 2.5 to find an integer $Q(1) \geqslant m_1$ such that

$$|N_m - EN_m| < \varepsilon_1 EN_m \quad (m \geqslant Q(1)). \tag{2.18}$$

And then we will define an increasing sequence $Q(1)$, $Q(2), \ldots$ inductively and define for $k \geqslant 1$

$$\begin{aligned} &\{I_{k,i}; 1 \leqslant i \leqslant M_k\} = \{I(b) \in \mathscr{F}^+_{Q(k)}; I(b) \subset E_{k-1}\}, \\ &E_0 = [0,1], \quad E_k = \bigcup_{i=1}^{M_k} I_{k,i}, \\ &M_k(J) = \#\{i; I_{k,i} \subset J\} \quad \text{for } J \subset [0,1], \quad M_k = M_k([0,1]), \\ &\gamma(k) = \beta(Q(k)), \quad \delta(k) = 1 - \gamma(k), \quad h_k = |I_{k,i}| = b\rho_{Q(k)}. \end{aligned} \tag{2.19}$$

For $k \geqslant 2$, suppose that $Q(k-1)$ has been defined we can define $Q(k)$ large enough to ensure

$$\begin{aligned} &Q(k) \geqslant m_0(\varepsilon, h_{k-1}^{2\delta(k-1)/\varepsilon}), \quad Q(k) \geqslant m_0(\varepsilon_k, h_{k-1}), \\ &Q(k) \geqslant 2Q(k-1), \quad \rho_{Q(k)} \leqslant \rho^2_{Q(k-1)}, \end{aligned} \tag{2.20}$$

where $m_0(\varepsilon, \tau)$ is the integer determined in Lemma 2.5 to invalidate (2.16), and

$$h_k^{1/2\varepsilon} \leqslant b^{2\beta} \prod_{i=1}^{k-1} h_i^{\delta(i)} b^{\gamma(i)}. \tag{2.21}$$

Then

$$|N_m(J) - EN_m(J)| < \varepsilon_k EN_m(J) \tag{2.22}$$

for all $J \subset [0,1]$ satisfying $|J| \geqslant h_{k-1}$ and $m \geqslant Q(k)$.

The proof will be completed if the following is true.

$$M_{k+j}(J) \leqslant c\left(\prod_{i=1}^{k} A_{Q(i)}\right)|J|^{\beta-\varepsilon} M_{k+j} \leqslant c\left(\prod_{i=1}^{k} A_{Q(i)}\right) h_k^{\varepsilon}|J|^{\beta-2\varepsilon} M_{k+j}$$

$$\text{for all } h_{k+1} < |J| \leqslant h_k, \quad k \geqslant 1, j \geqslant 1. \tag{2.23}$$

In fact, if (2.23) is true, by noting that

$$\rho_{Q(k)}^2 \leqslant \rho_{Q(k)}^{1+\frac{1}{2}+\cdots+\frac{1}{2^{k-1}}} \leqslant \rho_{Q(k)}\rho_{Q(k-1)}\cdots\rho_{Q(1)},$$

and

$$\prod_{i=1}^{k} A_{Q(i)} \leqslant \left(\prod_{i=1}^{k}\rho_{Q(i)}\right)^{-\delta},$$

we conclude that

$$M_{k+j}(J) \leqslant cb^{\varepsilon}\rho_{Q(k)}^{\varepsilon-2\delta}|J|^{\beta-2\varepsilon}M_{k+j} \quad \text{for all } h_{k+1}<|J|\leqslant h_k,\ k\geqslant 1,\ j\geqslant 1,$$

which, together with Lemma 2.1 and the fact that $\rho_{Q(k)}^{\varepsilon-2\delta}\to 0$ $(k\to\infty)$, implies that

$$s^{\beta-2\varepsilon}\text{-mes}(K)>0.$$

Hence, we have proved (2.4) in the case I.

Now we show (2.23). It is sufficient to consider the class $\mathscr{D}_k$ of intervals $[ih_{k+1}, jh_{k+1}]$ with $0\leqslant i<j\leqslant h_{k+1}^{-1}$. By the fact that $1/b$ and $b\rho_m/\rho_{m+1}$ are integers, we can verify that for any $k\geqslant 1$, $j\geqslant 1$,

$$I(b)\in\mathscr{F}^{+}_{Q(k+j+1)}(b),\quad I(b)\subset I_{k+j,i} \text{ imply } I\in\mathscr{F}^{+}_{Q(k+j+1)},\ I\subset I_{k+j,i}; \tag{2.24}$$

and for any $J\in\mathscr{D}_k$,

$$\text{interior } (I_{k+j,i}\cap J)\neq\emptyset \quad \text{implies } I_{k+j,i}\subset J. \tag{2.25}$$

From (2.24) and (2.25), it follows that for $J\in\mathscr{D}_k$,

$$M_{k+j+1}(J)=\#\{I(b)\in\mathscr{F}^{+}_{Q(k+j+1)}(b); I(b)\subset\bigcup_{i=1}^{M_{k+j}} I_{k+j,i}, I(b)\subset J\}$$

$$=\sum_{\{i;\,I_{k+j,i}\subset J\}} \mathrm{N}_{Q(k+j+1)}(I_{k+j+1}). \tag{2.26}$$

Note that $EN_{Q(k+j+1)}(I_{k+j,i})=l_{Q(k+j+1)}(I_{k+j,i})\rho_{Q(k+j+1)}^{1-\gamma(k+j+1)}$ and

$$l_{Q(k+j+1)}(I_{k+j,i})=\frac{b\rho_{Q(k+j)}}{A_{Q(k+j+1)}\rho_{Q(k+j+1)}}(1+\delta_{k+j+1,i}), \tag{2.27}$$

where

$$|\delta_{k+j+1,i}| \leqslant 8A_{Q(k+j+1)}\rho_{Q(k+j+1)}/(b\rho_{Q(k+j)})$$

$$\leqslant 8\rho_{Q(k+j+1)}^{1-\delta}/(b\rho_{Q(k+j)})\leqslant 8b^{-1}\rho_{Q(k+j+1)}^{3/4}/\rho_{Q(k+j)}$$

$$\leqslant 8b^{-1}\rho_{Q(k+j+1)}^{1/4}\leqslant b^{Q(k+j+1)/4},$$

from (2.22) and (2.26) we have

$$M_{k+j+1}(J) = M_{k+j}(J)\frac{1}{A_{Q(k+j+1)}}h_{k+j}h_{k+j+1}^{-\gamma(k+j+1)}b^{\gamma(k+j+1)}(1+\delta(J))(1+\eta(J)), \tag{2.28}$$

where $|\delta(J)| \leqslant b^{Q(k+j+1)/4}$, $|\eta(J)| \leqslant \varepsilon_{k+j+1}$. Following the lines of the proof of Theorem 2 of Orey and Taylor (1974, p. 183 and 184), we can verify from (2.21), (2.28), Lemmas 2.5 and 2.6 that for all $j \geqslant 1$, $k \geqslant 1$, $m \geqslant 1$,

$$M_{k+j+1} \approx \left(\prod_{i=1}^{k+j+1}\frac{1}{A_{Q(i)}}\right)h_{k+j+1}^{-\gamma(k+j+1)}b^{\gamma(k+j+1)}\prod_{i=1}^{k+j}h_i^{\delta(i)}b^{\gamma(i)}, \tag{2.29}$$

$$M_{k+j+1}(J) \leqslant c|J|^{\beta-\varepsilon}\left(\prod_{i=k+1}^{k+j+1}\frac{1}{A_{Q(i)}}\right)h_{k+j+1}^{-\gamma(k+j+1)}b^{\gamma(k+j+1)}\prod_{i=1}^{k+j}h_i^{\delta(i)}b^{\gamma(i)}, \quad J \in \mathscr{D}_k \tag{2.30}$$

Hence, we have proved (2.23). □

Case II: $p \geqslant 2$. Take j_m such that $\sigma_{j_m}(\rho_m) = \sigma^*(\rho_m)$. In this case, for any $I = [t_m(i), t_m(i) + \rho_m] \in \mathscr{F}_m$ we also have

$$\frac{\|Y(I)\|_{\ell^p}}{\tilde{\sigma}(p,\rho_m)(2\log\rho_m^{-1})^{1/2}} \geqslant \frac{X_{j_m}(I)}{\sigma_{j_m}(\rho_m)(2\log\rho_m^{-1})^{1/2}}.$$

Following the lines of the proof in case I, we conclude that (2.4) remains true in this case as well.

Proof of Theorem 1.2.

Since $E^*(\alpha) \subset E(\alpha)$, we only need to show that $\dim E^*(\alpha) \geqslant 1-\alpha^2$. We may suppose $P > 1$. By tightening the argument used in the proof of (2.4), we may show that for any $\alpha \in [0,1)$, ϕ-mes $E(\alpha) > 0$ a.s., where $\phi(s) = s^{1-\alpha^2}(\log s^{-1})^{6P+4}$. Since

$$E(\alpha) = E^*(\alpha)\bigcup_{n=1}^{\infty}E(\alpha+1/n),$$

and $\dim E(\alpha+1/n) = 1-(\alpha+1/n)^2$ implies ϕ-mes $E(\alpha+1/n) = 0$, we have

$$\phi\text{-mes}\,E^*(\alpha) \geqslant \phi\text{-mes}\,E(\alpha) > 0 \quad a.s.$$

which implies $\dim E^*(\alpha) \geqslant 1-\alpha^2$ a.s. Hence, we have proved Theorem 1.2. □

3. Some applications

Using the conclusions of Theorems 1.1 and 1.2, we can obtain some consequences.

3.1. Gaussian processes with regularly varying variance functions

Suppose $\{X(t);\ t\geqslant 0\}$ be a centered Gaussian process with stationary increments $\sigma^2(h)=E(X(t+h)-X(t))^2$. Assume that $\sigma(t)$ is increasing and regularly varying at zero with index $0<\gamma<1$, given in the canonical form

$$\sigma(h)=h^{\gamma}\exp\left(\int_h^1 \frac{\varepsilon(y)}{y}\,\mathrm{d}y\right), \tag{3.1}$$

where $\varepsilon(h)\to 0$ as $h\to 0$. The Lévy moduli of continuity for this kind of processes were due to Csáki et al. (1991). Clearly, $\{Y(t);\ t\geqslant 0\}$ is a fractional Brownian motion of order γ if $\varepsilon(h)\equiv 0$.

Let $p\geqslant 1$, $\{c_n;\ n\geqslant 1\}$ be non-negative numbers. Put

$$c(p)=\left(\sum_{k=1}^{\infty} c_k^p\right)^{1/p},$$

$$\tilde{c}(p)=\begin{cases} c(\frac{2p}{2-p}) & \text{if } 1\leqslant p<2,\\ \max_{k\geqslant 1} c_k & \text{if } p\geqslant 2. \end{cases} \tag{3.3}$$

Let $\{Y(t);\ t\geqslant 0\}=\{c_kX_k(t);\ t\geqslant 0\}$, where $X_k(\cdot)$, $k=1,2\ldots$ are independent copies of $X(\cdot)$. Set $\sigma_k^2(h)=c_k^2EX_k^2(h)=c_k^2\sigma^2(h)$. Define $\sigma(p,h)$ and $\tilde{\sigma}(p,h)$ as in (1.6) and (1.8), respectively. Clearly, we have

$$\sigma(p,h)=c(p)\sigma(h),\qquad \tilde{\sigma}(p,h)=\tilde{c}(p)\sigma(h).$$

Assume

$$0<\sum_{k=1}^{\infty} c_k^p<\infty. \tag{3.5}$$

As consequences of Theorems 1.1 and 1.2, we have the following results.

Theorem 3.1. *Let $p\geqslant 1$, $\{Y(t);\ t\geqslant 0\}=\{c_kX_k(t);\ t\geqslant 0\}_{k=1}^{\infty}$ be defined as above. If for any $\varepsilon>0$ there exists a constant C_ε such that $-h^{1+\varepsilon}\varepsilon'(h)\leqslant C_\varepsilon$ for all $h\in[0,1]$, then for any $\alpha\in[0,1]$, we have almost surely*

$$\dim\left\{t\in[0,1];\ \frac{\|Y(t+h)-Y(t)\|_{\ell^p}}{\tilde{c}_p\sigma(h)(2\log h^{-1})^{1/2}}\geqslant\alpha\right\}=1-\alpha^2. \tag{3.6}$$

If there exists a constant C such that $-h\varepsilon'(h)\leqslant C(\log e/h)^C$ for all $h\in[0,1]$, then for any $\alpha\in[0,1]$, we have almost surely

$$\dim\left\{t\in[0,1];\ \frac{\|Y(t+h)-Y(t)\|_{\ell^p}}{\tilde{c}_p\sigma(h)(2\log h^{-1})^{1/2}}=\alpha\right\}=1-\alpha^2. \tag{3.7}$$

Particularly, we have the following corollary.

Corollary 3.1. *Let $p\geqslant 1$, $\{\xi(t),\xi_k(t);\ t\geqslant 0,k\geqslant 1\}$ be independent fractional Brownian motions of order γ, $0<\gamma<1$. Define $\{Y(t);\ t\geqslant 0\}=\{c_k\xi_k(t);\ t\geqslant 0\}_{k=1}^{\infty}$. Assume that (3.5) is satisfied. Then for any $\alpha\in[0,1]$, we have almost surely*

$$\dim E(\alpha)=\dim E^*(\alpha)=1-\alpha^2,$$

where

$$E(\alpha)=\left\{t\in[0,1];\ \frac{\|Y(t+h)-Y(t)\|_{\ell^p}}{\tilde{c}_p h^{\gamma}(2\log h^{-1})^{1/2}}\geqslant\alpha\right\},$$

$$E^*(\alpha)=\left\{t\in[0,1];\ \frac{\|Y(t+h)-Y(t)\|_{\ell^p}}{\tilde{c}_p h^{\gamma}(2\log h^{-1})^{1/2}}=\alpha\right\}.$$

Proof of Theorem 3.1. To show (3.6) and (3.7), we need only to verify (1.15) and (1.18), respectively. Let

$$L(x)=\exp\left(2\int_x^1\frac{\varepsilon(y)}{y}\,\mathrm{d}y\right).$$

Then

$$(\sigma^2(x))''=2x^{2\gamma-2}L(x)\{\gamma(2\gamma-1)+(1-4\gamma)\varepsilon(x)+2\varepsilon^2(x)-x\varepsilon'(x)\}.$$

It is easy to see that $|\varepsilon(x)|\leqslant C$ for some $C>0$ and all $x\in[0,1]$, and also for any $\delta>0$ there is a constant C_δ such that $L(jx)/L(x)\leqslant C_\delta j^\delta$ for all x, $jx\in[0,1]$. Suppose $-x^{1+\delta}\varepsilon'(x)\leqslant C_\delta$. Then

$$\frac{(\sigma^2(jx))''x^2}{\sigma^2(x)}\leqslant c_\delta j^{2\gamma-2+\delta}x^{-\delta}.$$

Note that

$$\frac{\mathbf{E}(X(h)-X(0))(X((j+1)h)-X(jh))}{\sigma^2(h)}$$
$$=\frac{\sigma^2((j+1)h)+\sigma^2((j-1)h)-2\sigma^2(jh)}{2\sigma^2(h)}=\frac{(\sigma^2(\xi))''h^2}{2\sigma^2(h)}$$

for every $1>h\geqslant 0$, $j\geqslant 6$ and some $(j-2)h\leqslant\xi\leqslant jh$, we conclude that

$$\frac{\mathbf{E}(X(h)-X(0))(X((j+1)h)-X(jh))}{\sigma^2(h)}$$
$$\leqslant C\frac{(\sigma^2(\xi))''(\xi/j)^2}{2\sigma^2(\xi/j)}\leqslant c_\delta j^{2\gamma-2+\delta}\left(\frac{\xi}{j}\right)^{-\delta}\leqslant c_\delta j^{2\gamma-2+\delta}h^{-\delta},$$

which implies (1.15) immediately.

Suppose $-x\varepsilon'(x)\leqslant C(\log\frac{e}{x})^C$, then

$$\frac{(\sigma^2(jx))''x^2}{\sigma^2(x)}\leqslant c_\delta j^{2\gamma-2+\delta/2}\left(\log\frac{e}{jx}\right)^C\leqslant c_\delta j^{2\gamma-2+\delta}\left(\log\frac{e}{x}\right)^C.$$

We also conclude that

$$\frac{E(X(h)-X(0))(X((j+1)h)-X(jh))}{\sigma^2(h)} \leqslant c_\delta j^{2\gamma-2+\delta}(\log h^{-1})^C$$

which implies (1.18) immediately. □

3.2. ℓ^p-valued fractional Ornstein–Uhlenbeck processes and the infinite series of fractional Ornstein–Uhlenbeck processes

Let $\{Y(t);\ t\geqslant 0\}=\{X_k(t);\ t\geqslant 0\}_{k=1}^{\infty}$ be a sequence of independent fractional Ornstein–Uhlenbeck processes of orders β_k with coefficients γ_k and λ_k, where $0<\beta_k<1, \gamma_k\geqslant 0,\ \lambda_k>0$, i.e.,

$$\{X_k(t);\ t\geqslant 0\} \quad \text{and} \quad \left\{\left(\frac{\gamma_k}{\lambda_k}\right)^{1/2}\frac{\xi_k(e^{2\lambda_k t})}{e^{2\beta_k\lambda_k t}};\ t\geqslant 0\right\}$$

have the same distribution, where $\{\xi_k(t);\ t\geqslant 0\}$ is a fractional Brownian motion of order β_k. It is easily seen that $EX_k(t)=0$,

$$EX_k(t)X_k(s)=\frac{\gamma_k}{2\lambda_k}(e^{2\beta_k\lambda_k(t-s)}+e^{2\beta_k\lambda_k(s-t)}-|e^{\lambda_k(t-s)}-e^{\lambda_k(s-t)}|^{2\beta_k})$$

for all t, $s\geqslant 0$, and

$$\begin{aligned}\sigma_k^2(h)&=E(X_k(t+h)-X_k(t))^2\\&=\frac{\gamma_k}{\lambda_k}\{(e^{\lambda_k h}-e^{-\lambda_k h})^{2\beta_k}-(e^{\beta_k\lambda_k h}-e^{-\beta_k\lambda_k h})^2\}.\end{aligned}$$

Clearly, $\{X_k(t)\}$ is the usual Ornstein–Uhlenbeck processes if $\beta_k=\frac{1}{2}$ for all $k\geqslant 1$. The Lévy moduli of continuity for the infinite series of Ornstein–Uhlenbeck processes were due to Csáki et al. (1991). Similar quantities for l^p-valued fractional Ornstein–Uhlenbeck processes were studied by Csörgő and Shao (1993) in the case that all $X_k(\cdot)$ have the same order γ, i.e., $\beta_k=\gamma$ for all $k\geqslant 1$. In this section, as consequences of Theorems 1.1 and 1.2, we have the following results about the fractal nature.

Let $p\geqslant 1$. Define $\sigma(p,h)$ and $\tilde\sigma(p,h)$ as in (1.6) and (1.8), respectively.

Theorem 3.2. *Assume that $\tilde\sigma(p,h)/h^{\Lambda}$ is quasi-increasing on $(0,\Lambda)$ for some $\Lambda>0$. Suppose that $0<\beta_k\leqslant\beta_0<1$ for all $k\geqslant 1$. If*

$$\sigma(p,h)=o\left(\tilde\sigma(p,h)\left(\log\frac{1}{h}\right)^{1/2}\right) \quad \text{as } h\to 0, \tag{3.8}$$

then for any $\alpha\in[0,1]$, we have almost surely

$$\dim E(\alpha)=\dim E^*(\alpha)=1-\alpha^2,$$

where

$$E(\alpha)=\left\{t\in[0,1];\ \frac{\|Y(t+h)-Y(t)\|_{\ell^p}}{\tilde\sigma(p,h)(2\log h^{-1})^{1/2}}\geqslant\alpha\right\},$$
$$E^*(\alpha)=\left\{t\in[0,1];\ \frac{\|Y(t+h)-Y(t)\|_{\ell^p}}{\tilde\sigma(p,h)(2\log h^{-1})^{1/2}}=\alpha\right\}. \tag{3.9}$$

Theorem 3.3. *Suppose that* $\{X(t)=\sum_{k=1}^{\infty}X_k(t);\ t\geqslant 0\}$ *is the infinite series of* $\{X_k(t)\}_{k=1}^{\infty}$ *and* $0<\beta_k\leqslant\beta_0<1$ *for all* $k\geqslant 1$. *Assume that* $\sigma(2,h)/h^{\Lambda}$ *is quasi-increasing on* $(0,\Lambda)$ *for some* $\Lambda>0$. *Then, for any* $\alpha\in[0,1]$, *we have almost surely*

$$\dim E(\alpha)=\dim E^*(\alpha)=1-\alpha^2,$$

where

$$E(\alpha)=\left\{t\in[0,1];\ \frac{|X(t+h)-X(t)|}{\sigma(2,h)(2\log h^{-1})^{1/2}}\geqslant\alpha\right\},$$
$$E^*(\alpha)=\left\{t\in[0,1];\ \frac{|X(t+h)-X(t)|}{\sigma(2,h)(2\log h^{-1})^{1/2}}=\alpha\right\}. \tag{3.10}$$

Proof of Theorem 3.2. We need only to verify (1.18). For any fixed $0<\gamma\leqslant\beta_0<1$, let

$$f(x)=:f(\gamma,x)=(\mathrm{e}^{x}-\mathrm{e}^{-x})^{2\gamma}-(\mathrm{e}^{\gamma x}-\mathrm{e}^{-\gamma x})^{2}. \tag{3.11}$$

It is easy to see that

$$\begin{aligned}f'(x)&=2\gamma\{(\mathrm{e}^{x}-\mathrm{e}^{-x})^{2\gamma-1}(\mathrm{e}^{x}+\mathrm{e}^{-x})-(\mathrm{e}^{\gamma x}-\mathrm{e}^{-\gamma x})(\mathrm{e}^{\gamma x}+\mathrm{e}^{-\gamma x})\}\\&>0\quad\text{for all } x>0,\end{aligned} \tag{3.12}$$

$$\begin{aligned}f''(x)&=2\gamma\{(2\gamma-1)(\mathrm{e}^{x}-\mathrm{e}^{-x})^{2\gamma-2}(\mathrm{e}^{x}+\mathrm{e}^{-x})^{2}\\&\quad+(\mathrm{e}^{x}-\mathrm{e}^{-x})^{2\gamma}-2\gamma\mathrm{e}^{2\gamma x}-2\gamma\mathrm{e}^{-2\gamma x}\}\\&<0\quad\text{for all } x\geqslant 0 \text{ if } 0<\gamma\leqslant\tfrac{1}{2},\end{aligned} \tag{3.13}$$

$$\begin{aligned}f''(x)&\leqslant 16\gamma(2\gamma-1)(\mathrm{e}^{x}-\mathrm{e}^{-x})^{2\gamma-2}\\&\leqslant 16(\mathrm{e}^{x}-\mathrm{e}^{-x})^{2\gamma-2}\\&=:g(x)=:g(\gamma,x)\quad\text{for all } x>0 \text{ if } \gamma>\tfrac{1}{2}.\end{aligned} \tag{3.14}$$

Hence, in any case we have

$$f''(x)\leqslant g(x)=g(\gamma,x)\quad\text{for all } x>0. \tag{3.15}$$

We also have

$$f'(x) = 2\gamma \left\{ (e^x - e^{-x})^{2\gamma} \frac{e^x + e^{-x}}{e^x - e^{-x}} - (e^{\gamma x} - e^{-\gamma x})^2 \frac{e^{\gamma x} + e^{-\gamma x}}{e^{\gamma x} - e^{-\gamma x}} \right\}$$

$$< 2\gamma \left\{ (e^x - e^{-x})^{2\gamma} \frac{e^x + e^{-x}}{e^x - e^{-x}} - (e^{\gamma x} - e^{-\gamma x})^2 \frac{e^x + e^{-x}}{e^x - e^{-x}} \right\}$$

$$= 2\gamma f(x) \frac{e^x + e^{-x}}{e^x - e^{-x}},$$

which implies

$$\frac{f(x)}{(e^x - e^{-x})^{2\gamma}} \quad \text{is decreasing.}$$

So, we have that for $x \geqslant 1$,

$$f(x) \geqslant f(1) = e^{2\gamma}\{(1 - e^{-2})^{2\gamma} - (1 - e^{-2\gamma})^2\}$$

$$\geqslant (1 - e^{-2})^{2\beta_0} - (1 - e^{-2\beta_0})^2 = c_{\beta_0} > 0,$$

and for $0 < x \leqslant 1$,

$$f(x) \geqslant \frac{f(1)}{(e - e^{-1})^{2\gamma}} (e^x - e^{-x})^{2\gamma} \geqslant \frac{f(1)}{(e - e^{-1})^{2\beta_0}} (e^x - e^{-x})^{2\gamma} = c_{\beta_0} (e^x - e^{-x})^{2\gamma}.$$

Now we estimate

$$\frac{g(jx)x^2}{f(x)}.$$

Since $(e^{jx} - e^{-jx}) \geqslant j(e^x - e^{-x})$ for all $x \geqslant 0$, we have that for $0 < x \leqslant 1$,

$$\frac{g(jx)x^2}{f(x)} \leqslant c_{\beta_0} \left(\frac{e^{jx} - e^{-jx}}{e^x - e^{-x}} \right)^{2\gamma - 2} \left(\frac{x}{e^x - e^{-x}} \right)^2 \leqslant c_{\beta_0} j^{2\gamma - 2};$$

and for $x \geqslant 1$,

$$\frac{g(jx)x^2}{f(x)} \leqslant c_{\beta_0} g(jx) x^2 = c_{\beta_0} \left(\frac{e^{jx} - e^{-jx}}{e^x - e^{-x}} \right)^{2\gamma - 2} \frac{x^2}{(e^x - e^{-x})^{2 - 2\gamma}}$$

$$\leqslant c_{\beta_0} j^{2\gamma - 2} \frac{x^2}{(e^x - e^{-x})^{2 - 2\beta_0}} \leqslant c_{\beta_0} j^{2\gamma - 2}.$$

So, we conclude that for all $x > 0$,

$$\frac{g(jx)x^2}{f(x)} \leqslant c_{\beta_0} j^{2\beta_0 - 2}. \tag{3.16}$$

Now we verify (1.18). Define $f_k(x)=f(\beta_k,x)$. We have that

$$\frac{E(X_k(h)-X_k(0))(X_k((j+1)h)-X_k(jh))}{\sigma_k^2(h)}$$

$$=\frac{f_k((j+1)\lambda_k h)+f_k((j-1)\lambda_k h)-2f_k(j\lambda_k h)}{2f_k(\lambda_k h)}$$

$$=\frac{f_k''(\xi)(\lambda_k h)^2}{2f_k(\lambda_k h)} \tag{3.17}$$

for every $h>0$, $j\geqslant 6$, $k\geqslant 1$ and some $(j-1)\lambda_k h\leqslant\xi\leqslant(j+1)\lambda_k h$. Note from (3.15) that

$$f_k''(\xi)\leqslant g(\beta_k,\xi)\leqslant g(\beta_k,(j-1)\lambda_k h),$$

we deduce from (3.16) that

$$\frac{E(X_k(h)-X_k(0))(X_k((j+1)h)-X_k(jh))}{\sigma_k^2(h)}\leqslant 1/2\cdot c_{\beta_0}(j-1)^{2\beta_0-2}\leqslant c_{\beta_0}j^{2\beta_0-2}, \tag{3.18}$$

which implies (1.18) immediately. □

Proof of Theorem 3.3. From (3.18), we deduce that for every $j\geqslant 6$, $h\geqslant 0$,

$$E(X(h)-X(0))(X((j+1)h)-X(jh))$$

$$=\sum_{k=1}^{\infty}E(X_k(h)-X_k(0))(X_k((j+1)h)-X_k(jh))$$

$$\leqslant\sum_{k=1}^{\infty}\sigma_k^2(h)\cdot c_{\beta_0}j^{2\beta_0-2}=c_{\beta_0}j^{2\beta_0-2}\sigma^2(2,h),$$

which implies (1.18). Hence we have proved Theorem 3.3. □

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