Donsker's invariance principle under the sub-linear expectation with an application to Chung's law of the iterated logarithm*

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Abstract

We prove a new Donsker's invariance principle for independent and identically distributed random variables under the sub-linear expectation. As applications, the small deviations and Chung's law of the iterated logarithm are obtained.

Keywords: sub-linear expectation; capacity; central limit theorem; invariance principle; Chung's law of the iterated logarithm; small deviation

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1 Introduction

Let $\{X_n; n \geq 1\}$ be a sequence of independent and identically distributed random variables on a probability space $(\Omega, \mathscr{F}, P)$ such that $EX_1 = 0$ and $EX_1^2 = \sigma^2$. Set $S_n = \sum_{j=1}^n X_j$. In his classical paper Chung (1948) proved the following remarkable result: if $E|X_1|^3 < \infty$, then

$$P\left(\liminf_{n\to\infty} \sqrt{\frac{8\log\log n}{n\pi^2}} \max_{k\leq n} |S_k| = \sigma\right) = 1. \tag{1.1}$$

The result (1.1) is refereed to as the other law of the iterated logarithm or Chung's law of the iterated logarithm, in contrast to the Hartman-Winter law:

$$P\left(\limsup_{n\to\infty} \frac{\max_{k\leq n}|S_k|}{\sqrt{2n\log\log n}} = \limsup_{n\to\infty} \frac{|S_n|}{\sqrt{2n\log\log n}} = \sigma\right) = 1. \tag{1.2}$$

Jian and Pruitt (1975) were the first to prove that (1.1) is still true of only the existence of second moments is assumed. The main tool of Jian and Pruitt (1975) is the following

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inequality: for any $c < \sigma\pi/\sqrt{8}$, there is a $\eta > 1$ such that

$$P\left(\max_{i\le n}|S_i| < c\Big(\frac{n}{\log\log n}\Big)^{1/2}\right) \le C\frac{1}{(\log n)^\eta} \tag{1.3}$$

for all n sufficient large. A more general inequality of this type is the following small deviation obtained by Mogul'skiĭ (1974): if $0 < x_n \to 0$ and $n^{1/2}x_n \to \infty$, then

$$\log P\big(\max_{i\le n}|S_i| \le n^{1/2}x_n\big) \sim -\frac{\pi^2}{8x_n^2}. \tag{1.4}$$

A small deviation theorem and Chung' law of the iterated logarithm for general independent but non-identically distributed random variables was established by Shao (1995) under a uniform Lindeberg's condition. The key for establishing the small deviations as (1.3),(1.4) and Shao's is the remarkable Donsker's invariance principle as follows: let

$$W_n(t) = \frac{S_{[nt]}}{\sqrt{n}} + (nt-[nt])\frac{X_{[nt]+1}}{\sqrt{n}},\ \ 0\le t\le 1,$$

then $W_n \overset{d}{\to} \sigma W$ in $C[0,1]$ in the sense that

$$E_P\left[\varphi(W_n)\right] \to E_P\left[\varphi(\sigma W)\right] \tag{1.5}$$

for any bounded continuous map φ from $C[0,1]$ to $\mathbb{R}$, where W is a standard Browian motion under P and $C[0,1]$ is the space of all continuous function $x:[0,1]\to\mathbb{R}$ (c.f, Donsker (1951), Billingsley (1968)).

The key in the proof of the these classical results is the additivity of the probability and the expectation. Under the sub-linear expectation, the Hartman-Winter law of the iterated logarithm were recently established by Chen and Hu (2014) for bounded random variables, and large deviations were derived by Gao and Xu (2011, 2012). The main purpose of this paper is to show that (1.5) is still true under the sub-linear expectation and to establish the small deviations similar to (1.4) and Chung's law of the iterated logaritm similar to (1.1) under the capacities related to the sub-linear expectation.

The general framework of the sub-linear expectation of random variables in a general function space was introduced by Peng (2006, 2008a, 2008b) and is a natural extension of the classical linear expectation with the linear property being replaced by the sub-additivity and positive homogeneity (c.f Definition 2.1 below). This simple generalization provides a very flexible framework to model non-additive probability and expectation problems. Take the hedge pricing in fiance as an example. The famous Black-Shores's formula states that, if a market is complete and self-financial, then there exists a neutral probability measure P such that the pricing of any discounted contingent claim ξ in this market is given by $E_P[\xi]$. However, if the market is incomplete, such a neutral probability measure is no longer unique, but a set $\mathscr{P}$ of probability measures. In that case, one can define superhedge pricing $\widehat{\mathbb{E}}[\xi] = \sup_{Q\in\mathscr{P}} E_Q[\xi]$ and subhedge pricing $\widehat{\mathcal{E}}[\xi] = -\widehat{\mathbb{E}}[-\xi] = \inf_{Q\in\mathscr{P}} E_Q[\xi]$, respectively. Then $\widehat{\mathbb{E}}[\xi]$ is sub-linear, as a functional operator of random variables, and the related capacity $\mathbb{V}(A) = \sup_{Q\in\mathscr{P}} Q(A)$ is non-additive, as a function operator of events. The extension of linear expectations to sub-linear expectations also produces many interesting properties different from the classic ones. For examples, the limit in the law of large numbers is no longer a contact, and, comparing to the classical one-dimensional normal distribution which is characterized by the Stein equation, an ordinary differential equation (ODE), a normal distribution under the sub-linear expectation is characterized by a time-space parabolic partial different equation (PDE). Recently, Hu and Li (2014) showed that the characteristic function cannot determine the distribution of random variables on the sub-linear expectation space. Roughly speaking, a sub-linear expectation is related to a group of unknown linear expectations and the distribution under a sub-linear expectation is related to a group of probabilities (c.f. Lemma 2.4 of Peng (2008b)). For more properties of the sub-linear expectations, one can refer to Peng (2008b,2009), where the notion of independent and identically distributed random variables under the sub-linear expectations was introduced and a new central limit theorem was estiblished. It is a natural and interesting problem whether (1.4) is true when the linear expectation E is replaced by the sub-linear expectation.

In the classical probability space, (1.4) is equivalent to the weak convergence of related probability measures in the metric space $C[0,1]$ equipped with the super-metric $\|x-y\| = \sup_{0\le t\le 1} |x(t)-y(t)|$. Classically, the weak convergence of probability measures in $C[0,1]$ is showed by verifying the convergence of the finite-dimensional distributions and the tightness of the probability measures. We will find that this way is also valid for proving Donsker's invariance principle in the sub-linear expectation space, though there is no longer any one-to-one relationship between the convergence of sub-linear expectations and the convergence of related capacities. In the next section, we state basic settings in a sub-linear expectation space including, capacity, independence, identical distribution, G-Brownian motion etc. The main result on Donsker's invariance principle is established in Section 3 by assuming that the convergence of finite-dimensional distributions and the tightness are proved. And also, Chung's law of the iterated logarithm is established by assuming the small deviations. We consider the convergence of the finite-dimensional distributions in Section 4, and consider the tightness in Section 5. In Section 6, we establish the small deviations by applying Donsker's invariance principle. All of the results are established of only the existence of second moments is assumed.

2 Basic Settings

We use the framework and notations of Peng (2008b). Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathscr{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, \ldots, X_n \in \mathscr{H}$ then $\varphi(X_1, \ldots, X_n) \in \mathscr{H}$ for each $\varphi \in C_b(\mathbb{R}_n) \bigcup C_{l,Lip}(\mathbb{R}_n)$, where $C_b(\mathbb{R}_n)$ denote the space of all bounded continuous functions and $C_{l,Lip}(\mathbb{R}_n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{y})| \le C(1 + |\boldsymbol{x}|^m + |\boldsymbol{y}|^m)|\boldsymbol{x} - \boldsymbol{y}|, \ \ \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_n,$$
$$\text{for some } C > 0, m \in \mathbb{N} \text{ depending on } \varphi.$$

$\mathscr{H}$ is considered as a space of "random variables". In this case we denote $X \in \mathscr{H}$. Further, we let $C_{b,Lip}(\mathbb{R}_n)$ denote the space of all bounded and Lipschitz functions on $\mathbb{R}_n$.

Remark 2.1 *It is easily seen that if $\varphi_1, \varphi_2 \in C_{l,Lip}(\mathbb{R}_n)$, then $\varphi_1 \vee \varphi_2, \varphi_1 \wedge \varphi_2 \in C_{l,Lip}(\mathbb{R}_n)$ because $\varphi_1 \vee \varphi_2 = \frac{1}{2}(\varphi_1 + \varphi_2 + |\varphi_1 - \varphi_2|)$, $\varphi_1 \wedge \varphi_2 = \frac{1}{2}(\varphi_1 + \varphi_2 - |\varphi_1 - \varphi_2|)$.*

2.1 Sub-linear expectation and capacity

Deriniton 2.1 *A* **sub-linear expectation** $\widehat{\mathbb{E}}$ *on $\mathscr{H}$ is a functional $\widehat{\mathbb{E}} : \mathscr{H} \to \overline{\mathbb{R}}$ satisfying the following properties: for all $X, Y \in \mathscr{H}$, we have*

(a) **Monotonicity**: *If $X \ge Y$ then $\widehat{\mathbb{E}}[X] \ge \widehat{\mathbb{E}}[Y]$;*

(b) **Constant preserving** : *$\widehat{\mathbb{E}}[c] = c$;*

(c) **Sub-additivity**: *$\widehat{\mathbb{E}}[X + Y] \le \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ whenever $\widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;*

(d) **Positive homogeneity**: $\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}[X]$, $\lambda \geq 0$.

Here $\overline{\mathbb{R}} = [-\infty, \infty]$. *The triple* $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ *is called a sub-linear expectation space. Give a sub-linear expectation* $\widehat{\mathbb{E}}$*, let us denote the conjugate expectation* $\widehat{\mathcal{E}}$ *of* $\widehat{\mathbb{E}}$ *by*

$$\widehat{\mathcal{E}}[X] := -\widehat{\mathbb{E}}[-X], \ \ \forall X \in \mathscr{H}.$$

From the definition, it is easily shown that $\widehat{\mathcal{E}}[X] \leq \widehat{\mathbb{E}}[X]$, $\widehat{\mathbb{E}}[X + c] = \widehat{\mathbb{E}}[X] + c$ and $\widehat{\mathbb{E}}[X - Y] \geq \widehat{\mathbb{E}}[X] - \widehat{\mathbb{E}}[Y]$ for all $X, Y \in \mathscr{H}$ with $\widehat{\mathbb{E}}[Y]$ being finite. Further, if $\widehat{\mathbb{E}}[|X|]$ is finite, then $\widehat{\mathcal{E}}[X]$ and $\widehat{\mathbb{E}}[X]$ are both finite.

Next, we introduce the capacities corresponding to the sub-linear expectations. Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \to [0, 1]$ is called a capacity if

$$V(\emptyset) = 0, \ V(\Omega) = 1 \ \text{ and } V(A) \leq V(B) \ \ \forall \ A \subset B, \ A, B \in \mathcal{G}.$$

It is called to be sub-additive if $V(A \bigcup B) \leq V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \bigcup B \in \mathcal{G}$.

Let $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ be a sub-linear space, and $\widehat{\mathcal{E}}$ be the conjugate expectation of $\widehat{\mathbb{E}}$. It is natural to define the capacity of a set A to be the sub-linear expectation of the indicator function I_A of A. However, I_A may be not in $\mathscr{H}$. So, we denote a pair $(\mathbb{V}, \mathcal{V})$ of capacities by

$$\mathbb{V}(A) := \inf\{\widehat{\mathbb{E}}[\xi] : I_A \leq \xi, \xi \in \mathscr{H}\}, \ \ \mathcal{V}(A) := 1 - \mathbb{V}(A^c), \ \ \forall A \in \mathcal{F},$$

where A^c is the complement set of A. Then

$$\begin{aligned} &\mathbb{V}(A) = \widehat{\mathbb{E}}[I_A], \ \ \mathcal{V}(A) = \widehat{\mathcal{E}}[I_A], \quad \text{if } I_A \in \mathscr{H} \\ &\widehat{\mathbb{E}}[f] \leq \mathbb{V}(A) \leq \widehat{\mathbb{E}}[g], \ \ \widehat{\mathcal{E}}[f] \leq \mathcal{V}(A) \leq \widehat{\mathcal{E}}[g], \quad \text{if } f \leq I_A \leq g, f, g \in \mathscr{H}. \end{aligned} \tag{2.1}$$

It is obvious that $\mathbb{V}$ is sub-additive. But $\mathcal{V}$ and $\widehat{\mathcal{E}}$ may be not sub-additive. However, we have

$$\mathcal{V}(A \bigcup B) \leq \mathcal{V}(A) + \mathbb{V}(B) \quad \text{and} \quad \widehat{\mathcal{E}}[X + Y] \leq \widehat{\mathcal{E}}[X] + \widehat{\mathbb{E}}[Y] \tag{2.2}$$

due to the fact that $\mathbb{V}(A^c \bigcap B^c) = \mathbb{V}(A^c \backslash B) \geq \mathbb{V}(A^c) - \mathbb{V}(B)$ and $\widehat{\mathbb{E}}[-X - Y] \geq \widehat{\mathbb{E}}[-X] - \widehat{\mathbb{E}}[Y]$.

Further, we define an extension of $\widehat{\mathbb{E}}^*$ of $\widehat{\mathbb{E}}$ by

$$\widehat{\mathbb{E}}^*[X] = \inf\{\widehat{\mathbb{E}}[Y] : X \leq Y, \ Y \in \mathscr{H}\}, \ \ \forall X : \Omega \to \mathbb{R},$$

where $\inf \emptyset = +\infty$. Then

$$\widehat{\mathbb{E}}^*[X] = \widehat{\mathbb{E}}[X] \ \text{ if } X \in \mathscr{H}, \quad \mathbb{V}(A) = \widehat{\mathbb{E}}^*[I_A],$$

$$\widehat{\mathbb{E}}[f] \leq \widehat{\mathbb{E}}^*[X] \leq \widehat{\mathbb{E}}[g] \quad \text{if } f \leq X \leq g, f, g \in \mathscr{H}.$$

Deriniton 2.2 (I) *A sub-linear expectation* $\widehat{\mathbb{E}}: \mathscr{H} \to \mathbb{R}$ *is called to be countably sub-additive if it satisfies*

(e) **Countable sub-additivity**: $\widehat{\mathbb{E}}[X] \le \sum_{n=1}^{\infty} \widehat{\mathbb{E}}[X_n]$, *whenever* $X \le \sum_{n=1}^{\infty} X_n$, $X, X_n \in \mathscr{H}$ *and* $X \ge 0, X_n \ge 0$, $n = 1, 2, \ldots$;

It is called to continuous if it satisfies

(f) **Continuity from below**: $\widehat{\mathbb{E}}[X_n] \uparrow \widehat{\mathbb{E}}[X]$ *if* $0 \le X_n \uparrow X$, *where* $X_n, X \in \mathscr{H}$;

(g) **Continuity from above**: $\widehat{\mathbb{E}}[X_n] \downarrow \widehat{\mathbb{E}}[X]$ *if* $0 \le X_n \downarrow X$, *where* $X_n, X \in \mathscr{H}$.

(II) *A function* $V : \mathcal{F} \to [0, 1]$ *is called to be countably sub-additive if*

$$V\Big(\bigcup_{n=1}^{\infty} A_n\Big) \le \sum_{n=1}^{\infty} V(A_n) \ \ \forall A_n \in \mathcal{F}.$$

(III) *A capacity* $V : \mathcal{F} \to [0, 1]$ *is called a continuous capacity if it satisfies*

(III1) **Ccontinuity from below**: $V(A_n) \uparrow V(A)$ *if* $A_n \uparrow A$, *where* $A_n, A \in \mathcal{F}$;

(III2) **Continuity from above**: $V(A_n) \downarrow V(A)$ *if* $A_n \downarrow A$, *where* $A_n, A \in \mathcal{F}$.

It is obvious that a continuous sub-additive capacity V (resp. a sub-linear expectation $\widehat{\mathbb{E}}$) is countably sub-additive.

2.2 Independence and distribution

Deriniton 2.3 *(*Peng (2006, 2008b)*)*

(i) *(***Identical distribution***) Let* $\boldsymbol{X}_1$ *and* $\boldsymbol{X}_2$ *be two* n*-dimensional random vectors defined respectively in sub-linear expectation spaces* $(\Omega_1, \mathscr{H}_1, \widehat{\mathbb{E}}_1)$ *and* $(\Omega_2, \mathscr{H}_2, \widehat{\mathbb{E}}_2)$. *They are called identically distributed, denoted by* $\boldsymbol{X}_1 \overset{d}{=} \boldsymbol{X}_2$ *if*

$$\widehat{\mathbb{E}}_1[\varphi(\boldsymbol{X}_1)] = \widehat{\mathbb{E}}_2[\varphi(\boldsymbol{X}_2)], \ \ \forall \varphi \in C_{l,Lip}(\mathbb{R}_n),$$

whenever the sub-expectations are finite. A sequence $\{X_n; n \ge 1\}$ *of random variables is said to be identically distributed if* $X_i \overset{d}{=} X_1$ *for each* $i \ge 1$.

(ii) (**Independence**) *In a sub-linear expectation space* $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$, *a random vector* $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$, $Y_i \in \mathscr{H}$ *is said to be independent to another random vector* $\boldsymbol{X} = (X_1, \ldots, X_m)$, $X_i \in \mathscr{H}$ *under* $\widehat{\mathbb{E}}$ *if for each test function* $\varphi \in C_{l,Lip}(\mathbb{R}_m \times \mathbb{R}_n)$ *we have*

$$\widehat{\mathbb{E}}[\varphi(\boldsymbol{X}, \boldsymbol{Y})] = \widehat{\mathbb{E}}\big[\widehat{\mathbb{E}}[\varphi(\boldsymbol{x}, \boldsymbol{Y})]\big|_{\boldsymbol{x}=\boldsymbol{X}}\big],$$

whenever $\overline{\varphi}(\boldsymbol{x}) := \widehat{\mathbb{E}}\left[|\varphi(\boldsymbol{x}, \boldsymbol{Y})|\right] < \infty$ *for all* $\boldsymbol{x}$ *and* $\widehat{\mathbb{E}}\left[|\overline{\varphi}(\boldsymbol{X})|\right] < \infty$.

(iii) (**IID random variables**) *A sequence of random variables* $\{X_n; n \geq 1\}$ *is said to be independent and identically distributed (IID), if* $X_i \overset{d}{=} X_1$ *and* X_{i+1} *is independent to* $(X_1, \ldots, X_i)$ *for each* $i \geq 1$.

2.3 G-normal distribution and G-Brownian motion

Let $0 \leq \underline{\sigma} \leq \overline{\sigma} < \infty$ and $G(\alpha) = \frac{1}{2}(\overline{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-)$. X is call a normal $N\big(0, [\underline{\sigma}^2, \overline{\sigma}^2]\big)$ distributed random variable (write $X \sim N\big(0, [\underline{\sigma}^2, \overline{\sigma}^2]\big)$) under $\widehat{\mathbb{E}}$, if for any $\varphi \in C_b(\mathbb{R})$, the function $u(x,t) = \widehat{\mathbb{E}}\left[\varphi\left(x + \sqrt{t}X\right)\right]$ $(x \in \mathbb{R}, t \geq 0)$ satisfies the following heat equation:

$$\partial_t u - G\left(\partial^2_{xx} u\right) = 0, \;\; u(0,x) = \varphi(x).$$

Let $C[0,1]$ be a function space of continuous functions on $[0,1]$ equipped with the supernorm $\|x\| = \sup\limits_{0 \leq t \leq 1} |x(t)|$ and $C_b\big(C[0,1]\big)$ is the set of bounded continuous functions $h(x) : C[0,1] \to \mathbb{R}$.The modulus of the continuity of an element $x \in C[0,1]$ is defined by

$$\omega_\delta(x) = \sup_{|t-s|<\delta} |x(t) - x(s)|.$$

Denis, Hu and Peng (2011) showed that there is a sub-linear expectation space $\big(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathbb{E}}\big)$ with $\widetilde{\Omega} = C[0,1]$ and $C_b\big(C[0,1]\big) \subset \widetilde{\mathscr{H}}$ such that $\widetilde{\mathbb{E}}$ is countably sub-additive, and the canonical process $W(t)(\omega) = \omega_t (\omega \in \widetilde{\Omega})$ is a G-Brownian motion with $W(1) \sim N\big(0, [\underline{\sigma}^2, \overline{\sigma}^2]\big)$ under $\widetilde{\mathbb{E}}$, i.e., for all $0 \leq t_1 < \ldots < t_n \leq 1$, $\varphi \in C_{b,lip}(\mathbb{R}^n)$,

$$\widetilde{\mathbb{E}}\left[\varphi\big(W(t_1), \ldots, W(t_{n-1}), W(t_n) - W(t_{n-1})\big)\right] = \widetilde{\mathbb{E}}\left[\psi\big(W(t_1), \ldots, W(t_{n-1})\big)\right], \tag{2.3}$$

where $\psi(x_1, \ldots, x_{n-1})\big) = \widetilde{\mathbb{E}}\left[\varphi\big(x_1, \ldots, x_{n-1}, \sqrt{t_n - t_{n-1}} W(1)\big)\right]$.

Denis, Hu and Peng (2011) also showed the following representation of the G-Brownian motion (c.f, Theorem 52).

Lemma 2.1 *Let $(\Omega, \mathscr{F}, P)$ be a probability measure space and $\{B(t)\}_{t\geq 0}$ is a P-Brownian motion. Then for all $\varphi \in C_b(\widetilde{\Omega})$,*

$$\widetilde{\mathbb{E}}\left[\varphi(W)\right] = \sup_{\theta\in\Theta} \mathsf{E}_P\left[\varphi\left(W_\theta\right)\right], \quad W_\theta(t) = \int_0^t \theta(s)dB(s),$$

where

$$\Theta = \left\{\theta : \theta(t) \text{ is } \mathscr{F}_t\text{-adapted process such that } \underline{\sigma} \leq \theta(t) \leq \overline{\sigma}\right\},$$

$$\mathscr{F}_t = \sigma\{B(s) : 0 \leq s \leq t\} \vee \mathscr{N}, \quad \mathscr{N} \text{ is the collection of } P\text{-null subsets.}$$

We we denote a pair of capacities corresponding to the sub-linear expectation $\widetilde{\mathbb{E}}$ by $(\widetilde{\mathbb{V}}, \widetilde{\mathcal{V}})$, and the extension of $\widetilde{\mathbb{E}}$ by $\widetilde{\mathbb{E}}^*$. By using Lemma 2.1, one can show that

$$\widetilde{\mathbb{E}}^*\left[\sup_{0\leq t\leq 1}|W(t)|^p\right] = \overline{\sigma}^p E_P\left[\sup_{0\leq t\leq 1}|B(t)|^p\right]$$

and for $x \geq 0$

$$\widetilde{\mathbb{V}}\left(\sup_{0\leq t\leq 1}|W(t)| \geq x\right) = P\left(\sup_{0\leq t\leq 1}\overline{\sigma}|B(t)| \geq x\right), \tag{2.4}$$

$$\widetilde{\mathcal{V}}\left(\sup_{0\leq t\leq 1}|W(t)| \geq x\right) = P\left(\sup_{0\leq t\leq 1}\underline{\sigma}|B(t)| \geq x\right),$$

$$\widetilde{\mathbb{V}}\left(\sup_{0\leq t\leq 1}W(t) \geq x\right) = 2P\left(\overline{\sigma}B(1) \geq x\right),$$

$$\widetilde{\mathcal{V}}\left(\sup_{0\leq t\leq 1}W(t) \geq x\right) = 2P\left(\underline{\sigma}B(1) \geq x\right)$$

(c.f, Lemma 6.1 below).

In the sequel of this paper, the sequence $\{X_n; n \geq 1\}$ of the random variables are considered in $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ and Brownian motions are considered in $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathbb{E}})$. We suppose $\{X_n; n \geq 1\}$ is a sequence of independent and identically distributed random variables in $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ with $\widehat{\mathbb{E}}[X_1] = \widehat{\mathcal{E}}[X_1] = 0$, $\widehat{\mathbb{E}}[X_1^2] = \overline{\sigma}^2$ and $\widehat{\mathcal{E}}[X_1^2] = \underline{\sigma}^2$, and suppose $W(t)$ is a G-Brownian motion on $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathbb{E}})$ with $W(1) \sim N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$. Denote $S_0 = 0$, $S_n = \sum_{k=1}^n X_k$.

3 Main results

Define the $C[0,1]$-valued random variable W_n by setting

$$W_n(t) = \begin{cases} S_k/\sqrt{n}, & \text{if } t = k/n \ (k = 0, 1, \ldots, n); \\ \text{extended by linear interpolation in each interval} \\ \qquad \big[[k-1]n^{-1}, kn^{-1}\big]. \end{cases}$$

Our first result is the following Donsker's invariance principle, or called the functional central limit theorem.

Theorem 3.1 *Suppose* $\widehat{\mathbb{E}}[(X_1^2-b)^+]\to 0$ *as* $b\to\infty$. *Then for all bounded continuous function* $\varphi: C[0,1]\to\mathbb{R}$,

$$\widehat{\mathbb{E}}\left[\varphi(W_n)\right]\to\widetilde{\mathbb{E}}\left[\varphi(W)\right]. \tag{3.1}$$

Remark 3.1 *Note* $C_b\left(C[0,1]\right)\subset\widetilde{\mathscr{H}}$. *The expectation on the right hands of (3.1) is well-defined. On the other hand, the map* $g:(S_1/\sqrt{n},\ldots,S_n/\sqrt{n})\to W_n(t)$ *is a continuous one-to-one map of* $(S_1/\sqrt{n},\ldots,S_n/\sqrt{n})$. *So,* $\varphi\circ g\in C_b(\mathbb{R}^n)$. *It follows that* $\varphi(W_n)\in\mathscr{H}$. *So, even though* $\widehat{\mathbb{E}}$ *has no definition on* $C_b\left(C[0,1]\right)$, *the expectation on the left hand of (3.1) is well-defined.*

There are there immediate corollaries of Theorem 3.1.

Corollary 3.1 *Suppose* $\widehat{\mathbb{E}}[(X_1^2-b)^+]\to 0$ *as* $b\to\infty$. *If* $h_n(x,y)$ *and* $h(x,y)$: $C[0,1]\times C[0,1]\to\mathbb{R}$ *are bounded continuous functions for which*

$$|h_n(x,y)|\le M,\ \ |h(x,y)|\le M,$$

$$h_n(x_n,y_n)\to h(x,y)\ \text{whenever}\ x_n\to x, y_n\to y,$$

then

$$\widehat{\mathbb{E}}\left[h_n(W_n,y)\right]\to\widetilde{\mathbb{E}}\left[h(W,y)\right]\quad\text{uniformly in } y\in K, \tag{3.2}$$

for any compact set $K\subset C[0,1]$.

Corollary 3.2 *Suppose* $p\ge 2$ *and* $\widehat{\mathbb{E}}[(|X_1|^p-b)^+]\to 0$ *as* $b\to\infty$. *Then for all continuous function* $\varphi: C[0,1]\to\mathbb{R}$ *with* $|\varphi(x)|\le C(1+\|x\|^p)$,

$$\widehat{\mathbb{E}}^*\left[\varphi(W_n)\right]\to\widetilde{\mathbb{E}}^*\left[\varphi(W)\right] \tag{3.3}$$

where $\widehat{\mathbb{E}}^*$ *and* $\widetilde{\mathbb{E}}^*$ *are extensions of* $\widehat{\mathbb{E}}$ *and* $\widetilde{\mathbb{E}}$, *respectively. In particular,*

$$\widehat{\mathbb{E}}\left[\max_{k\le n}\left|\frac{S_k}{\sqrt{n}}\right|^p\right]\to\overline{\sigma}^pE_P[\sup_{0\le t\le 1}|B(t)|^p],\ \ \widehat{\mathcal{E}}\left[\max_{k\le n}\left|\frac{S_k}{\sqrt{n}}\right|^p\right]\to\underline{\sigma}^pE_P[\sup_{0\le t\le 1}|B(t)|^p].$$

Corollary 3.3 *Suppose* $\widehat{\mathbb{E}}[(X_1^2-b)^+]\to 0$ *as* $b\to\infty$. *Then for all* $x>0$,

$$\mathbb{V}\left(\max_{k\le n}|S_k|/\sqrt{n}\ge x\right)\to P\left(\sup_{0\le t\le 1}\overline{\sigma}|B(t)|\ge x\right),$$

$$\mathcal{V}\left(\max_{k\le n}|S_k|/\sqrt{n}|\ge x\right)\to P\left(\sup_{0\le t\le 1}\underline{\sigma}|B(t)|\ge x\right),$$

$$\mathbb{V}\left(\max_{k\le n}S_k/\sqrt{n}\ge x\right)\to 2P\left(\overline{\sigma}B(1)\ge x\right),$$

$$\mathcal{V}\left(\max_{k\le n}S_k/\sqrt{n}\ge x\right)\to 2P\left(\underline{\sigma}B(t)\ge x\right).$$

Note

$$\lim_{x\to 0+}x^2\log P\left(\sup_{0\le t\le 1}|B(t)|\le x\right)=-\frac{\pi^2}{8}.$$

From Corollary 3.3, we conclude that

$$\lim_{x\to 0+}x^2\lim_{n\to\infty}\log\mathbb{V}\left(\max_{k\le n}|S_k|/\sqrt{n}\le x\right)=-\frac{\pi^2\underline{\sigma}^2}{8},$$

$$\lim_{x\to 0+}x^2\lim_{n\to\infty}\log\mathcal{V}\left(\max_{k\le n}|S_k|/\sqrt{n}\le x\right)=-\frac{\pi^2\overline{\sigma}^2}{8}.$$

In Section 6, we will prove more accurate results refereed to as the small deviations:

$$\log\mathbb{V}\left(\max_{k\le n}|S_k|\le n^{1/2}x_n\right)\sim-\frac{\pi^2\underline{\sigma}^2}{8x_n^2}, \tag{3.4}$$

$$\log\mathcal{V}\left(\max_{k\le n}|S_k|\le n^{1/2}x_n\right)\sim-\frac{\pi^2\overline{\sigma}^2}{8x_n^2} \tag{3.5}$$

whenever $0<x_n\to 0$ and $\sqrt{n}x_n\to\infty$. By the small deviations, we obtain our another main result which gives Chung's law of the iterated logarithm.

Theorem 3.2 *Suppose* $\widehat{\mathbb{E}}[(X_1^2-b)^+]\to 0$ *as* $b\to\infty$ *and* $\mathbb{V}$ *is continuous. Then*

$$\mathcal{V}\left(\underline{\sigma}\le\liminf_{n\to\infty}\sqrt{\frac{8\log\log n}{n\pi^2}}\max_{k\le n}|S_k|\le\overline{\sigma}\right)=1 \tag{3.6}$$

and

$$\mathbb{V}\left(\liminf_{n\to\infty}\sqrt{\frac{8\log\log n}{n\pi^2}}\max_{k\le n}|S_k|=\underline{\sigma}\right)=1. \tag{3.7}$$

Next, we give a sketch of the proof of Theorems 3.1, 3.2 and Corollaries 3.1- 3.3. We need the following Rosenthal type inequalities under $\widehat{\mathbb{E}}$ which have been obtained by Zhang (2014a).

Lemma 3.1 (Rosnethal's inequality) *Let $\{X_1, \ldots, X_n\}$ be a sequence of independent random variables in $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$. Then*

$$\begin{aligned}\widehat{\mathbb{E}}\left[\max_{k\le n}|S_k|^p\right] \le & C_p\left\{\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] + \left(\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^2]\right)^{p/2}\right. \\ & \left. + \left(\sum_{k=1}^n \left\{\left(\widehat{\mathcal{E}}[X_k]\right)^- + \left(\widehat{\mathbb{E}}[X_k]\right)^+\right\}\right)^p\right\}, \quad \text{for } p\ge 2. \end{aligned} \tag{3.8}$$

In particular, if $\widehat{\mathbb{E}}[X_k] = \widehat{\mathbb{E}}[-X_k] = 0$, $k = 1, \ldots, n$, then

$$\widehat{\mathbb{E}}\left[\max_{k\le n}|S_k|^p\right] \le C_p\left\{\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^p] + \left(\sum_{k=1}^n \widehat{\mathbb{E}}[|X_k|^2]\right)^{p/2}\right\}, \quad \text{for } p\ge 2.$$

Proof of Theorem 3.1. The proof is based on the convergence of the finite-dimensional distributions of W_n under $\widehat{\mathbb{E}}$ and the tightness of W_n under $\mathbb{V}$ which are given in Section 4 and Section 5, respectively. Here, we give the proof of theorem after assuming these results.

For $0 < t_1 < t_2 \ldots < t_d \le 1$, we define the projection $\pi_{t_1,\ldots,t_d}$ from $C[0,1]$ to $\mathbb{R}^d$ by

$$\pi_{t_1,\ldots,t_d} x = (x(t_1), \ldots, x(t_d)),$$

and define a map $\Pi^{-1}_{t_1,\ldots,t_d}$ from $\mathbb{R}^d$ to $C[0,1]$ by

$$\Pi^{-1}_{t_1,\ldots,t_d}(x_1,\ldots,x_d) = \begin{cases} 0, \ \text{ if } t=0; \ \ x_k, \ \text{ if } t = t_k \ (k=0,1,\ldots,n); \\ \text{extended by linear interpolation in each interval} \\ \qquad [t_{k-1}, t_k]. \end{cases}$$

Then $\pi_{t_1,\ldots,t_d}$ and $\Pi^{-1}_{t_1,\ldots,t_d}$ are both continuous maps. Denote $\widetilde{\pi}_{t_1,\ldots,t_d} = \Pi^{-1}_{t_1,\ldots,t_d} \circ \pi_{t_1,\ldots,t_d}$. Then $\widetilde{\pi}_{t_1,\ldots,t_d} : C[0,1] \to C[0,1]$ is continuous and

$$W_n = \Pi^{-1}_{1/n,2/n,\ldots,n/n}\big(S_1/\sqrt{n}, S_2/\sqrt{n}, \ldots, S_n/\sqrt{n}\big).$$

Let $\varphi \in C_b\big(C[0,1]\big)$. Then $\varphi(\widetilde{\pi}_{t_1,\ldots,t_d} x) = \varphi \circ \Pi^{-1}_{t_1,\ldots,t_d}(x(t_1),\ldots,x(t_d))$ and $\varphi\circ\Pi^{-1}_{t_1,\ldots,t_d} \in C_b(\mathbb{R}^d)$. By Theorem 4.1 on the convergence of the finite-dimensional distributions of W_n, it follows that

$$\begin{aligned}\lim_{n\to\infty}\widehat{\mathbb{E}}\left[\varphi\left(\widetilde{\pi}_{t_1,\ldots,t_d}W_n\right)\right] &= \lim_{n\to\infty}\widehat{\mathbb{E}}\left[\varphi\circ\Pi^{-1}_{t_1,\ldots,t_d}\left(W_n(t_1),\ldots,W_n(t_d)\right)\right] \\ &= \widehat{\mathbb{E}}\left[\varphi\circ\Pi^{-1}_{t_1,\ldots,t_d}\left(W(t_1),\ldots,W(t_d)\right)\right] = \widehat{\mathbb{E}}\left[\varphi\left(\widetilde{\pi}_{t_1,\ldots,t_d}W\right)\right].\end{aligned}$$

Now, let $t_0 = 0$, $t_{d+1} = 1$, and suppose that $t_{i+1} - t_i < \delta$ for $i = 0, \ldots, d$. Recall $\omega_\delta(x) = \sup_{|t-s|<\delta} |x(t) - x(s)|$ and $\|x\| = \sup_{0\le t\le 1} |x(t)|$. It is easily seen that $\|\widetilde{\pi}_{t_1,\ldots,t_d} x - x\| \le \omega_\delta(x)$. Let $\epsilon > 0$ be given. Since φ is a continuous function, for each x, there is an $\epsilon_x > 0$ such that

$$|\varphi(x) - \varphi(y)| < \epsilon \text{ whenever } \|y - x\| < \epsilon_x.$$

Let $K \subset C[0,1]$ is a compact set. Then it can be covered by a union of finite many of the sets $\{y : \|y - x\| < \epsilon_x\}$, $x \in K$. So, there is an $\epsilon_K > 0$ such that $|\varphi(x) - \varphi(y)| < \epsilon$ whenever $\|y - x\| < \epsilon_K$ and $x \in K$. Denote $M = \sup_x |\varphi(x)|$. It follows that

$$|\varphi(\widetilde{\pi}_{t_1,\ldots,t_d} x) - \varphi(x)| < \epsilon + 2MI\{\omega_\delta(x) \ge \epsilon_K\} + 2MI\{x \notin K\}.$$

By Theorems 5.1 and 5.2 on the tightness of $\{W_n\}$ and W, respectively, we can choose K and δ such that

$$\sup_n \mathbb{V}(\omega_\delta(W_n) \ge \epsilon_K) + \sup_n \mathbb{V}(W_n \notin K) \le \frac{\epsilon}{4M} \quad \text{and}$$
$$\widetilde{\mathbb{V}}(\omega_\delta(W) \ge \epsilon_K) + \widetilde{\mathbb{V}}(W \notin K) \le \frac{\epsilon}{4M}.$$

Hence

$$\begin{aligned}
&\left|\widehat{\mathbb{E}}[\varphi(W_n)] - \widetilde{\mathbb{E}}[\varphi(W)]\right| \\
\le &\left|\widehat{\mathbb{E}}\left[\varphi(\widetilde{\pi}_{t_1,\ldots,t_d} W_n)\right] - \widetilde{\mathbb{E}}\left[\varphi(\widetilde{\pi}_{t_1,\ldots,t_d} W)\right]\right| \\
&+ \left|\widehat{\mathbb{E}}[\varphi(W_n)] - \widehat{\mathbb{E}}[\varphi(\widetilde{\pi}_{t_1,\ldots,t_d} W_n)]\right| + \left|\widetilde{\mathbb{E}}\left[\varphi(\widetilde{\pi}_{t_1,\ldots,t_d} W)\right] - \widetilde{\mathbb{E}}[\varphi(W)]\right| \\
\le &\left|\widehat{\mathbb{E}}\left[\varphi(\widetilde{\pi}_{t_1,\ldots,t_d} W_n)\right] - \widetilde{\mathbb{E}}\left[\varphi(\widetilde{\pi}_{t_1,\ldots,t_d} W)\right]\right| \\
&+ 2\epsilon + 2M\mathbb{V}(\omega_\delta(W_n) \ge \epsilon_K) + 2M\mathbb{V}(W_n \notin K) \\
&+ 2M\widetilde{\mathbb{V}}(\omega_\delta(W) \ge \epsilon_K) + 2M\widetilde{\mathbb{V}}(W \notin K) \\
\le &\left|\widehat{\mathbb{E}}\left[\varphi(\widetilde{\pi}_{t_1,\ldots,t_d} W_n)\right] - \widetilde{\mathbb{E}}\left[\varphi(\widetilde{\pi}_{t_1,\ldots,t_d} W)\right]\right| + 3\epsilon.
\end{aligned}$$

Letting $n \to \infty$ and then $\epsilon \to 0$ completes the proof of (3.1). □

Remark 3.2 *Here we give a direct proof of Theorem 3.1. In Section 5, we will give another proof by using the results of Peng (2010).*

Proof of Corollary 3.1. Let $\{\delta_n\}$ be a sequence of positive numbers with $\delta_n \downarrow 0$, and let $y_n \in K$ such that

$$\left|\widehat{\mathbb{E}}[h_n(W_n, y_n)] - \widetilde{\mathbb{E}}[h(W, y_n)]\right| \ge \sup_{y\in K} \left|\widehat{\mathbb{E}}[h_n(W_n, y)] - \widetilde{\mathbb{E}}[h(W, y)]\right| - \delta_n.$$

Since K is compact, any subsequence of $\{y_n\}$ has a further convergent subsequence. Without loss of generality, we assume $y_n \to y \in K$. Let $g_n(x) = h_n(x, y_n)$ and $g(x) = h(x, y)$. Then $g_n(x_n) \to g(x)$ whenever $x_n \to x$, which implies that for any compact set $K_1 \subset C[0,1]$, $\sup_{x\in K_1} |g_n(x) - g(x)| \to 0$. For any $\epsilon > 0$, by Theorem 5.1, one can choose K_1 such that $\sup_n \mathbb{V}(W_n \notin K_1) < \epsilon/(2M)$. It follows that

$$\sup_{y\in K} \left| \widehat{\mathbb{E}}\left[h_n(W_n, y)\right] - \widetilde{\mathbb{E}}\left[h(W, y)\right] \right| \le \left| \widehat{\mathbb{E}}\left[g_n(W_n)\right] - \widetilde{\mathbb{E}}\left[g(W)\right] \right| + \delta_n$$

$$\le \left| \widehat{\mathbb{E}}\left[g_n(W_n)\right] - \widehat{\mathbb{E}}\left[g(W_n)\right] \right| + \left| \widehat{\mathbb{E}}\left[g(W_n)\right] - \widetilde{\mathbb{E}}\left[g(W)\right] \right| + \delta_n$$

$$\le \left| \widehat{\mathbb{E}}\left[g(W_n)\right] - \widetilde{\mathbb{E}}\left[g(W)\right] \right| + \sup_{x\in K_1} |g_n(x) - g(x)| + 2M \cdot \epsilon/(2M) + \delta_n.$$

From (5.1), it follows that

$$\limsup_{n\to\infty} \sup_{y\in K} \left| \widehat{\mathbb{E}}\left[h_n(W_n, y)\right] - \widetilde{\mathbb{E}}\left[h(W, y)\right] \right| \le \epsilon.$$

The proof is now completed. □

Proof of Corollary 3.2. For $\lambda > 0$, $\varphi_\lambda = (-\lambda) \vee (\varphi(x) \wedge \lambda) \in C_b(C[0,1])$. So, by Theorem 3.1,

$$\widehat{\mathbb{E}}^*\left[\varphi_\lambda(W_n)\right] = \widehat{\mathbb{E}}\left[\varphi_\lambda(W_n)\right] \to \widetilde{\mathbb{E}}\left[\varphi_\lambda(W)\right] = \widetilde{\mathbb{E}}^*\left[\varphi_\lambda(W)\right].$$

On the other hand,

$$\left| \widetilde{\mathbb{E}}^*\left[\varphi_\lambda(W)\right] - \widetilde{\mathbb{E}}^*\left[\varphi(W)\right] \right| \le \widetilde{\mathbb{E}}^*\left[(|\varphi_\lambda(W)| - \lambda)^+\right]$$

$$\le C\widetilde{\mathbb{E}}^*\left[(\|W\|^p - \lambda/C + 1)^+\right] \to 0 \text{ as } \lambda \to 0$$

and

$$\left| \widehat{\mathbb{E}}^*\left[\varphi_\lambda(W_n)\right] - \widehat{\mathbb{E}}^*\left[\varphi(W_n)\right] \right| \le C\widehat{\mathbb{E}}^*\left[(\|W_n\|^p - \lambda/C + 1)^+\right]$$

$$= C\widehat{\mathbb{E}}\left[\left(\max_{k\le n}\left|\frac{S_k}{\sqrt{n}}\right|^p - \lambda/C + 1\right)^+\right].$$

It is sufficient to show

$$\lim_{\lambda\to\infty} \limsup_{n} \widehat{\mathbb{E}}\left[\left(\max_{k\le n}\left|\frac{S_k}{\sqrt{n}}\right|^p - \lambda\right)^+\right] = 0, \tag{3.9}$$

i.e., the sequence $\{\max_{k\le n} |S_k/\sqrt{n}|^p; n \ge 1\}$ is uniformly integrable under $\widehat{\mathbb{E}}$. Let $Y_j = (-\sqrt{n}) \vee (X_j \wedge \sqrt{n})$, $\widehat{X}_j = X_j - Y_j$, $T_j = \sum_{i=1}^j Y_i$, $\widehat{S}_j = \sum_{i=1}^j \widehat{X}_i$, $j = 1, \ldots, n$. Then

$\max_{k\le n}|S_k|\le \max_{k\le n}|\widehat{S}_k|+\max_{k\le n}|T_k|$. Note $\widehat{\mathbb{E}}[X_1]=\widehat{\mathcal{E}}[X_1]=0$. So, $|\widehat{\mathcal{E}}[Y_1]|=|\widehat{\mathcal{E}}[X_1]-\widehat{\mathcal{E}}[Y_1]|\le \widehat{\mathbb{E}}|\widehat{X}_1|=\widehat{\mathbb{E}}[(|X_1|^2-n)^+]n^{-1/2}$ and $|\widehat{\mathbb{E}}[Y_1]|=|\widehat{\mathbb{E}}[X_1]-\widehat{\mathbb{E}}[Y_1]|\le \widehat{\mathbb{E}}|\widehat{X}_1|=\widehat{\mathbb{E}}[(|X_1|^2-n)^+]n^{-1/2}$. By Rosnethal's inequality (c.f, (3.8)),

$$\begin{aligned}\widehat{\mathbb{E}}\left[\max_{k\le n}|T_k|^{2p}\right]&\le C_p\left\{n\widehat{\mathbb{E}}[|Y_1|^{2p}]+\left(n\widehat{\mathbb{E}}[|Y_1|^2]\right)^p+\left(n\left[\left(\widehat{\mathcal{E}}[Y_1]\right)^-+\left(\widehat{\mathbb{E}}[Y_1]\right)^+\right]\right)^{2p}\right\}\\ &\le C_p\left\{nn^{p/2}\widehat{\mathbb{E}}[|X_1|^p]+n^p\left(\widehat{\mathbb{E}}[X_1^2]\right)^p+\left(nn^{-1/2}\widehat{\mathbb{E}}\left[(X_1^2-n)^+\right]\right)^{2p}\right\}\le Cn^p\end{aligned}$$

and

$$\begin{aligned}\widehat{\mathbb{E}}\left[\max_{k\le n}\left|\widehat{S}_k\right|^p\right]\le &C_p\left\{n\widehat{\mathbb{E}}[|\widehat{X}_1|^p]+\left(n\widehat{\mathbb{E}}[|\widehat{X}_1|^2]\right)^{p/2}+\left(n\left[\left(\widehat{\mathcal{E}}[\widehat{X}_1]\right)^-+\left(\widehat{\mathbb{E}}[\widehat{X}_1]\right)^+\right]\right)^p\right\}\\ \le &C_p\left\{n\widehat{\mathbb{E}}\left[\left(|X_1|^p-n^{p/2}\right)^+\right]+n^{p/2}\left(\widehat{\mathbb{E}}\left[(X_1^2-n)^+\right]\right)^p\right\}.\end{aligned}$$

It follows that

$$\begin{aligned}&\lim_{\lambda\to\infty}\limsup_n\widehat{\mathbb{E}}\left[\left(\max_{k\le n}\left|\frac{T_k}{\sqrt{n}}\right|^p-\lambda\right)^+\right]\\ \le&\lim_{\lambda\to\infty}\limsup_n\frac{1}{\lambda}\widehat{\mathbb{E}}\left[\max_{k\le n}\left|\frac{T_k}{\sqrt{n}}\right|^{2p}\right]\le\lim_{\lambda\to\infty}\frac{C}{\lambda}=0\end{aligned}$$

and

$$\limsup_n\widehat{\mathbb{E}}\left[\max_{k\le n}\left|\frac{\widehat{S}_k}{\sqrt{n}}\right|^p\right]=0.$$

The proof of (3.9) is completed. □

Proof of Corollary 3.3. We only give the proof of the first result. Let $\varphi(y)$ be a Lipschitz function such that $I\{y\ge x\}\le\varphi(y)\le I\{y\ge x(1+\delta)\}$. Then by Theorem 3.1,

$$\begin{aligned}\limsup_{n\to\infty}\mathbb{V}\left(\max_{k\le n}|S_k|/\sqrt{n}\ge x\right)=&\limsup_{n\to\infty}\mathbb{V}\left(\sup_{0\le t\le 1}|W_n(t)|\ge x\right)\\ \le\limsup_{n\to\infty}\widehat{\mathbb{E}}\left[\varphi\left(\sup_{0\le t\le 1}|W_n(t)|\right)\right]=&\widetilde{\mathbb{E}}\left[\varphi\left(\sup_{0\le t\le 1}|W(t)|\right)\right]\\ \le\widetilde{\mathbb{V}}\left(\sup_{0\le t\le 1}|W(t)|\ge x(1+\delta)\right)=&P\left(\sup_{0\le t\le 1}\overline{\sigma}|B(t)|\ge x(1+\delta)\right),\end{aligned}$$

where the last inequality is due to (2.4). Letting $\delta\to 0$ yields

$$\limsup_{n\to\infty}\mathbb{V}\left(\max_{k\le n}|S_k|/\sqrt{n}\ge x\right)\le P\left(\sup_{0\le t\le 1}\overline{\sigma}|B(t)|\ge x\right).$$

By considering a function $\varphi(y)$ with $I\{y\ge x(1-\delta)\}\le\varphi(y)\le I\{y\ge x\}$ instead, we can show that

$$\liminf_{n\to\infty}\mathbb{V}\left(\max_{k\le n}|S_k|/\sqrt{n}\ge x\right)\ge P\left(\sup_{0\le t\le 1}\overline{\sigma}|B(t)|\ge x\right).$$

The proof is completed. □.

Proof of Theorem 3.2. The proof is based on small deviations for $\max_{i\le n}|S_i|$ which are proved in Section 6. Let $0<\epsilon<1/2$, $\beta(n)=\sqrt{\frac{n\pi^2}{8\log\log n}}$ and $x_n=(1+\epsilon)^{-1}\underline{\sigma}\beta(n)/\sqrt{n}$. Then by (3.4) (c.f, Theorem 6.1),

$$\log\mathbb{V}\Big(\max_{i\le n}|S_i|\le\beta(n)(1+\epsilon)^{-1}\underline{\sigma}\Big)\sim-\frac{\pi^2\underline{\sigma}^2}{8x_n^2}\sim-(1+\epsilon)^2\log\log n.$$

Let $n_k=[e^{k/\log k}]$. Then $n_{k-1}/n_k\to 1$, $\beta(n_{k-1})/\beta(n_k)\to 1$ and $\log\log n_k\sim\log k$. It follows that

$$\sum_{k=1}^{\infty}\mathbb{V}\Big(\max_{i\le n_k}|S_i|\le\beta(n_k)(1+\epsilon)^{-1}\underline{\sigma}\Big)\le C\sum_{k=1}^{\infty}k^{-(1+\epsilon)}<\infty.$$

Hence by the countable sub-additivity of $\mathbb{V}$,

$$\begin{aligned}&\mathbb{V}\Big(\max_{i\le n_k}|S_i|\le\beta(n_k)(1+\epsilon)^{-1}\underline{\sigma}\ \ i.o.\Big)\\ \le&\sum_{k=K}^{\infty}\mathbb{V}\Big(\max_{i\le n_k}|S_i|\le\beta(n_k)(1+\epsilon)^{-1}\underline{\sigma}\Big)\to 0\ \text{ as } K\to\infty.\end{aligned}$$

So,

$$\mathbb{V}\Big(\liminf_{k\to\infty}\frac{\max_{i\le n_k}|S_i|}{\beta(n_k)}\le(1+\epsilon)^{-1}\underline{\sigma}\Big)=0.$$

Note for $n_k\le n\le n_{k+1}$,

$$\frac{\max_{i\le n}|S_i|}{\beta(n)}\ge\frac{\max_{i\le n_k}|S_i|}{\beta(n_k)}\cdot\frac{\beta(n_k)}{\beta(n_{k+1})}.$$

It follows that

$$\mathbb{V}\Big(\liminf_{n\to\infty}\frac{\max_{i\le n}|S_i|}{\beta(n)}\le(1+\epsilon)^{-1}\underline{\sigma}\Big)=0.$$

Note the continuity of $\mathbb{V}$. Letting $\epsilon\to 0$ yields

$$\mathbb{V}\Big(\liminf_{n\to\infty}\frac{\max_{i\le n}|S_i|}{\beta(n)}<\underline{\sigma}\Big)=0. \tag{3.10}$$

Next, we consider the lower bound. Let $n_k=[e^{k(\log k)^2}]$, then $n_{k-1}/n_k\sim e^{-(\log k)^2}\to 0$ and $\log\log n_k\sim\log k$. Let $x_{n_k}=(1-\epsilon)^{-1}\underline{\sigma}\beta(n_k)/\sqrt{n_k-n_{k-1}}$. Then by (3.4) (c.f, Theorem 6.1),

$$\log\mathbb{V}\Big(\max_{n_{k-1}<i\le n_k}|S_i-S_{n_{k-1}}|\le\beta(n_k)(1-\epsilon)^{-1}\underline{\sigma}\Big)\sim-\frac{\pi^2\underline{\sigma}^2}{8x_{n_k}^2}\sim-(1-\epsilon)^{-2}\log k.$$

So,

$$\sum_{k=1}^{\infty}\mathbb{V}\Big(\max_{n_{k-1}<i\le n_k}|S_i-S_{n_{k-1}}|\le\beta(n_k)(1-\epsilon)^{-1}\underline{\sigma}\Big)=\infty.$$

Let $\xi_k = \max_{n_{k-1}<i\le n_k}|S_i - S_{n_{k-1}}| - \beta(n_k)(1-\epsilon)^{-1}\underline{\sigma}$ and $\varphi(y)$ be a Lipschitz function such that $I\{y\le 0\}\le\varphi(y)\le I\{y\le\epsilon\}$. Then $\sum_{k=1}^\infty\widehat{\mathbb{E}}[\varphi(\xi_k)]=\infty$. Note the independence and the continuity of $\mathcal{V}$. We have

$$\mathcal{V}\left(\bigcap_{k=n}^{\infty}\{\xi_k>\epsilon\}\right)=\lim_{N\to\infty}\mathcal{V}\left(\bigcap_{k=n}^{N}\{\xi_k>\epsilon\}\right)\le\lim_{N\to\infty}\widehat{\mathcal{E}}\left[\prod_{k=n}^{N}(1-\varphi(\xi_k))\right]$$
$$=\lim_{N\to\infty}\prod_{k=n}^{N}\widehat{\mathcal{E}}\left[(1-\varphi(\xi_k))\right]\le\lim_{N\to\infty}\prod_{k=n}^{N}\left(1-\widehat{\mathbb{E}}[\varphi(\xi_k)]\right)\le\exp\left\{-\sum_{k=n}^{\infty}\widehat{\mathbb{E}}[\varphi(\xi_k)]\right\}=0.$$

Hence

$$\mathbb{V}\left(\{\xi_k\le\epsilon\}\ i.o.\right)=1-\mathcal{V}\left(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}\{\xi_k>\epsilon\}\right)=1-\lim_{n\to\infty}\mathcal{V}\left(\bigcap_{k=n}^{\infty}\{\xi_k>\epsilon\}\right)=1,$$

i.e.,

$$\mathbb{V}\Big(\max_{n_{k-1}<i\le n_k}|S_i-S_{n_{k-1}}|\le\beta(n_k)(1-\epsilon)^{-1}\underline{\sigma}+\epsilon\ \ i.o.\Big)=1.$$

It follows that

$$\mathbb{V}\Big(\liminf_{k\to\infty}\frac{\max_{n_{k-1}<i\le n_k}|S_i-S_{n_{k-1}}|}{\beta(n_k)}\le(1-\epsilon)^{-1}\underline{\sigma}+\epsilon\Big)=1. \tag{3.11}$$

On the other hand,

$$\sum_{k=1}^{\infty}\mathbb{V}\Big(\max_{i\le n_{k-1}}|S_i|\ge\beta(n_k)/\sqrt{\log\log n_k}\Big)\le\sum_{k=1}^{\infty}\frac{\log\log n_k}{\beta^2(n_k)}\widehat{\mathbb{E}}\left[\max_{i\le n_{k-1}}|S_i|^2\right]$$
$$\le\sum_{k=1}^{\infty}C\frac{(\log\log n_k)^2}{n_k}n_{k-1}\widehat{\mathbb{E}}[X_1^2]\le\sum_{k=1}^{\infty}C\frac{(\log k)^3}{e^{(\log k)^2}}<\infty,$$

where the second inequality is due to the Rosenthal type inequality (c.f. Lemma 3.1). It follows that

$$\mathbb{V}\Big(\limsup_{k\to\infty}\frac{\max_{i\le n_{k-1}}|S_i|}{\beta(n_k)}>0\Big)=0,$$

which, together with (3.11), implies

$$\mathbb{V}\Big(\liminf_{k\to\infty}\frac{\max_{i\le n_k}|S_i|}{\beta(n_k)}\le(1-\epsilon)^{-1}\underline{\sigma}+\epsilon\Big)=1. \tag{3.12}$$

By the continuity of $\mathbb{V}$, letting $\epsilon\to 0$ yields

$$\mathbb{V}\Big(\liminf_{k\to\infty}\frac{\max_{i\le n_k}|S_i|}{\beta(n_k)}\le\underline{\sigma}\Big)=1.$$

Hence

$$\mathbb{V}\Big(\liminf_{n\to\infty}\frac{\max_{i\le n}|S_i|}{\beta(n)}\le\underline{\sigma}\Big)=1. \tag{3.13}$$

Finally, let $x_{n_k}=(1-\epsilon)^{-1}\overline{\sigma}\beta(n_k)/\sqrt{n_k-n_{k-1}}$. Then by (3.5) (c.f, Theorem 6.1),

$$\log \mathcal{V}\Big(\max_{n_{k-1}<i\le n_k}|S_i-S_{n_{k-1}}|\le \beta(n_k)(1-\epsilon)^{-1}\underline{\sigma}\Big)\sim -\frac{\pi^2\overline{\sigma}^2}{8x_{n_k}^2}\sim -(1-\epsilon)^2\log k,$$

which implies

$$\sum_{k=1}^{\infty}\mathcal{V}\Big(\max_{n_{k-1}<i\le n_k}|S_i-S_{n_{k-1}}|\le \beta(n_k)(1-\epsilon)^{-1}\overline{\sigma}\Big)=\infty.$$

So, similar to (3.13) we have

$$\mathcal{V}\Big(\liminf_{n\to\infty}\frac{\max_{i\le n}|S_i|}{\beta(n)}\le\overline{\sigma}\Big)=1. \tag{3.14}$$

It is obvious that (3.6) follows from (3.10), (3.14) and the fact $\mathcal{V}(A\bigcap B)\ge\mathcal{V}(A)-\mathbb{V}(B^c)$, and (3.7) follows from (3.10), (3.13) and the fact $\mathbb{V}(A\bigcap B)\ge\mathbb{V}(A)-\mathbb{V}(B^c)$. □

4 Convergence of the finite-dimensional distributions

The purpose of this section is to prove the following theorem on the convergence of finite-dimensional distributions of W_n under $\widehat{\mathbb{E}}$.

Theorem 4.1 *Suppose $\widehat{\mathbb{E}}[(X_1^2-b)^+]\to 0$ as $b\to\infty$. Then for any $0\le t_1<\ldots<t_d\le 1$ and any $\varphi\in C_b(\mathbb{R}^d)$, we have*

$$\widehat{\mathbb{E}}\left[\varphi\left(W_n(t_1),\ldots,W_n(t_d)\right)\right]\to\widetilde{\mathbb{E}}\left[\varphi\left(W(t_1),\ldots,W(t_d)\right)\right]. \tag{4.1}$$

For proving this theorem, we need some lemmas. The first is the central limit theorem which was firstly obtained by Peng (2008b) under the condition $\widehat{\mathbb{E}}[|X_1|^{2+\epsilon}]<\infty$. The condition was relaxed to only the existence of the second moments by Zhang (2014b).

Lemma 4.1 *(CLT) Suppose $\widehat{\mathbb{E}}[(X_1^2-b)^+]\to 0$ as $b\to\infty$. Then for any bounded continuous function φ,*

$$\lim_{n\to\infty}\widehat{\mathbb{E}}\left[\varphi\left(\frac{S_n}{\sqrt{n}}\right)\right]=\widetilde{\mathbb{E}}[\varphi(W(1))]. \tag{4.2}$$

For random vectors $\boldsymbol{X}_n=(X_n^1,\cdots,X_n^d)$ in $\mathscr{H}^d$ and $\boldsymbol{X}=(X^1,\cdots,X^d)$ in $\widetilde{\mathscr{H}}^d$, we write $\boldsymbol{X}_n\overset{d}{\to}\boldsymbol{X}$ if $\widehat{\mathbb{E}}[\varphi(\boldsymbol{X}_n)]\to\widetilde{\mathbb{E}}[\varphi(\boldsymbol{X})]$ for any bounded continuous function $\varphi:\mathbb{R}^d\to\mathbb{R}$; and write $\boldsymbol{X}_n\overset{\mathbb{V}}{\to}\boldsymbol{a}$ if $\mathbb{V}\left(\|\boldsymbol{X}_n-\boldsymbol{a}\|>\epsilon\right)\to 0$ for all $\epsilon>0$. It is obvious that for any continuous function $f(\boldsymbol{x})$, $\boldsymbol{X}_n\overset{d}{\to}\boldsymbol{X}$ implies $f(\boldsymbol{X}_n)\overset{d}{\to}f(\boldsymbol{X})$, and $\boldsymbol{X}_n\overset{\mathbb{V}}{\to}\boldsymbol{X}$ implies $f(\boldsymbol{X}_n)\overset{\mathbb{V}}{\to}f(\boldsymbol{X})$.

The following lemma is Slutsky's theorem. The proof is standard and ommited.

Lemma 4.2 *Suppose $\boldsymbol{X}_n \xrightarrow{d} \boldsymbol{X}$, $\boldsymbol{Y}_n \xrightarrow{\mathbb{V}} \boldsymbol{y}$, $\eta_n \xrightarrow{\mathbb{V}} a$, where a is a constant and $\boldsymbol{y}$ is a constant vector, and $\widetilde{\mathbb{V}}(\|\boldsymbol{X}\| > \lambda) \to 0$ as $\lambda \to \infty$. Then $(\boldsymbol{X}_n, \boldsymbol{Y}_n, \eta_n) \xrightarrow{d} (\boldsymbol{X}, \boldsymbol{y}, a)$, and as a result, $\eta_n \boldsymbol{X}_n + \boldsymbol{Y}_n \xrightarrow{d} a\boldsymbol{X}$.*

Lemma 4.3 *Suppose $\boldsymbol{X}_n \xrightarrow{d} \boldsymbol{X}$ and $\widetilde{\mathbb{V}}(\|\boldsymbol{X}\| > \lambda) \to 0$ as $\lambda \to \infty$. Assume that $g_n(\boldsymbol{x})$ and $g(x)$ are continuous functions for which*

$$|g_n(\boldsymbol{x})| \le M, \quad |g(\boldsymbol{x})| \le M,$$

$$g_n(\boldsymbol{x}_n) \to g(\boldsymbol{x}) \text{ whenever } x_n \to x.$$

Then

$$\widehat{\mathbb{E}}[g_n(\boldsymbol{X}_n)] \to \widetilde{\mathbb{E}}[g(\boldsymbol{X})].$$

Proof. The conditions for g_n imply that $\sup_{\|\boldsymbol{x}\| \le \lambda} |g_n(\boldsymbol{x}) - g(\boldsymbol{x})| \to 0$. It is obvious that

$$\begin{aligned} &\left|\widehat{\mathbb{E}}[g_n(\boldsymbol{X}_n)] - \widetilde{\mathbb{E}}[g(\boldsymbol{X})]\right| \le \left|\widehat{\mathbb{E}}[g_n(\boldsymbol{X}_n)] - \widehat{\mathbb{E}}[g(\boldsymbol{X}_n)]\right| + \left|\widehat{\mathbb{E}}[g(\boldsymbol{X}_n)] - \widetilde{\mathbb{E}}[g(\boldsymbol{X})]\right| \\ \le& \left|\widehat{\mathbb{E}}[g(\boldsymbol{X}_n)] - \widetilde{\mathbb{E}}[g(\boldsymbol{X})]\right| + \sup_{\|\boldsymbol{x}\| \le \lambda} |g_n(\boldsymbol{x}) - g(\boldsymbol{x})| + 2M\mathbb{V}\left(\|\boldsymbol{X}_n\| > \lambda\right). \end{aligned}$$

Choose a Lipschitz function $\varphi(x)$ such that $I\{x > \lambda\} \ge \varphi(x) \ge I\{x > \lambda/2\}$. Letting $n \to \infty$ yields that

$$\begin{aligned} &\limsup_{n\to\infty} \left|\widehat{\mathbb{E}}[g_n(\boldsymbol{X}_n)] - \widetilde{\mathbb{E}}[g(\boldsymbol{X})]\right| \\ \le& 2M \limsup_{n\to\infty} \mathbb{V}\left(\|\boldsymbol{X}_n\| > \lambda\right) \le 2M \limsup_{n\to\infty} \widehat{\mathbb{E}}\left[\varphi(\|\boldsymbol{X}_n\|)\right] \\ =& 2M\widetilde{\mathbb{E}}\left[\varphi(\|\boldsymbol{X}\|)\right] \le 2M\widetilde{\mathbb{V}}\left(\|\boldsymbol{X}\| > \lambda/2\right) \to \text{ as } \lambda \to \infty. \end{aligned}$$

The proof is completed. $\Box$

Lemma 4.4 *Suppose that $\boldsymbol{X}_n \xrightarrow{d} \boldsymbol{X}$, $\boldsymbol{Y}_n \xrightarrow{d} \boldsymbol{Y}$, $\boldsymbol{Y}_n$ is independent to $\boldsymbol{X}_n$ under $\widehat{\mathbb{E}}$, $\boldsymbol{Y}$ is independent to $\boldsymbol{X}$ under $\widetilde{\mathbb{E}}$, and $\widetilde{\mathbb{V}}(\|\boldsymbol{X}\| > \lambda) \to 0$ and $\widetilde{\mathbb{V}}(\|\boldsymbol{Y}\| > \lambda) \to 0$ as $\lambda \to 0$. Then $(\boldsymbol{X}_n, \boldsymbol{Y}_n) \xrightarrow{d} (\boldsymbol{X}, \boldsymbol{Y})$.*

Proof. Suppose $\varphi(\boldsymbol{x}, \boldsymbol{y})$ is a bounded continuous function. We want to show that

$$\widehat{\mathbb{E}}\left[\varphi(\boldsymbol{X}_n, \boldsymbol{Y}_n)\right] \to \widetilde{\mathbb{E}}\left[\varphi(\boldsymbol{X}, \boldsymbol{Y})\right]. \tag{4.3}$$

First we assume that $\varphi(\boldsymbol{x},\boldsymbol{y})$ is a bounded Lipschitz function. Then $\varphi\in C_{l,Lip}$. By the definition of the independence,

$$\widehat{\mathbb{E}}\left[\varphi(\boldsymbol{X}_n,\boldsymbol{Y}_n)\right]=\widehat{\mathbb{E}}\left[g_n(\boldsymbol{X}_n)\right],\ \ \widetilde{\mathbb{E}}\left[\varphi(\boldsymbol{X},\boldsymbol{Y})\right]=\widetilde{\mathbb{E}}\left[g(\boldsymbol{X})\right],$$

where $g_n(\boldsymbol{x})=\widehat{\mathbb{E}}[\varphi(\boldsymbol{x},\boldsymbol{Y}_n)]$, $g(\boldsymbol{x})=\widetilde{\mathbb{E}}[\varphi(\boldsymbol{x},\boldsymbol{Y})]$. Suppose $\boldsymbol{x}_n\to\boldsymbol{x}$. It follows that

$$\begin{aligned}&|g_n(\boldsymbol{x}_n)-g(\boldsymbol{x})|=\left|\widehat{\mathbb{E}}[\varphi(\boldsymbol{x}_n,\boldsymbol{Y}_n)]-\widetilde{\mathbb{E}}[\varphi(\boldsymbol{x},\boldsymbol{Y})]\right|\\ \le&\left|\widehat{\mathbb{E}}[\varphi(\boldsymbol{x}_n,\boldsymbol{Y}_n)]-\widehat{\mathbb{E}}[\varphi(\boldsymbol{x},\boldsymbol{Y}_n)]\right|+\left|\widehat{\mathbb{E}}[\varphi(\boldsymbol{x},\boldsymbol{Y}_n)]-\widetilde{\mathbb{E}}[\varphi(\boldsymbol{x},\boldsymbol{Y})]\right|\\ \le&\sup_{\boldsymbol{y}}|\varphi(\boldsymbol{x}_n,\boldsymbol{y})-\varphi(\boldsymbol{x},\boldsymbol{y})|+\left|\widehat{\mathbb{E}}[\varphi(\boldsymbol{x},\boldsymbol{Y}_n)]-\widetilde{\mathbb{E}}[\varphi(\boldsymbol{x},\boldsymbol{Y})]\right|\to 0\end{aligned}$$

by noting that $\varphi(\boldsymbol{x},\boldsymbol{y})$ is uniformly continuous and $\boldsymbol{Y}_n\overset{d}{\to}\boldsymbol{Y}$. By the uniform continuity of φ, $g_n(\boldsymbol{x})$ and $g(\boldsymbol{x})$ are continuous functions. So, $g_n(\boldsymbol{x})$ and $g(\boldsymbol{x})$ satisfy the conditions in Lemma 4.3. It follows that

$$\widehat{\mathbb{E}}[\varphi(\boldsymbol{X}_n,\boldsymbol{Y}_n)]=\widehat{\mathbb{E}}[g_n(\boldsymbol{X}_n)]\to\widetilde{\mathbb{E}}[g(\boldsymbol{X})]=\widetilde{\mathbb{E}}[\varphi(\boldsymbol{X},\boldsymbol{Y})].$$

Next, we assume that $\varphi(\boldsymbol{x},\boldsymbol{y})$ is a bounded uniformly continuous function. Then for any $\epsilon>0$, there is bounded Lipschitz function $\varphi_\epsilon(\boldsymbol{x},\boldsymbol{y})$ such that $|\varphi(\boldsymbol{x},\boldsymbol{y})-\varphi_\epsilon(\boldsymbol{x},\boldsymbol{y})|<\epsilon$. It follows that

$$\begin{aligned}&\limsup_{n\to\infty}\left|\widehat{\mathbb{E}}[\varphi(\boldsymbol{X}_n,\boldsymbol{Y}_n)]-\widetilde{\mathbb{E}}[\varphi(\boldsymbol{X},\boldsymbol{Y})\right|\\ \le&\limsup_{n\to\infty}\left|\widehat{\mathbb{E}}[\varphi_\epsilon(\boldsymbol{X}_n,\boldsymbol{Y}_n)]-\widetilde{\mathbb{E}}[\varphi_\epsilon(\boldsymbol{X},\boldsymbol{Y})]\right|+2\epsilon=2\epsilon.\end{aligned}$$

So, (4.3) is proved for a bounded uniformly continuous function. Finally, let $\varphi(\boldsymbol{x},\boldsymbol{y})$ be a bounded continuous function with $|\varphi(\boldsymbol{x},\boldsymbol{y})|\le M$. Let $\lambda>0$. For $\boldsymbol{x}=(x_1,\ldots,x_d)$, denote $\boldsymbol{x}_\lambda=\big((-\lambda)\vee(x_1\wedge\lambda)\lambda,\ldots,(-\lambda)\vee(x_d\wedge\lambda)\big)$ and define $\varphi_\lambda(\boldsymbol{x},\boldsymbol{y})=\varphi(\boldsymbol{x}_\lambda,\boldsymbol{y}_\lambda)$. Then φ_λ is a bounded uniformly continuous function with

$$|\varphi_\lambda(\boldsymbol{x},\boldsymbol{y})-\varphi(\boldsymbol{x},\boldsymbol{y})|\le 2MI\{\|\boldsymbol{x}\|>\lambda\}+2MI\{\|\boldsymbol{y}\|>\lambda\}.$$

It follows that

$$
\begin{aligned}
&\limsup_{n\to\infty}\left|\widehat{\mathbb{E}}[\varphi(\boldsymbol{X}_n,\boldsymbol{Y}_n)]-\widetilde{\mathbb{E}}[\varphi(\boldsymbol{X},\boldsymbol{Y})\right|\\
\le&\limsup_{n\to\infty}\left|\widehat{\mathbb{E}}[\varphi_\lambda(\boldsymbol{X}_n,\boldsymbol{Y}_n)]-\widetilde{\mathbb{E}}[\varphi_\lambda(\boldsymbol{X},\boldsymbol{Y})]\right|\\
&+2M\limsup_{n\to\infty}\left\{\mathbb{V}(\|\boldsymbol{X}_n\|>\lambda)+\mathbb{V}(\|\boldsymbol{Y}_n\|>\lambda)\right\}\\
&+2M\left\{\widetilde{\mathbb{V}}(\|\boldsymbol{X}\|>\lambda)+\widetilde{\mathbb{V}}(\|\boldsymbol{Y}\|>\lambda)\right\}\\
\le&4M\left\{\widetilde{\mathbb{V}}(\|\boldsymbol{X}\|>\lambda/2)+\widetilde{\mathbb{V}}(\|\boldsymbol{Y}\|>\lambda/2)\right\}\to 0\ \text{ as } \lambda\to\infty.
\end{aligned}
$$

The proof is completed. □

Proof of Theorem 4.1. Note

$$
\sup_{0\le t\le 1}\left|W_n(t)-\frac{S_{[nt]}}{\sqrt{n}}\right|\le\frac{\max_{k\le n}|X_k|}{\sqrt{n}},
$$

and for any $\epsilon>0$,

$$
\mathbb{V}\left(\max_{k\le n}|X_k|\ge\epsilon\sqrt{n}\right)\le n\frac{2}{\epsilon^2 n}\widehat{\mathbb{E}}\left[\left(X_1^2-\frac{\epsilon^2 n}{2}\right)^+\right]\to 0.
$$

It is sufficient to show that

$$
\frac{1}{\sqrt{n}}\left(S_{[nt_1]},\ldots,S_{[nt_d]}\right)\overset{d}{\to}(W(t_1),\ldots,W(t_d))
$$

by Lemma 4.2, or equivalently,

$$
\begin{aligned}
&\frac{1}{\sqrt{n}}\left(S_{[nt_1]},S_{[nt_2]}-S_{[nt_1]},\ldots,S_{[nt_d]}-S_{[nt_{d-1}]}\right)\\
&\quad\overset{d}{\to}\left(W(t_1),W(t_2)-W(t_1),\ldots,W(t_d)-W(t_{d-1})\right).
\end{aligned}
\tag{4.4}
$$

By Lemmas 4.1 and 4.3,

$$
\frac{S_{[nt_i]}-S_{[nt_{i-1}]}}{\sqrt{n}}=\frac{\sqrt{[nt_i]-[nt_{i-1}]}}{\sqrt{n}}\frac{S_{[nt_i]}-S_{[nt_{i-1}]}}{\sqrt{[nt_i]-[nt_{i-1}]}}\overset{d}{\to}W(t_i)-W(t_{i-1}).
$$

Hence, by noting the independence, (4.4) follows from Lemma 4.4 and the induction. The proof is now completed. □

5 Tightness

Recall $\omega_\delta(x)=\sup_{|t-s|<\delta}|x(t)-x(s)|$. It is known that $\{W_n\}$ is tight under the probability measure P in the following sense: for any $\eta>0$, there a compact set $K\subset C[0,1]$ such that

$\sup_n P(W_n \notin K) < \eta$. This is also equivalent to $\lim_{\delta\to 0}\sup_n P(\omega_\delta(W_n)\geq \epsilon) = 0$ for any $\epsilon > 0$. In this section, we will prove the following theorem on the tightness of $\{W_n\}$ under capacities.

Theorem 5.1 *Suppose* $\widehat{\mathbb{E}}[(X_1^2 - b)^+] \to 0$ *as* $b \to \infty$. *Then*

(a) *for any* $\epsilon > 0$,

$$\lim_{\delta\to 0}\sup_n \mathbb{V}\left(\omega_\delta(W_n)\geq \epsilon\right) = 0; \tag{5.1}$$

(b) *for any* $\eta > 0$, *there exists a compact set* $K \subset C[0,1]$ *such that*

$$\sup_n \mathbb{V}\left(W_n \notin K\right) < \eta. \tag{5.2}$$

Proof. We first show (a). With the same argument of Billingsley (1968, Pages 56-59, c.f., Theorem 8.4), it is sufficient to show that

$$\lim_{\lambda\to\infty}\lambda^2\sup_n\sup_k \mathbb{V}\left(\max_{i\leq n}|S_{k+i}-S_k|\geq \lambda\sqrt{n}\right) = 0. \tag{5.3}$$

Note that for each fixed n,

$$\begin{aligned}&\lim_{\lambda\to\infty}\lambda^2\sup_k \mathbb{V}\left(\max_{i\leq n}|S_{k+i}-S_k|\geq \lambda\sqrt{n}\right)\\ \leq&\sum_{i=1}^n\lim_{\lambda\to\infty}\sup_k \mathbb{V}\left(|X_{k+i}|\geq \lambda/\sqrt{n}\right)\leq 2n^2\lim_{\lambda\to\infty}\widehat{\mathbb{E}}\left[\left(X_1^2-\frac{\lambda^2}{2n}\right)^+\right]=0.\end{aligned}$$

So, it is sufficient to show that

$$\lim_{\lambda\to\infty}\lambda^2\limsup_{n\to\infty}\sup_k \mathbb{V}\left(\max_{i\leq n}|S_{k+i}-S_k|\geq \lambda\sqrt{n}\right) = 0. \tag{5.4}$$

Now,

$$\mathbb{V}\left(\max_{i\leq n}|S_{k+i}-S_k|\geq \lambda\sqrt{n}\right) = \mathbb{V}\left(\max_{i\leq n}|S_i|\geq \lambda\sqrt{n}\right)\leq \frac{2}{\lambda^2}\widehat{\mathbb{E}}\left[\left(\max_{i\leq n}\frac{|S_i|^2}{n}-\frac{\lambda^2}{2}\right)^+\right].$$

By (3.9) where $p = 2$, (5.4) follows.

Now, we show (5.2). By (5.1), choose $\delta_k \downarrow 0$ such that, if

$$A_k = \left\{x : \omega_{\delta_k}(x) < \frac{1}{k}\right\},$$

then $\sup_n \mathbb{V}\left(W_n \in A_k^c\right) \leq \eta/2^{k+1}$. Let $A = \{x : |x(0)| \leq a\}$, $K = A\bigcap_{k=1}^\infty A_k$. Then by the Arzelá-Ascoli theorem, K is compact. It is obvious that $\{W_n \notin A\} = \emptyset$ since $W_n(0) = 0$. Next, we show that

$$\mathbb{V}(W_n \in K^c) \leq \mathbb{V}(W_n \in A^c) + \sum_{k=1}^\infty \mathbb{V}(W_n \in A_k^c),$$

which is obvious when $\mathbb{V}$ is countably sub-additive. Note that when $\delta < 1/(2n)$,

$$\omega_\delta(W_n) \le 2|t-s| \max_{i\le n} \frac{|X_i|/\sqrt{n}}{1/n} \le 2\sqrt{n}\delta \max_{i\le n} |X_i|.$$

Choose a k_0 such that $\delta_k < 1/(2Mk)$ and $\delta_k < 1/(2n)$ for $k \ge k_0$. Then on the event $E = \{\sqrt{n}\max_{i\le n}|X_i| \le M\}$, $\{W_n \in A_k^c\} = \emptyset$ for $k \ge k_0$. So, by the (finite) sub-additivity of $\mathbb{V}$,

$$\begin{aligned}\mathbb{V}\big(E\bigcap\{W_n \in K^c\}\big) \le & \mathbb{V}\big(E\bigcap\{W_n \in A^c\}\big) + \sum_{k=1}^{k_0} \mathbb{V}\big(E\bigcap\{W_n \in A_k^c\}\big) \\ \le & \mathbb{V}(W_n \in A^c) + \sum_{k=1}^{\infty} \mathbb{V}(W_n \in A_k^c).\end{aligned}$$

On the other hand,

$$\mathbb{V}(E^c) \le \frac{\sqrt{n}\widehat{\mathbb{E}}[\max_{i\le n}|X_i|]}{M} \le \frac{n^2\widehat{\mathbb{E}}[|X_1|]}{M}.$$

It follows that

$$\mathbb{V}\big(W_n \in K^c\big) \le \mathbb{V}(W_n \in A^c) + \sum_{k=1}^{\infty} \mathbb{V}(W_n \in A_k^c) + \frac{n^2\widehat{\mathbb{E}}[|X_1|]}{M}.$$

Letting $M \to \infty$ yields

$$\mathbb{V}\big(W_n \in K^c\big) \le \mathbb{V}(W_n \in A^c) + \sum_{k=1}^{\infty} \mathbb{V}(W_n \in A_k^c) < 0 + \sum_{k=1}^{\infty} \frac{\eta}{2^{k+1}} < \eta.$$

The proof of (5.2) is now completed. $\square$.

For the G-Brownian motion $W(t)$ on $(\widetilde{\Omega}, \widetilde{\mathscr{H}}, \widetilde{\mathbb{E}})$ we have a similar result.

Theorem 5.2 *We have*

(a) *for any* $\epsilon > 0$, $\lim_{\delta\to 0} \widetilde{\mathbb{V}}\left(\omega_\delta(W) \ge \epsilon\right) = 0$;

(b) *for any* $\eta > 0$, *there exists a compact set* $K \subset C[0,1]$ *such that* $\widetilde{\mathbb{V}}\left(W \not\in K\right) < \eta$.

Proof. Note

$$\begin{aligned}\widetilde{\mathbb{V}}\left(\omega_\delta(W) \ge \epsilon\right) \le & \sum_{k=0}^{[1/\delta]} \widetilde{\mathbb{V}}\left(\sup_{0\le s\le\delta} |W(k/\delta+s) - W(k/\delta)| \ge \epsilon\right) \\ \le & \sum_{k=0}^{[1/\delta]} \frac{1}{\epsilon^4} \widetilde{\mathbb{E}}\left[\sup_{0\le s\le\delta} |W(k/\delta+s) - W(k/\delta)|^4\right] \\ \le & \frac{2}{\delta\epsilon^4}\delta^2 \widetilde{\mathbb{E}}\left[\sup_{0\le s\le 1} |W(s)|^4\right] = \delta\frac{2\overline{\sigma}^4}{\epsilon^4} E_P\left[\sup_{0\le s\le 1} |B(s)|^4\right].\end{aligned}$$

Hence (a) follows. The proof of (b) is similar to that of Theorem 5.1 (b) by noting that $\widetilde{\mathbb{V}}$ is countably sub-additive. □

In the end of this section, we give another proof of Theorem 3.1. Define

$$\mathbb{F}_n[\varphi] = \widehat{\mathbb{E}}\left[\varphi(W_n)\right], \ \mathbb{F}[\varphi] = \sup_n \widehat{\mathbb{E}}\left[\varphi(W_n)\right], \ \ \varphi \in C_b\left(C[0,1]\right)$$

and $\mathbb{F}_n^*[\varphi]$, $\mathbb{F}^*$ be their extensions. By Theorem 5.1, $\mathbb{F}$ is tight and hence the family of sub-linear expectations $\{\mathbb{F}_n; n \geq 1\}$ on $(C[0,1], C_b\left(C[0,1]\right))$ is tight in the sense of Definition 7 of Peng (2010). Hence, by Theorem 9 of Peng (2010), $\{\mathbb{F}_n\}$ is weakly compact, namely, for each subsequence $\{\mathbb{F}_{n'}\}$ there exists a further subsequence $\{\mathbb{F}_{n''}\}$ such that, for each $\varphi \in C_b\left(C[0,1]\right)$, $\{\mathbb{F}_{n''}[\varphi]\}$ is a Cauchy sequence. Define

$$\widetilde{\mathbb{F}}[\varphi] = \lim_{n''\to\infty} \mathbb{F}_{n''}[\varphi], \ \varphi \in C_b\left(C[0,1]\right).$$

Then $\widetilde{\mathbb{F}}$ satisfies (2.3) by Theorem 4.1. So, under the sub-linear expectation $\widetilde{\mathbb{F}}$, the canonical process $W(t) = \omega_t$ is a G-Browinian with $W(1) \sim N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$. The proof is completed.

6 Small deviations

The purpose of this section is to establish the following theorem on the small deviations under $\widehat{\mathbb{E}}$.

Theorem 6.1 *Suppose $\widehat{\mathbb{E}}[(X_1^2 - b)^+] \to 0$ as $b \to \infty$, $0 < x_n \to 0$ and $n^{1/2}x_n \to \infty$. Then*

$$\lim_{n\to\infty} x_n^2 \log \mathbb{V}\left(\max_{k\leq n} |S_k| \leq n^{1/2}x_n\right) = -\frac{\pi^2\underline{\sigma}^2}{8}, \tag{6.1}$$

$$\lim_{n\to\infty} x_n^2 \log \mathcal{V}\left(\max_{k\leq n} |S_k| \leq n^{1/2}x_n\right) = -\frac{\pi^2\overline{\sigma}^2}{8}. \tag{6.2}$$

To prove Theorem 6.1, we need some lemmas on the properties of G-Brownian motions.

Lemma 6.1 *We have for all $x > 0$,*

$$\widetilde{\mathbb{V}}\left(\sup_{0\leq t\leq 1} |W(t)| \leq x\right) = \sup_{\underline{\sigma}\leq\sigma\leq\overline{\sigma}} P\left(\sup_{0\leq t\leq 1} |\sigma B(t)| \leq x\right) = P\left(\sup_{0\leq t\leq 1} |\underline{\sigma} B(t)| \leq x\right) \tag{6.3}$$

$$\widetilde{\mathcal{V}}\left(\sup_{0\leq t\leq 1} |W(t)| \leq x\right) = \inf_{\underline{\sigma}\leq\sigma\leq\overline{\sigma}} P\left(\sup_{0\leq t\leq 1} |\sigma B(t)| \leq x\right) = P\left(\sup_{0\leq t\leq 1} |\overline{\sigma} B(t)| \leq x\right) \tag{6.4}$$

$$\widetilde{\mathbb{V}}\left(\sup_{0\leq t\leq 1} W(t) \leq x\right) = \sup_{\underline{\sigma}\leq\sigma\leq\overline{\sigma}} P\left(\sup_{0\leq t\leq 1} \sigma B(t) \leq x\right) = P\left(\sup_{0\leq t\leq 1} \underline{\sigma} B(t) \leq x\right) \tag{6.5}$$

$$\widetilde{\mathcal{V}}\left(\sup_{0\leq t\leq 1} W(t) \leq x\right) = \inf_{\underline{\sigma}\leq\sigma\leq\overline{\sigma}} P\left(\sup_{0\leq t\leq 1} \sigma B(t) \geq x\right) = P\left(\sup_{0\leq t\leq 1} \overline{\sigma} B(t) \leq x\right). \tag{6.6}$$

In particular, for $x > 0$,

$$\frac{2}{\pi}\exp\Big\{-\frac{\pi^2\underline{\sigma}^2}{8x^2}\Big\} \le \widetilde{\mathbb{V}}\left(\sup_{0\le t\le 1}|W(t)| \le x\right) \le \frac{4}{\pi}\exp\Big\{-\frac{\pi^2\underline{\sigma}^2}{8x^2}\Big\}, \tag{6.7}$$

$$\frac{2}{\pi}\exp\Big\{-\frac{\pi^2\overline{\sigma}^2}{8x^2}\Big\} \le \widetilde{\mathcal{V}}\left(\sup_{0\le t\le 1}|W(t)| \le x\right) \le \frac{4}{\pi}\exp\Big\{-\frac{\pi^2\overline{\sigma}^2}{8x^2}\Big\}. \tag{6.8}$$

Proof. Let $\varphi(y)$ be a non-increasing Lipschitz function for which $I\{y \le x\} \le \varphi(y) \le I\{y \le x(1+\delta)\}$. Then by Lemma 2.1,

$$\begin{aligned}&\widetilde{\mathbb{V}}\left(\sup_{0\le t\le 1}|W(t)| \le x\right) \le \widetilde{\mathbb{E}}\left[\varphi\big(\sup_{0\le t\le 1}|W(t)|\big)\right]\\
=&\sup_{\theta\in\Theta} E_P\left[\varphi\left(\sup_{0\le t\le 1}\Big|\int_0^t \theta(s)dB(s)\Big|\right)\right] \le \sup_{\theta\in\Theta} P\left(\sup_{0\le t\le 1}\Big|\int_0^t \theta(s)dB(s)\Big| \le x(1+\delta)\right).\end{aligned}$$

Note that $W_\theta(t) = \int_0^t \theta(s)dB(s)$ is a continuous martingale with quadratic variation process $\langle W_\theta, W_\theta\rangle(t) = \int_0^t \theta^2(s)ds$. By the Dambis-Dubins-Schwarz theorem, there is a standard Brownian motion $\widetilde{B}$ under P such that $W_\theta(t) = \widetilde{B}\left(\langle W_\theta, W_\theta\rangle(t)\right)$. On the other hand, $\langle W_\theta, W_\theta\rangle(t)$ is continuous. So,

$$\sup_{0\le t\le \overline{\sigma}^2}|\widetilde{B}(s)| \ge \sup_{0\le t\le 1}\left|\widetilde{B}\left(\langle W_\theta, W_\theta\rangle(t)\right)\right| = \sup_{0\le t\le \langle W_\theta, W_\theta\rangle(1)}|\widetilde{B}(s)| \ge \sup_{0\le t\le \underline{\sigma}^2}|\widetilde{B}(s)|.$$

It follows that

$$\widetilde{\mathbb{V}}\Big(\sup_{0\le t\le 1}|W(t)| \le x\Big) \le P\Big(\sup_{0\le t\le \underline{\sigma}^2}|B(s)| \le x(1+\delta)\Big).$$

Letting $\delta \to 0$ yields

$$\widetilde{\mathbb{V}}\Big(\sup_{0\le t\le 1}|W(t)| \le x\Big) \le P\Big(\sup_{0\le t\le \underline{\sigma}^2}|\widetilde{B}(s)| \le x\Big) = P\Big(\sup_{0\le t\le 1}\underline{\sigma}|B(s)| \le x\Big).$$

On the other hand, for $\theta(s) \equiv \underline{\sigma}$, $W_\theta(t)$ is a Brownian motion under P with $W(1) \sim N(0, \underline{\sigma}^2)$. So

$$\widetilde{\mathbb{V}}\Big(\sup_{0\le t\le 1}|W(t)| \le x\Big) \ge P\Big(\sup_{0\le t\le \underline{\sigma}^2}|B(s)| \le x\Big).$$

It follows that

$$\widetilde{\mathbb{V}}\Big(\sup_{0\le t\le 1}|W(t)| \le x\Big) = P\Big(\sup_{0\le t\le 1}\underline{\sigma}|B(s)| \le x\Big).$$

Hence, (6.3) is proved. The proof of (6.4)-(6.6) is similar. The proof is completed by noting

$$\frac{2}{\pi}\exp\Big\{-\frac{\pi^2}{8x^2}\Big\} \le P\Big(\sup_{0\le t\le 1}|B(t)| \le x\Big) \le \frac{4}{\pi}\exp\Big\{-\frac{\pi^2}{8x^2}\Big\}. \ \square$$

Lemma 6.2 *We have for all* y,

$$\widetilde{\mathbb{V}}\left(\sup_{0\le t\le 1}|W(t)+y|\le x\right)\le \frac{4}{\pi}\exp\Big\{-\frac{\pi^2\underline{\sigma}^2}{8x^2}\Big\} \tag{6.9}$$

$$\widetilde{\mathcal{V}}\left(\sup_{0\le t\le 1}|W(t)+y|\le x\right)\le \frac{4}{\pi}\exp\Big\{-\frac{\pi^2\overline{\sigma}^2}{8x^2}\Big\}. \tag{6.10}$$

Proof. The proof is similar to that of Lemma 6.1 by noting

$$P\Big(\sup_{0\le t\le 1}|B(t)+y|\le x\Big)\le P\Big(\sup_{0\le t\le 1}|B(t)|\le x\Big)\le \frac{4}{\pi}\exp\Big\{-\frac{\pi^2}{8x^2}\Big\}$$

according to the Anderson inequality. □

Lemma 6.3 *We have for any* $\alpha>0$, $0<\epsilon<\alpha/2$ *and* $\delta>0$,

$$\liminf_{x\to 0^+} x^2\log\left[\inf_{|y|\le \epsilon x}\mathbb{V}\Big(\sup_{0\le t\le 1}|W(t)|\le \alpha x, |y+W(1)|\le \delta x\Big)\right]\ge -\frac{\pi^2\underline{\sigma}^2}{8}(\alpha-2\epsilon)^{-2}, \tag{6.11}$$

$$\liminf_{x\to 0^+} x^2\log\left[\inf_{|y|\le \epsilon x}\mathcal{V}\Big(\sup_{0\le t\le 1}|W(t)|\le \alpha x, |y+W(1)|\le \delta x\Big)\right]\ge -\frac{\pi^2\overline{\sigma}^2}{8}(\alpha-2\epsilon)^{-2}, \tag{6.12}$$

Proof. Note

$$\mathbb{V}\Big(\sup_{0\le t\le 1}|W(t)|\le \alpha x, |y+W(1)|\le \delta x\Big)\ge P\Big(\sup_{0\le t\le 1}|B(t\underline{\sigma}^2)|\le \alpha x, |y+W(\underline{\sigma}^2)|\le \delta x\Big).$$

By Lemma 3.2 of Acosta (1983),

$$\liminf_{x\to 0^+} x^2\log\left[\inf_{|y|\le \epsilon x}P\Big(\sup_{0\le t\le 1}|B(t\underline{\sigma}^2)|\le \alpha x, |y+W(\underline{\sigma}^2)|\le \delta x\Big)\right]\ge -\frac{\pi^2\underline{\sigma}^2}{8}(\alpha-2\epsilon)^{-2}.$$

The proof of (6.11) is completed.

The proof of (6.12) is more technical and similar to that of Lemma 3.2 of Acosta (1983) after smoothing $I\{y\le r\}$ by a Lipschitz function $\varphi(y)$, and so omitted. □

Proof of Theorem 6.1. Let $\delta>0$ be a small number. Denote $T=\delta^{-2}$, $m=m_n=[Tnx_n^2]$, $l=l_n=[n/m]$. Then $l\sim\frac{1}{Tx_n^2}$, $nx_n^2/m\sim T^{-1}$. Let $\phi(y)$ be a non-decreasing Lipschitz function such that $I\{y\le 1\}\le\phi(y)\le I\{y\le 1+\delta/32\}$. Then by the definition of independence,

$$\begin{aligned}
&\mathbb{V}\Big(\max_{k\le n}|S_k|\le\sqrt{n}x_n\Big)\le\mathbb{V}\Big(\max_{j\le l}\max_{m(j-1)<k\le mj}|S_k|\le\sqrt{n}x_n\Big)\\
\le&\widehat{\mathbb{E}}\left[\prod_{j=1}^{l}\phi\left(\max_{m(j-1)<k\le mj}|S_k|/(\sqrt{n}x_n)\right)\right]\\
=&\widehat{\mathbb{E}}\left[\prod_{j=1}^{l-1}\phi\left(\max_{m(j-1)<k\le mj}|S_k|/(\sqrt{n}x_n)\right)\right.\\
&\qquad\left.\cdot\widehat{\mathbb{E}}\left[\phi\left(\max_{m(l-1)<k\le ml}|S_k-S_{m(l-1)}+S_{m(l-1)}|/(\sqrt{n}x_n)\right)\Big|S_k:k\le m(l-1)\right]\right].
\end{aligned}$$

When $\phi\left(\max_{m(j-2)<k\le m(j-1)}|S_k|/(\sqrt{n}x_n)\right)\ne 0$, we have $|S_{m(j-1)}|\le(1+\delta/32)\sqrt{n}x_n$, and then

$$\begin{aligned}&\widehat{\mathbb{E}}\left[\phi\left(\max_{m(j-1)<k\le mj}|S_k-S_{m(j-1)}+S_{m(j-1)}|/(\sqrt{n}x_n)\right)\Big|S_k:k\le m(j-1)\right]\\ \le&\sup_{|y|\le(1+\delta/32)\sqrt{n}x_n}\widehat{\mathbb{E}}\left[\phi\left(\max_{k\le m}|S_k+y|/(\sqrt{n}x_n)\right)\right]\\ \le&\sup_{|y|\le 2T^{-1/2}}\widehat{\mathbb{E}}\left[\phi\left(\frac{\sup_{0\le t\le 1}|W_m(t)+y|}{\sqrt{n}x_n/\sqrt{m}}\right)\right],\end{aligned}$$

by noting that $\{S_k-S_{m(j-1)};k=m(j-1)+1,\ldots,mj\}$ and $\{S_k;k=1,\ldots,m\}$ are identically distributed under $\widehat{\mathbb{E}}$. Note $\sqrt{n}x_n/\sqrt{m}\to T^{-1/2}$. Choose $h_m(x,y)=\phi\left(\frac{\sup_{0\le t\le 1}|x(t)+y(t)|}{\sqrt{n}x_n/\sqrt{m}}\right)$, $h(x,y)=\phi\left(\sup_{0\le t\le 1}|x(t)+y(t)|/T^{-1/2}\right)$ and $K=\{y(t)\equiv y:|y|\le 2T^{-1/2}\}$ in (3.2). By Corollary 3.1, uniformly in $|y|\le 2T^{-1/2}$,

$$\begin{aligned}&\lim_{n\to\infty}\widehat{\mathbb{E}}\left[\phi\left(\frac{\sup_{0\le t\le 1}|W_m(t)+y|}{\sqrt{n}x_n/\sqrt{m}}\right)\right]\\ &=\widehat{\mathbb{E}}\left[\phi\left(\sup_{0\le t\le 1}|W(t)+y|/T^{-1/2}\right)\right]\\ &\le\mathbb{V}\Big(\sup_{0\le t\le 1}|W(t)+y|\le(1+\delta/32)T^{-1/2}\Big)\\ &\le\frac{4}{\pi}\exp\Big\{-\frac{\pi^2T\overline{\sigma}^2}{8x^2(1+\delta/32)^2}\Big\}\ \text{(by Lemma 6.1).}\end{aligned}$$

So, there is a n_0 such that for all $n\ge n_0$,

$$\begin{aligned}&\sup_{|y|\le 2T^{-1/2}}\widehat{\mathbb{E}}\left[\phi\left(\frac{\sup_{0\le t\le 1}|W_m(t)+y|}{\sqrt{n}x_n/\sqrt{m}}\right)\right]\\ &\le\frac{8}{\pi}\exp\Big\{-\frac{\pi^2T\overline{\sigma}^2}{8x^2(1+\delta/32)^2}\Big\}\le\exp\Big\{-\frac{\pi^2T\overline{\sigma}^2}{8x^2(1+\delta/8)^2}\Big\}\end{aligned}$$

if $T=\delta^{-2}$ is large enough. Hence

$$\log\mathbb{V}\Big(\max_{k\le n}|S_k|\le\sqrt{n}x_n\Big)\le -l\frac{\pi^2T\overline{\sigma}^2}{8(1+\delta/8)^2}\sim-\frac{\pi^2\overline{\sigma}^2}{8(1+\delta/8)^2}x_n^{-2}.$$

It follows that

$$\limsup_{n\to\infty}x_n^2\log\mathbb{V}\Big(\max_{k\le n}|S_k|\le\sqrt{n}x_n\Big)\le-\frac{\pi^2\overline{\sigma}^2}{8}.$$

Next, we consider the lower bound. Let ϕ be defined as above, and ϕ_1 be a non-decreasing Lipschitz function such that $I\{y\le 1-\delta/32\}\le\phi(y)\le I\{y\le 1\}$. Let $\epsilon>0$ be a number whose value will be given later. Then

$$\mathbb{V}\Big(\max_{k\le n}|S_k|\le\sqrt{n}x_n\Big)\ge\mathbb{V}\Big(\max_{j\le l+1}\max_{m(j-1)<k\le mj}|S_k|\le\sqrt{n}x_n\Big)$$

$$\ge\widehat{\mathbb{E}}\left[\prod_{j=1}^{l+1}\phi_1\left(\max_{m(j-1)<k\le mj}|S_k|/(\sqrt{n}x_n)\right)\phi\left(|S_{jm}|/(\epsilon\sqrt{n}x_n)\right)\right]$$

$$=\widehat{\mathbb{E}}\left[\prod_{j=1}^{l}\phi_1\left(\frac{\max_{m(j-1)<k\le mj}|S_k|}{\sqrt{n}x_n}\right)\phi\left(\frac{|S_{jm}|}{\epsilon\sqrt{n}x_n}\right)\right.$$

$$\left.\cdot\widehat{\mathbb{E}}\left[\phi_1\left(\frac{\max_{ml<k\le m(l+1)}|S_k|}{\sqrt{n}x_n}\right)\phi\left(\frac{|S_{(l+1)m}|}{\epsilon\sqrt{n}x_n}\right)\Big|S_k:k\le m(l-1)\right]\right].$$

When $\phi\left(\frac{|S_{(j-1)m}|}{\epsilon\sqrt{n}x_n}\right)\ne 0$, we have $|S_{(j-1)m}|\le(1+\delta/32)\epsilon\sqrt{n}x_n$, and then

$$\widehat{\mathbb{E}}\left[\phi_1\left(\frac{\max_{m(j-1)<k\le mj}|S_k|}{\sqrt{n}x_n}\right)\phi\left(\frac{|S_{jm}|}{\epsilon\sqrt{n}x_n}\right)\Big|S_k:k\le m(j-1)\right]$$

$$\ge\inf_{|y|\le(1+\delta/32)\epsilon\sqrt{n}x_n}\widehat{\mathbb{E}}\left[\phi_1\left(\frac{\max_{k\le m}|S_k+y|}{\sqrt{n}x_n}\right)\phi\left(\frac{|S_m+y|}{\epsilon\sqrt{n}x_n}\right)\right]$$

$$\ge\inf_{|y|\le 2\epsilon T^{-1/2}}\widehat{\mathbb{E}}\left[\phi_1\left(\frac{\sup_{0\le t\le 1}|W_n(t)+y|}{\sqrt{n}x_n/\sqrt{m}}\right)\phi\left(\frac{|W_m(1)+y|}{\epsilon\sqrt{n}x_n/\sqrt{m}}\right)\right].$$

By Corollary 3.1, uniformly in $|y|\le 2\epsilon T^{-1/2}$,

$$\lim_{n\to\infty}\widehat{\mathbb{E}}\left[\phi_1\left(\frac{\sup_{0\le t\le 1}|W_n(t)+y|}{\sqrt{n}x_n/\sqrt{m}}\right)\phi\left(\frac{|W_m(1)+y|}{\epsilon\sqrt{n}x_n/\sqrt{m}}\right)\right]$$

$$=\widetilde{\mathbb{E}}\left[\phi_1\left(\frac{\sup_{0\le t\le 1}|W(t)+y|}{T^{-1/2}}\right)\phi\left(\frac{|W(1)+y|}{\epsilon T^{-1/2}}\right)\right]$$

$$\ge\widetilde{\mathbb{V}}\Big(\sup_{0\le t\le 1}|W(t)+y|\le(1-\delta/32)T^{-1/2},|W(1)+y|\le\epsilon T^{-1/2}\Big).$$

Choose $\epsilon=\frac{\delta}{128}$. By Lemma 6.3, if $T=\delta^{-2}$ is large enough,

$$T^{-1}\log\left(\inf_{|y|\le 2\epsilon T^{-1/2}}\widetilde{\mathbb{V}}\Big(\sup_{0\le t\le 1}|W(t)+y|\le(1-\delta/32)T^{-1/2},|W(1)+y|\le\epsilon T^{-1/2}\Big)\right)$$

$$\ge-\frac{T\pi^2\underline{\sigma}^2}{8}(1-\delta/32-4\epsilon)^{-2}(1-\delta/16)^{-1}\ge-\frac{T\pi^2\underline{\sigma}^2}{8}(1-\delta/16)^{-3}.$$

Hence, there is a n_0 such that for all $n\ge n_0$,

$$\widehat{\mathbb{E}}\left[\phi\left(\frac{\max_{mj-1<k\le mj}|S_k|}{\sqrt{n}x_n}\right)\phi_1\left(\frac{|S_{jm}|}{\epsilon\sqrt{n}x_n}\right)\Big|S_k:k\le m(j-1)\right]\ge\exp\left\{-\frac{T\pi^2\underline{\sigma}^2}{8}(1-\delta/16)^{-4}\right\}.$$

It follows that

$$\log\mathbb{V}\Big(\max_{k\le n}|S_k|\le\sqrt{n}x_n\Big)\ge-(l+1)T\frac{\pi^2\underline{\sigma}^2}{8}(1-\delta/16)^{-4}\sim-\frac{\pi^2\underline{\sigma}^2}{8}(1-\delta/16)^{-4}x_n^{-2}.$$

Hence,

$$\liminf_{n\to\infty}x_n^2\log\mathbb{V}\Big(\max_{k\le n}|S_k|\le\sqrt{n}x_n\Big)\ge-\frac{\pi^2\underline{\sigma}^2}{8}.$$

The proof of (6.1) is completed. The proof of (6.1) is similar. □.

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