2.5 Stochastic differential equation with examples

2.5.1 Stochastic exponential

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Let X have a stochastic differential. Find U satisfy

$$dU(t) = U(t)dX(t), \quad U(0) = 1. \tag{2.5.1}$$

If $X(t)$ is of finite variation, then the equation is the ordinary differential equation (ODE) and the solution is $U(t) = e^{X(t)}$ and $\log U(t) = X(t)$. Now, let $f(x) = \log x$, then $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$. From (2.5.1), it follows that

$d[U,U](t) = U^2(t)d[X,X](t)$.

So

$$\begin{aligned} d\log U(t) =& \frac{1}{U(t)} dU(t) - \frac{1}{2}\frac{1}{U^2(t)} d[U,U](t) \\ =& dX(t) - \frac{1}{2} d[X,X](t) = d\left(X(t) - \frac{1}{2}[X,X](t) \right). \end{aligned}$$

So

$$
\begin{aligned}
d\log U(t) =& \frac{1}{U(t)}dU(t) - \frac{1}{2}\frac{1}{U^2(t)}d[U,U](t) \\
=& dX(t) - \frac{1}{2}d[X,X](t) = d\left(X(t) - \frac{1}{2}[X,X](t)\right).
\end{aligned}
$$

Hence

$$
\log U(t) - \log U(0) = X(t) - \frac{1}{2}[X,X](t) - \left(X(0) - \frac{1}{2}[X,X](0)\right).
$$

So

$$
\begin{aligned}
d\log U(t) =& \frac{1}{U(t)} dU(t) - \frac{1}{2}\frac{1}{U^2(t)} d[U,U](t) \\
=& dX(t) - \frac{1}{2} d[X,X](t) = d\left(X(t) - \frac{1}{2}[X,X](t)\right).
\end{aligned}
$$

Hence

$$
\log U(t) - \log U(0) = X(t) - \frac{1}{2}[X,X](t) - \left(X(0) - \frac{1}{2}[X,X](0)\right).
$$

That is

$$
\log U(t) = X(t) - \frac{1}{2}[X,X](t),
$$

$$
U(t) = \exp\left\{X(t) - \frac{1}{2}[X,X](t)\right\}.
$$

On the other hand, if $U(t) = \exp\left\{X(t) - \frac{1}{2}[X,X](t)\right\}$, then by Itô formula with $f(x) = e^x$,

$$
\begin{aligned}
dU(t) = &\exp\left\{X(t) - \frac{1}{2}[X,X](t)\right\} d\Big(X(t) - \frac{1}{2}[X,X](t)\Big) \\
&+\frac{1}{2}\exp\left\{X(t) - \frac{1}{2}[X,X](t)\right\} d\Big[X - \frac{1}{2}[X,X], X - \frac{1}{2}[X,X]\Big](t) \\
=&U(t)dX(t) - \frac{1}{2}U(t)d[X,X](t) + \frac{1}{2}U(t)d[X,X](t) \\
=&U(t)dX(t).
\end{aligned}
$$

Indeed, $U(t) = \exp\left\{X(t) - \frac{1}{2}[X,X](t)\right\}$ is a solution of (2.5.1).

Finally, if $V(t)$ is also a solution. Then by Itô formula with $f(x,y)=\frac{x}{y}$, $\frac{\partial f}{\partial x}=\frac{1}{y}$, $\frac{\partial f}{\partial y}=-\frac{x}{y^2}$, $\frac{\partial^2 f}{\partial x^2}=0$, $\frac{\partial^2 f}{\partial y^2}=2\frac{x}{y^3}$ $\frac{\partial^2 f}{\partial x\partial y}=-\frac{1}{y^2}$,

$$
\begin{aligned}
d(V(t)/U(t)) = & \frac{1}{U(t)}dV(t)-\frac{V(t)}{U^2(t)}dU(t) \\
& +\frac{1}{2}\cdot 0d[V,V](t)+\frac{1}{2}2\frac{V(t)}{U^3(t)}d[U,U](t)-\frac{1}{U^2(t)}d[V,U](t) \\
=&\frac{1}{U(t)}V(t)dX(t)-\frac{V(t)}{U^2(t)}U(t)dX(t) \\
& +\frac{1}{2}2\frac{V(t)}{U^3(t)}U^2(t)d[X,X](t)-\frac{1}{U^2(t)}V(t)U(t)d[X,X](t) \\
\equiv & 0.
\end{aligned}
$$

Hence the solution is unique.

Theorem

Let $X(t)$ be an Itô process. The equation

$$dU(t) = U(t)dX(t), \quad U(0) = 1$$

has an unique solution

$$U(t) =: \mathscr{E}(X)(t) = \exp\left\{X(t) - \frac{1}{2}[X,X](t)\right\},$$

this process is called the stochastic exponential of X.

Example

The stochastic exponential of $\alpha B(t)$ is $e^{\alpha B(t)-\frac{\alpha^2}{2}t}$.

Example

(Stock processes and its return process.) Let $S(t)$ denotes the price of stock and assume that it has a stochastic differential. Return on stock $R(t)$ is defined by the relation

$$dR(t) = \frac{dS(t)}{S(t)}.$$

Thus

$$dS(t) = S(t)dR(t).$$

The return is the value of $\$1$ invested at time t after a unit time.

If return is a constant rate r, then $dS(t) = S(t)rdt$ is the ordinary differential equation and the solution is $S(t) = S(0)e^{rt}$.

In Black-Scholes model, the return is uncertain and assumed to be a constant rate r plus a white noise, that is

$$R(t) = r + \sigma \dot{B}(t),$$

which means that

$$R(t)\Delta t \approx r\Delta t + \sigma(B(t+\Delta t) - B(t)).$$

So

$$dR(t) = rdt + \sigma dB(t).$$

The stock price $S(t)$ is the stochastic exponential of the return $R(t)$,

$$\begin{aligned} S(t) =& S(0) \exp\{R(t) - \frac{1}{2}[R, R](t)\} \\ =& S(0) \exp\left\{\sigma B(t) + \left(r - \frac{1}{2}\sigma^2\right) t\right\}. \end{aligned}$$

2.5.2 Definition of the stochastic differential equations

Definition

A equation of the form

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dB(t), \tag{2.5.2}$$

where $\mu(x, t)$ and $\sigma(x, t)$ are given and $X(t)$ is the unknown process, is called a stochastic differential equation (SDE) driven by Brownian motion.

Definition

$X(t)$ is called a strong solution of the SDE (2.5.2) with initial value $X(0)$ if for all $t > 0$, $X(t)$ is a function $F(t, X(0), (B(s), s \leq t))$ of the given Brownian motion $B(t)$ and $X(0)$, integrals $\int_0^t \mu(X(s), s)ds$ and $\int_0^t \sigma(X(s), s)dB(s)$ exist, and the integral equation is satisfied

$$X(t) = X(0) + \int_0^t \mu\big(X(s), s\big)ds + \int_0^t \sigma\big(X(s), s\big)dB(s).$$

Example

(Kabgevin equation and Ornstein-Uhlenbeck process). Find the solution of the SDE

$$dX(t) = -\alpha X(t)dt + \sigma dB(t).$$

Solution. If $\alpha = 0$, then the solution is $X(t) = X(0) + \sigma \int_0^t dB(s) = X(0) + \sigma B(t)$. If $\sigma = 0$, then the solution is $X(t) = X(0)e^{-\alpha t}$. Now, let $Y(t) = X(t)e^{\alpha t}$.

Then

$$\begin{aligned} dY(t) =& e^{\alpha t} dX(t) + X(t)\alpha e^{\alpha t} dt + dX(t) de^{\alpha t} \\ =& - e^{\alpha t}\alpha X(t)dt + e^{\alpha t}\sigma dB(t) + X(t)\alpha e^{\alpha t} dt + 0 \\ =& \sigma e^{\alpha t} dB(t). \end{aligned}$$

This gives

$$Y(t) = Y(0) + \int_0^t \sigma e^{\alpha s} dB(s).$$

Hence

$$X(t) = e^{-\alpha t}\left(X(0) + \int_0^t \sigma e^{\alpha s} dB(s)\right).$$

Integration by parts yields

$$\int_0^t \sigma e^{\alpha s} dB(s) = \sigma e^{\alpha t} B(t) - \int_0^t B(s) d\left(\sigma e^{\alpha s}\right).$$

Hence $X(t)$ is a function of $B(s), s \leq t$ and so a strong solution of the equation.

Suppose $X(0) = 0$. Then $X(t)$ is a mean zero Gaussian process with

$$\begin{aligned}
\mathrm{Cov}\{X(t), X(s)\} =& \sigma^2 e^{-\alpha t} e^{-\alpha s} \mathsf{E}\left[\int_0^t e^{\alpha u} dB(u) \int_0^s e^{\alpha v} dB(v)\right] \\
=& \sigma^2 e^{-\alpha(t+s)} \int_0^{s\wedge t} e^{2\alpha u} du \\
=& \frac{\sigma^2}{2\alpha} e^{-\alpha(t+s)} e^{2\alpha(s\wedge t)} \\
=& \frac{\sigma^2}{2\alpha} e^{-\alpha|t-s|}.
\end{aligned}$$

So, $X(t)$ is a stationary Gaussian process.

2.5.3 Existence and uniqueness of strong solution

Theorem

(Existence and Uniqueness) For the SDE

$$dX(t) = \mu\big(X(t), t\big)dt + \sigma\big(X(t), t\big)dB(t),$$

if the following conditions are satisfied

Theorem

1. *Coefficients are locally Lipschitz in x uniformly in t, that is for every T and N, there is a constant K depending only on T and N such that for all $|x|, |y| \leq N$ and all $0 \leq t \leq T$,*

$$|\mu(x,t) - \mu(y,t)| + |\sigma(x,t) - \sigma(y,t)| \leq K|x - y|,$$

2. *Coefficients satisfy the linear growth condition*

$$|\mu(x,t)| + |\sigma(x,t)| \leq K(1 + |x|),$$

3. $X(0)$ *is independent of* $(B(t), 0 \leq t \leq T)$, *and* $\mathsf{E}X^2(0) < \infty$,

Theorem

then there exists a unique strong solution $X(t)$ of the SDE.

$X(t)$ has continuous paths, moreover

$$\mathsf{E}\left[\sup_{0\le t\le T} X^2(t)\right] \le C\left(1+\mathsf{E}X^2(0)\right),$$

where constant C depends only on K and T.

If the coefficients depend on x only, the conditions can be weakened.

Theorem

(Yamada-Watanabe) Suppose that $\mu(x)$ satisfies Lipschitz condition and $\sigma(x)$ satisfies a Hölder condition of order α, $\alpha \geq 1/2$, that is, there is constant K such that

$$|\sigma(x) - \sigma(y)| \leq K|x - y|^{\alpha}.$$

Then the strong solution exists and is unique.

The proof of the above theorems can be found in L.C.G. Rogers and D. Williams (1990), *Diffusion, Markov Processes, and Martingale. Volume 2 Itô Calculus*, Wiley.

Here we give the proof of the first Theorem.

Lemma

Let μ and σ satisfy the locally Lipschitz condition. Let X and Y be adapted processes, and define $\widetilde{X}$ and $\widetilde{Y}$ as follows:

$$\widetilde{X}(t) = \xi + \int_0^t \mu(X(u), u)du + \int_0^t \sigma(X(u), u)dB(u),$$

$$\widetilde{Y}(t) = \eta + \int_0^t \mu(Y(u), u)du + \int_0^t \sigma(Y(u), u)dB(u)$$

Then there is a constant C such that

$$\mathsf{E}[(\widetilde{X}-\widetilde{Y})_t^{*2}] \le C\left\{\mathsf{E}[|\xi-\eta|^2] + \mathsf{E}\left(\int_0^t (X-Y)_u^{*2}du\right)\right\}, \quad 0 \le t \le T,$$

where $f_t^* = \sup\{|f(s)| : s \le t\}$.

Proof. Let $a(u)=\mu\big(X(u),u\big)-\mu\big(Y(u),u\big)$, $b(u)=\sigma\big(X(u),u\big)-\sigma\big(Y(u),u\big)$. Then

$$\widetilde{X}(t)-\widetilde{Y}(t)=\xi-\eta+\int_0^t a(u)du+\int_0^t b(u)dB(u),$$

$$(\widetilde{X}-\widetilde{Y})_t^*\le|\xi-\eta|+\int_0^t|a(u)|du+\sup_{s\le t}\left|\int_0^s b(u)dB(u)\right|.$$

So

$$
\begin{aligned}
& \mathsf{E}[(\widetilde{X}-\widetilde{Y})_t^{*2}] \\
\le & 3\mathsf{E}[|\xi-\eta|^2] + 3\mathsf{E}\left[\int_0^t |a(u)|du\right]^2 + 3\mathsf{E}\left[\sup_{s\le t}\left(\int_0^s b(u)dB(u)\right)^2\right] \\
\le & 3\mathsf{E}[|\xi-\eta|^2] + 3t\mathsf{E}\left[\int_0^t a^2(u)du\right] + 6\mathsf{E}\left[\int_0^t b^2(u)du\right] \\
\le & 3\mathsf{E}[|\xi-\eta|^2] + 3TK^2\mathsf{E}\left(\int_0^t (X-Y)_u^{*2}du\right) \\
& + 3K^2\mathsf{E}\left(\int_0^t (X-Y)_u^{*2}du\right).
\end{aligned}
$$

The proof is completed. □

Proof of the Theorem. Define

$$X_0(t) = X(0), \quad 0 \leq t \leq T,$$

$$X_{n+1}(t) = (\mathscr{R}X_n)(t)$$

$$= X(0) + \int_0^t \mu\big(X_n(u), u\big)du + \int_0^t \sigma\big(X_n(u), u\big)dB(u), \quad 0 \leq t \leq T.$$

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$$X_{n+1}(t) = (\mathscr{R}X_n)(t)$$

$$= X(0) + \int_0^t \mu\big(X_n(u), u\big)du + \int_0^t \sigma\big(X_n(u), u\big)dB(u), \quad 0 \le t \le T.$$

By the condition in the theorem, X_0 and X_1 are in $L^2_{[0,T]}$.

Now, define

$$\Delta_{n+1}(t) = \mathsf{E}\left[(X_{n+1} - X_n)_t^{*2}\right] = \mathsf{E}\left[(\mathscr{R}X_n - \mathscr{R}X_{n-1})_t^{*2}\right].$$

By the Lemma, we find that

$$\Delta_{n+1}(t) \le C \int_0^t \Delta_n(u)du,$$

and, by induction, it follows that, for all $0 \leq t \leq T$,

$$\Delta_n(t) \leq \eta C^n t^n / n!,$$

where $\eta = \Delta_0(T)$ with

$$\begin{aligned}\Delta_0(t) \leq & 2K^2 t^2 \big(1 + \mathsf{E}[X^2(0)]\big) + 2K^2 t \big(1 + \mathsf{E}[X^2(0)]\big) \\ = & 2K^2 (t^2 + t) \big(1 + \mathsf{E}[X^2(0)]\big) < \infty\end{aligned}$$

by Condition 2.

Hence, for $m > n \geq 0$,

$$\left(\mathsf{E}[(X_m - X_n)_T^{*2}]\right)^{1/2} \leq \left(\mathsf{E}\left[\left(\sum_{k=n}^{m-1}(X_{k+1} - X_k)_T^*\right)^2\right]\right)^{1/2}$$

$$\leq \sum_{k=n}^{m-1}\left(\Delta_{k+1}(T)\right)^{1/2} \leq \sum_{k=n}^{\infty}\left(\eta\frac{C^{k+1}T^{k+1}}{(k+1)!}\right)^{1/2} \to 0 \text{ as } n \to \infty.$$

Hence, on $[0, T]$, the sequence X_n converges uniformly in L_2 to a adapted process X.

Since

$$\mathsf{E}[(X - X_n)_T^{*2}] \leq \eta \left(\sum_{k=n}^{\infty} \left(\frac{C^{k+1} T^{k+1}}{(k+1)!} \right)^{1/2} \right)^2 \to 0, \quad (2.5.3)$$

it follows by the Lemma that

$$\mathsf{E}[(\mathscr{R}X - X_{n+1})_T^{*2}] = \mathsf{E}[(\mathscr{R}X - \mathscr{R}X_n)_T^{*2}] \leq C\mathsf{E}[(X - X_n)_T^{*2}] \to 0.$$

Since

$$\mathsf{E}[(X - X_n)_T^{*2}] \le \eta \left(\sum_{k=n}^{\infty} \left(\frac{C^{k+1} T^{k+1}}{(k+1)!} \right)^{1/2} \right)^2 \to 0, \quad (2.5.3)$$

it follows by the Lemma that

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Hence, $X = \mathscr{R}X$, and X is a solution of the SDE. Since the integrals are continuous, so X is continuous.

Next, we show the uniqueness of the solution. Let X and X' be two solutions. By the Lemma it follows that

$$\mathsf{E}[(X-X')_t^{*2}] \le C\int_0^t \mathsf{E}[(X-X')_u^{*2}]du \quad 0 \le t \le T,$$

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which implies that $\mathsf{E}[(X-X')_t^{*2}]=0$ for all $0\le t\le T$, and hence that $X=X'$ on $[0,T]$.

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which implies that $\mathsf{E}[(X - X')_t^{*2}] = 0$ for all $0 \le t \le T$, and hence that $X = X'$ on $[0, T]$.

In fact, let $w(t) = \int_0^t \mathsf{E}[(X - X')_u^{*2}] du$ and $f(t) = w(t) e^{-Ct}$.

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$$\mathsf{E}[(X-X')_t^{*2}] \le C\int_0^t \mathsf{E}[(X-X')_u^{*2}]du \;\; 0\le t\le T,$$

which implies that $\mathsf{E}[(X-X')_t^{*2}]=0$ for all $0\le t\le T$, and hence that $X=X'$ on $[0,T]$.

In fact, let $w(t)=\int_0^t \mathsf{E}[(X-X')_u^{*2}]du$ and $f(t)=w(t)e^{-Ct}$.

Then $0\le \frac{d}{dt}w(t)\le Cw(t)$ and $\frac{d}{dt}f(t)\le 0$.

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$$\mathsf{E}[(X-X')_t^{*2}] \le C\int_0^t \mathsf{E}[(X-X')_u^{*2}]du \quad 0\le t\le T,$$

which implies that $\mathsf{E}[(X-X')_t^{*2}]=0$ for all $0\le t\le T$, and hence that $X=X'$ on $[0,T]$.

In fact, let $w(t)=\int_0^t \mathsf{E}[(X-X')_u^{*2}]du$ and $f(t)=w(t)e^{-Ct}$. Then $0\le \frac{d}{dt}w(t)\le Cw(t)$ and $\frac{d}{dt}f(t)\le 0$. It follows that $f(t)\le f(0)$, $w(t)\le w(0)e^{Ct}=0$, $\frac{d}{dt}w(t)=0$. $\square$

Example

(Gisanov's SDE)

$$dX(t) = |X(t)|^r dB(t), \quad X(0) = 0, 1/2 \leq r < 1.$$

By Yamada-Watanabe theorem, this SDE has a unique solution. Obviously, $X(t) \equiv 0$ is the solution.

Markov property of the solution:

Theorem

Under the theorem for the existence and uniqueness, the strong solution of the SDE

$$dX(t) = \mu\big(X(t), t\big)dt + \sigma\big(X(t), t\big)dB(t), \quad X(0) = X_0,$$

is a Markov process.

Proof. Let $f(x)$ be a bounded function, $s, t \geq 0$. It is sufficient to show that

$$\mathsf{E}\left[f(X(s+t))\middle|\mathscr{F}_s\right] = \mathsf{E}\left[f(X(s+t))\middle|(X(s)\right].$$

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$$\mathsf{E}\left[f(X(s+t))\middle|\mathscr{F}_s\right] = \mathsf{E}\left[f(X(s+t))\middle|(X(s)\right].$$

Let $\widetilde{B}(u) = B(s+u) - B(s)$. Then $\widetilde{B}$ is independent of $\mathcal{F}_s$.

Proof. Let $f(x)$ be a bounded function, $s, t \geq 0$. It is sufficient to show that

$$\mathsf{E}\left[f(X(s+t))\middle|\mathscr{F}_s\right] = \mathsf{E}\left[f(X(s+t))\middle|(X(s)\right].$$

Let $\widetilde{B}(u) = B(s+u) - B(s)$. Then $\widetilde{B}$ is independent of $\mathcal{F}_s$.

Denote $X^{s,x}(t)$ be the unique solution of the SDE

$$dX(t) = \mu\big(X(t), s+t\big)dt + \sigma\big(X(t), s+t\big)d\widetilde{B}(t), \quad X(0) = x,$$

i.e.,

$$X^{s,x}(t) = x + \int_0^t \mu\big(X^{s,x}(u), s+u\big)du + \int_0^t \sigma\big(X^{s,x}(u), s+u\big)d\widetilde{B}(u).$$

Then $X^{s,x}(t)$ is independent of $\mathcal{F}_s$. Denote

$F(x,s,t)=X^{s,x}(t)$.

Then $X^{s,x}(t)$ is independent of $\mathcal{F}_s$. Denote $F(x,s,t)=X^{s,x}(t)$. Note

$$
\begin{aligned}
& X(s+t)-X(s) \\
= & \int_s^{s+t} \mu\big(X(u),u\big)du+\int_s^{s+t}\sigma\big(X(u),u\big)dB(u) \\
= & \int_0^t \mu\big(X(s+u),s+u\big)du+\int_0^t\sigma\big(X(s+u),s+u\big)d\widetilde{B}(u).
\end{aligned}
$$

So

$$X(s+t)=X^{s,X(s)}(t)=F(X(s),s,t).$$

Let $G(x) = f\left(F(x, s, t)\right)$. Then $G(x)$ is independent of $\mathscr{F}_s$.

Let $G(x) = f(F(x, s, t))$. Then $G(x)$ is independent of $\mathscr{F}_s$.

It follows that

$$
\begin{aligned}
& \mathsf{E}\left[f(X(s+t))\middle|\mathscr{F}_s\right] = \mathsf{E}\left[f\big(F(X(s), s, t)\big)\middle|\mathscr{F}_s\right] \\
=& \mathsf{E}\left[G(X(s))\middle|\mathscr{F}_s\right] \\
=& \left(\mathsf{E}\left[G(y)\middle|\mathscr{F}_s\right]\right)\Big|_{y=X(s)} \\
=& \left(\mathsf{E}\left[G(y)\right]\right)\Big|_{y=X(s)} \\
=& \mathsf{E}\left[G(X(s))\middle|X(s)\right] = \mathsf{E}\left[f(X(s+t))\middle|X(s)\right].
\end{aligned}
$$

The proof is completed.

Strong Markov property of the solution:

Theorem

Under the theorem for the existence and uniqueness, the strong solution of the SDE

$$dX(t) = \mu\big(X(t)\big)dt + \sigma\big(X(t)\big)dB(t), \quad X(0) = X_0,$$

is a strong Markov process.

Proof. Let $f(x)$ be a bounded function, τ is a stopping time, $t \geq 0$. It is sufficient to show that

$$\mathsf{E}\left[f(X(\tau+t))\middle|\mathscr{F}_\tau\right] = \mathsf{E}\left[f(X(\tau+t))\middle|(X(\tau)\right].$$

Proof. Let $f(x)$ be a bounded function, τ is a stopping time, $t \geq 0$. It is sufficient to show that

$$\mathsf{E}\left[f(X(\tau+t))\middle|\mathscr{F}_\tau\right] = \mathsf{E}\left[f(X(\tau+t))\middle|(X(\tau)\right].$$

Let $\widetilde{B}(u) = B(\tau+u) - B(\tau)$. Then $\widetilde{B}$ is independent of $\mathcal{F}_\tau$. Denote $X^x(t)$ be the unique solution of the SDE

$$dX(t) = \mu\big(X(t)\big)dt + \sigma\big(X(t)\big)d\widetilde{B}(t), \quad X(0) = x,$$

i.e.,

$$X^x(t) = x + \int_0^t \mu\big(X^x(u)\big)du + \int_0^t \sigma\big(X^x(u)\big)d\widetilde{B}(u).$$

Then $X^x(t)$ is independent of $\mathcal{F}_\tau$. Denote $F(x,t)=X^x(t)$.

Then $X^x(t)$ is independent of $\mathcal{F}_\tau$. Denote $F(x,t) = X^x(t)$.

Note

$$\begin{aligned} & X(\tau+t) - X(\tau) \\ = & \int_\tau^{\tau+t} \mu\big(X(u)\big)du + \int_\tau^{\tau+t} \sigma\big(X(u)\big)dB(u) \\ = & \int_0^t \mu\big(X(\tau+u)\big)du + \int_0^t \sigma\big(X(\tau+u)\big)d\widetilde{B}(u). \end{aligned}$$

So

$$X(\tau+t) = X^{X(\tau)}(t) = F(X(\tau), t).$$

Let $G(x) = f\left(F(x,t)\right)$. Then $G(x)$ is independent of $\mathscr{F}_\tau$.

Let $G(x) = f\left(F(x,t)\right)$. Then $G(x)$ is independent of $\mathscr{F}_\tau$. It follows that

$$
\begin{aligned}
&\mathsf{E}\left[f(X(\tau+t))\middle|\mathscr{F}_\tau\right] = \mathsf{E}\left[f\Big(F(X(\tau),t)\Big)\middle|\mathscr{F}_\tau\right] \\
=&\mathsf{E}\left[G(X(\tau))\middle|\mathscr{F}_\tau\right] \\
=&\left(\mathsf{E}\left[G(y)\middle|\mathscr{F}_\tau\right]\right)\Big|_{y=X(\tau)} \\
=&\left(\mathsf{E}\left[G(y)\right]\right)\Big|_{y=X(s)} \\
=&\mathsf{E}\left[G(X(\tau))\middle|X(\tau)\right] = \mathsf{E}\left[f(X(\tau+t))\middle|X(\tau)\right].
\end{aligned}
$$

The proof is completed.

Example

(Tanaka's SDE)

$$dX(t) = \mathrm{sign}\big(X(t)\big)dB(t), \tag{2.5.4}$$

where

$$\mathrm{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x \le 0. \end{cases}.$$

Since $\sigma(x) = \text{sign}(x)$ is discontinuous, it is not Lipschitz, and conditions for strong existence fail. It can be shown that a strong solution to Tanaks's SDE does not exist, for example, one can refer to Gihman I. I. and Skorohod A. V. (1982), *Stochastic Differential Equations*, Springer-Verlag.

Consider the case $X(0) = 0$. Here we show that if the Tanaka's SDE has a strong solution, then the solution is not unique.

Suppose $X(t)$ is an unique strong solution, then

$$X(t) = \int_0^t \text{sign}\big(X(s)\big) dB(s)$$

is a function of $B(s), s \leq t$. Also the solution is a continuous martingale with $[X,X](t) = \int_0^t \left(\text{sign}\big(X(s)\big)\right)^2 ds = t$. So by Levy's theorem $X(t)$ is a Brownian motion.

Let $Y(t) = -X(t)$. Then

$$\begin{aligned} Y(t) &= \int_0^t -\mathrm{sign}\big(X(s)\big)dB(s) \\ &= \int_0^t \mathrm{sign}\big(-X(s)\big)dB(s) + 2\int_0^t I_{\{0\}}\big(X(s)\big)dB(s). \end{aligned}$$

Notice

$$\begin{aligned} & \mathsf{E}\sup_{t\le T}\left(\int_0^t I_{\{0\}}\big(X(s)\big)dB(s)\right)^2 \\ \le & 4\mathsf{E}\left(\int_0^T I_{\{0\}}\big(X(s)\big)dB(s)\right)^2 = 4\mathsf{E}\int_0^T I_{\{0\}}^2\big(X(s)\big)ds \\ = & 4\mathsf{E}\lambda(\{s\in[0,T]: X(s)=0\}) = 0. \end{aligned}$$

Here λ is the Lebegue measure. It follows that

$$\mathsf{P}\Big(\int_0^t I_{\{0\}}\big(X(s)\big)dB(s) = 0 \ \ \forall t\Big) = 1.$$

Hence

$$Y(t) = \int_0^t \mathsf{sign}\big(Y(s)\big) dB(s),$$

which means that $Y(t) = -X(t)$ is also a solution. By the uniqueness, we must have $\mathsf{P}(X(t) = -X(t)) = 1$ which is impossible because $X(t)$ is a Brownian motion.

2.5.4. Solutions to linear stochastic differential equations

Consider the general linear SDE in one dimension:

$$dX(t) = \big(\alpha(t) + \beta(t)X(t)\big)dt + \big(\gamma(t) + \delta(t)X(t)\big)dB(t), \tag{2.5.5}$$

where functions $\alpha(\cdot)$, $\beta(\cdot)$, $\gamma(\cdot)$ and $\delta(t)$ are given adapted processes, and are continuous functions of t.

First, consider the case when $\alpha(t) \equiv 0$ and $\gamma(t) \equiv 0$. The SDE becomes

$$dU(t) = \beta(t)U(t)dt + \delta(t)U(t)dB(t). \tag{2.5.6}$$

This SDE is of the form

$$dU(t) = U(t)dY(t),$$

where the Itô process $Y(t)$ is defined by

$$dY(t) = \beta(t)dt + \delta(t)dB(t).$$

So, the uniqueness solution of (2.5.6) is the stochastic exponential of $Y(t)$ given by

$$
\begin{aligned}
U(t) =& \mathcal{E}(Y)(t) \\
=& U(0)\exp\left\{Y(t)-Y(0)-\frac{1}{2}[Y,Y](t)\right\} \\
=& U(0)\exp\left\{\int_0^t \beta(s)ds+\int_0^t \delta(s)dB(s)-\frac{1}{2}\int_0^t \delta^2(s)ds\right\} \\
=& U(0)\exp\left\{\int_0^t \left(\beta(s)-\frac{1}{2}\delta^2(s)\right)ds+\int_0^t \delta(s)dB(s)\right\}.
\end{aligned} \tag{2.5.7}
$$

To find a solution in the general case with nonzero $\alpha(t)$ and $\gamma(t)$, look for a solution of the form

$$X(t) = U(t)V(t), \tag{2.5.8}$$

where

$$dU(t) = \beta(t)U(t)dt + \delta(t)U(t)dB(t) \tag{2.5.9}$$

and

$$dV(t) = a(t)dt + b(t)dB(t). \tag{2.5.10}$$

Set $U(0) = 1$ and $V(0) = X(0)$. Note that U is given by (2.5.7).

Taking the differential of the product it is easy seen that

$$
\begin{aligned}
dX(t) =& U(t)dV(t) + V(t)dU(t) + dU(t)dV(t) \\
=& U(t)\big(a(t)dt + b(t)dB(t)\big) + V(t)U(t)\big(\beta(t)dt + \delta(t)dB(t)\big) \\
&+ \delta(t)U(t)b(t)(dB(t))^2 \\
=& \Big(a(t)U(t) + \delta(t)b(t)U(t) + X(t)\beta(t)\Big)dt \\
&+ \Big(U(t)b(t) + X(t)\delta(t)\Big)dB(t).
\end{aligned}
$$

By comparing the coefficients in the above equation and the equation (2.5.5), it follows that

$$b(t)U(t) = \gamma(t) \quad \text{and} \quad a(t)U(t) = \alpha(t) - \delta(t)\gamma(t).$$

Hence

$$V(t)-V(0)=\int_0^t a(s)ds+\int_0^t b(s)dB(s)$$
$$=\int_0^t \frac{\alpha(s)-\delta(s)\gamma(s)}{U(s)}ds+\int_0^t \frac{\gamma(s)}{U(s)}dB(s).$$

Thus, $X(t)$ is found to be

$$X(t)=U(t)\left(X(0)+\int_0^t \frac{\alpha(s)-\delta(s)\gamma(s)}{U(s)}ds+\int_0^t \frac{\gamma(s)}{U(s)}dB(s)\right),$$

where

$$U(t)=\exp\left\{\int_0^t\left(\beta(s)-\frac{1}{2}\delta^2(s)\right)ds+\int_0^t \delta(s)dB(s)\right\}.$$

To show the uniqueness, suppose X_1 and X_2 are solutions of (2.5.5) with $X_1(0) = X_2(0)$,

$$\begin{aligned} dX_1(t) =& \big(\alpha(t) + \beta(t)X_1(t)\big)dt + \big(\gamma(t) + \delta(t)X_1(t)\big)dB(t), \\ dX_2(t) =& \big(\alpha(t) + \beta(t)X_2(t)\big)dt + \big(\gamma(t) + \delta(t)X_2(t)\big)dB(t). \end{aligned}$$

Then $Z(t) = X_1(t) - X_2(t)$ satisfies

$$dZ(t) = \beta(t)Z(t)dt + \delta(t)Z(t)dB(t), \quad Z(0) = 0.$$

So $Z(t)$ satisfies the equation (2.5.10). $Z(t) \equiv 0$ is the unique solution.

2.5.5 Examples

I. Interest models

Example

(Vasicek interest rate model) The vasicek model for the interest rate process $R(t)$ is

$$dR(t) = \Big(\alpha - \beta R(t)\Big)dt + \sigma dB(t), \tag{2.5.11}$$

where α, β and σ are positive constants.

To solve the equation, we first solute $dR(t) = -\beta R(t)dt$. Obviously, the solution is $R(t) = R(0)e^{-\beta t}$. Then write

$$R(t) = e^{-\beta t}V(t), \quad \text{with } V(0) = R(0).$$

Then

$$dR(t) = -\beta e^{-\beta t}V(t)dt + e^{-\beta t}dV(t) = -\beta R(t)dt + e^{-\beta t}dV(t).$$

So

$$e^{-\beta t}dV(t) = \alpha dt + \sigma dB(t).$$

Hence

$$\begin{aligned} V(t) =& V(0) + \alpha \int_0^t e^{\beta s} ds + \sigma \int_0^t e^{\beta s} dB(s) \\ =& R(0) + \frac{\alpha}{\beta}\left(e^{\beta t} - 1\right) + \sigma \int_0^t e^{\beta s} dB(s). \end{aligned}$$

It follows that

$$R(t) = e^{-\beta t} R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma e^{-\beta t} \int_0^t e^{\beta s} dB(s).$$

The term $\int_0^t e^{\beta s} dB(s)$ is a normal random variable with mean zero and variance

$$\int_0^s e^{2\beta s} ds = \frac{1}{2\beta}\left(e^{2\beta t} - 1\right).$$

Therefore, $R(t)$ is normally distributed with mean

$$e^{-\beta t} R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right)$$

and variance

$$\frac{\sigma^2}{2\beta}\left(1 - e^{-2\beta t}\right).$$

Desirable property The Vasicek model has the desirable property that the interest rate is *mean-reverting*.

1. When $R(t) = \frac{\alpha}{\beta}$, the drift term in (2.5.11) is zero.
2. When $R(t) > \frac{\alpha}{\beta}$, this term is negative, which pushes $R(t)$ back toward $\frac{\alpha}{\beta}$.
3. When $R(t) < \frac{\alpha}{\beta}$, this term is positive, which again pushes $R(t)$ back toward $\frac{\alpha}{\beta}$.

If $R(0) = \frac{\alpha}{\beta}$, then $\mathsf{E}R(t) = \frac{\alpha}{\beta}$ for all t. If $R(0) \neq \frac{\alpha}{\beta}$, then $\lim_{t\to\infty} \mathsf{E}R(t) = \frac{\alpha}{\beta}$.

Undesirable property Note that $R(t)$ is normal distributed, no matter how the parameter $\alpha > 0$, $\beta > 0$ and $\sigma > 0$ are chosen, there is positive probability that $R(t)$ is negative, an undesirable property for an interest rate model.

Example

(Cox-Ingesoll-Ross (CIR) interest rate model). The Cox-Ingesoll-Ross model the interest rate $R(t)$ is

$$dR(t) = \Big(\alpha - \beta R(t)\Big)dt + \sigma\sqrt{R(t)}dB(t), \tag{2.5.12}$$

where α, β and σ are positive constants.

Like the Vasicek model, the CIR model is mean-reverting. The advantage of the CIR model over the Vasicek model is that the interest rate in the CIR model does not become negative. If $R(t)$ reaches zero, the term multiplying $dB(t)$ vanishes and the positive drift term αdt in the equation (2.5.12) drives the interest rate back into positive territory.

Unlike the Vasicek equation (2.5.11), the CIR equation (2.5.12) does not have a closed-form solution. However, the distribution of $R(t)$ for each positive t can be determined. That computation would take us too far afield. We derive the mean and variance of $R(t)$ instead.

To do so, we also write

$$R(t) = e^{-\beta t} V(t), \quad \text{with } V(0) = R(0).$$

Then

$$dR(t) = -\beta e^{-\beta t} V(t) dt + e^{-\beta t} dV(t) = -\beta R(t) dt + e^{-\beta t} dV(t).$$

So

$$e^{-\beta t} dV(t) = \alpha dt + \sigma \sqrt{R(t)} dB(t).$$

Hence

$$\begin{aligned} V(t) =& V(0) + \alpha \int_0^t e^{\beta s} ds + \sigma \int_0^t e^{\beta s} \sqrt{R(s)} dB(s) \\ =& R(0) + \frac{\alpha}{\beta}\left(e^{\beta t} - 1\right) + \sigma \int_0^t e^{\beta s} \sqrt{R(s)} dB(s). \end{aligned}$$

It follows that

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + \sigma e^{-\beta t}\int_0^t e^{\beta s}\sqrt{R(s)}dB(s).$$

Notice the expectation of an Itô integral is zero, we obtain

$$\mathsf{E}R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right).$$

This is the same expectation as in the Vasicek model.

Also, by the Isometry property of the Itô integral, we obtain

$$
\begin{aligned}
\mathsf{Var}\{R(t)\} =& \sigma^2 e^{-2\beta t}\mathsf{Var}\left\{\int_0^t e^{\beta s}\sqrt{R(s)}dB(s)\right\}\\
=& \sigma^2 e^{-2\beta t}\int_0^t e^{2\beta s}\mathsf{E}[R(s)]ds\\
=& \sigma^2 e^{-2\beta t}\int_0^t e^{2\beta s}\left[e^{-\beta s}R(0)+\frac{\alpha}{\beta}\left(1-e^{-\beta s}\right)\right]ds\\
=& \frac{\sigma^2}{\beta}R(0)\left(e^{-\beta t}-e^{-2\beta t}\right)+\frac{\alpha\sigma^2}{2\beta^2}\left(1-2e^{-\beta t}+e^{-2\beta t}\right).
\end{aligned}
$$

In particular,

$$
\lim_{t\to\infty}\mathsf{Var}\{R(t)\}=\frac{\alpha\sigma^2}{2\beta^2}.
$$

The moment generating function of $R(t)$:

Let $m(u,t) = Ee^{uR(t)}$. Then

$$\frac{\partial m}{\partial t} = E\frac{\partial}{\partial t}e^{uR(t)}?$$

The moment generating function of $R(t)$:

Let $m(u,t)=Ee^{uR(t)}$. Then

$$\frac{\partial m}{\partial t}=E\frac{\partial}{\partial t}e^{uR(t)}?$$

$$\begin{aligned} de^{uR(t)} &= e^{uR(t)}udR(t)+\frac{1}{2}e^{uR(t)}u^2(dR(t))^2 \\ =&e^{uR(t)}u\left[(\alpha-\beta R(t))dt+\sigma\sqrt{R(t)}dB(t)\right]+\frac{1}{2}e^{uR(t)}u^2\sigma^2R(t)dt \\ =&\alpha ue^{uR(t)}dt+\left(\frac{1}{2}u^2\sigma^2-u\beta\right)e^{uR(t)}R(t)dt \\ &+e^{uR(t)}\sigma\sqrt{R(t)}dB(t). \end{aligned}$$

It follows that

$$e^{uR(t)} - e^{uR(0)} = \int_0^t (\cdot)(s)ds + \int_0^t e^{uR(s)}\sigma\sqrt{R(s)}dB(s).$$

Taking the expectation yields

$$m(u,t) - m(u,0) = \int_0^t \left[E(\cdot)(s)\right] ds.$$

So

$$\begin{aligned}\frac{\partial m}{\partial t} =& E(\cdot)(t) \\ =& \alpha u E e^{uR(t)} + \left(\frac{1}{2}u^2\sigma^2 - u\beta\right) E[e^{uR(t)}R(t)] \\ =& \alpha u m + \left(\frac{1}{2}u^2\sigma^2 - u\beta\right)\frac{\partial m}{\partial u}.\end{aligned}$$

We arrive the equation that

$$\begin{cases} \frac{\partial m}{\partial t} = \alpha u m + \left(\frac{1}{2}u^2\sigma^2 - u\beta\right)\frac{\partial m}{\partial u}, \\ m(0,t) = 1. \end{cases}$$

II. Black-Scholes-Merton equation

Evolution of portfolio value

Consider an agent who at each time t has a portfolio valued at $X(t)$. This portfolio invests in a money market account paying a constant rate of interest r and in stock modeled by

$$dS(t) = \alpha S(t)dt + \sigma S(t)dB(t).$$

Suppose at each time t, the investor holds $\Delta(t)$ shares of stock. The position $\Delta(t)$ can be random but must be adapted to the filtration associated with the Brownian motion $B(t)$. The remainder of the portfolio value, $X(t) - \Delta(t)S(t)$, is invested in the money market account.

The differential $dX(t)$ is due to two factors,

1. the capital gain $\Delta(t)dS(t)$ on the stock position and
2. the interest earnings $r\big(X(t)-\Delta(t)S(t)\big)dt$ on the cash position.

In the other words

$$\begin{aligned} dX(t) =& \Delta(t)dS(t)+r\big(X(t)-\Delta(t)S(t)\big)dt \\ =& \Delta(t)\big(\alpha S(t)dt+\sigma S(t)dB(t)\big)+r\big(X(t)-\Delta(t)S(t)\big)dt \\ =& rX(t)dt+\Delta(t)\big(\alpha-r\big)S(t)dt+\Delta(t)\sigma S(t)dB(t). \end{aligned} \tag{2.5.13}$$

The three terms appearing in the last line above can be understood as follows:

1. any average underlying rate of return r on the portfolio, which is reflected by the term $rX(t)dt$,
2. a risk premium $\alpha - r$ for investing in the stock, which is reflected by the term $\Delta(t)(\alpha - r)S(t)dt$, and
3. a volatility term proportional to the size of stock investment, which is the term $\Delta(t)\sigma S(t)dB(t)$.

Discount. We shall often consider the discounted stock price $e^{-rt}S(t)$ and the discounted portfolio value of agent, $e^{-rt}X(t)$. According the Itô formula,

$$\begin{aligned} d\big(e^{-rt}S(t)\big) =& S(t)de^{-rt} + e^{-rt}dS(t) + d[e^{-rs}, S(s)](t) \\ =& -re^{-rt}S(t)dt + e^{-rt}dS(t) \\ =& (\alpha - r)e^{-rt}S(t)dt + \sigma e^{-rt}S(t)dB(t) \end{aligned}$$

and

$$\begin{aligned}
d\big(e^{-rt}X(t)\big) =& X(t)de^{-rt} + e^{-rt}dX(t) + d[e^{-rs}, X(s)](t) \\
=& -re^{-rt}X(t)dt + e^{-rt}dX(t) \\
=& \Delta(t)(\alpha - r)e^{-rt}S(t)dt + \Delta(t)\sigma e^{-rt}S(t)dB(t) \\
=& \Delta(t)d\big(e^{-rt}S(t)\big).
\end{aligned}$$

The last line shows that change in the discounted portfolio value is solely due to change in the discounted stock price.

Evolution of option value

Consider a European call that pays $(S(T)-K)^+$ at time T. Black, Scholes and Merton argued that the value of this call at any time should depend on the time (more precisely, on the time to expiration) and on the value of the stock price at that time. Following this reasoning, we let $V(t,x)$ denote the value of the call at time t if the stock price at that time is $S(t)=x$. So the value of the call is $V(t)=V(t,S(t))$.

According to the Itô formula, the differential of $V(t)$ is

$$
\begin{aligned}
&dV(t,S)\\
=&V_t(t,S)dt+V_x(t,S)dS+\frac{1}{2}V_{xx}(t,S)(dS)^2\\
=&V_t(t,S)dt+V_x(t,S)\big(\alpha S dt+\alpha S dB\big)\\
&+\frac{1}{2}V_{xx}(t,S)\sigma^2S^2dt\\
=&\Big[V_t(t,S)+\alpha S V_x(t,S)+\frac{1}{2}\sigma^2S^2V_{xx}(t,S)\Big]dt\\
&+\sigma S V_x(t,S)dB. \qquad (2.5.14)
\end{aligned}
$$

Equating the Evolutions

A (short option) hedging portfolio starts with some initial capital $X(0)$ and invests in the stock and money market account so that the portfolio value $X(t)$ at each time $t \in [0, T]$ agrees with $V(t, S(t))$. This happens if and only if

$$dX(t) = dV(t, S(t)) \tag{2.5.15}$$

and $X(0) = V(0, S(0))$. Comparing (2.5.13) and (2.5.14), (2.5.15) holds if and only if

$$\begin{aligned}&\Big[rV(t,S)+\Delta(t)\big(\alpha-r\big)S\Big]dt+\Delta(t)\sigma S dB\\=&\Big[V_t(t,S)+\alpha S V_x(t,S)+\frac{1}{2}\sigma^2S^2V_{xx}(t,S)\Big]dt+\sigma S V_x(t,S)dB.\end{aligned}$$

It follows that

$$\begin{cases}\Delta(t)=V_x(t,S),\\ rV(t,S)+\Delta(t)\big(\alpha-r\big)S\\ \qquad\qquad =V_t(t,S)+\alpha S V_x(t,S)+\frac{1}{2}\sigma^2S^2V_{xx}(t,S).\end{cases}$$

So, $V(t,x)$ satisfies the *Black-Scholes-Merton* partial differential equation:

$$V_t + rxV_x + \frac{1}{2}\sigma^2 x^2 V_{xx} = rV \quad \text{for all } t \in [0,T), \quad x \geq 0, \tag{2.5.16}$$

and that satisfies the terminal equation

$$V(T,x) = (x-K)^+.$$

2.5.6 Weak solutions to the stochastic differential equations

Definition

If there exist a probability space with filtration, Brownian motion $\widehat{B}(t)$ adapted to that filtration, a process $\widehat{X}(t)$ adapted to that filtration, such that $\widehat{X}(0)$ has distribution F_0, and for all t integrals below are defined, and $\widehat{X}(t)$ satisfies

$$\widehat{X}(t) = \widehat{X}(0) + \int_0^t \mu\big(\widehat{X}(s), s\big)ds + \int_0^t \sigma\big(\widehat{X}(s), s\big)d\widehat{B}(s),$$

then $\widehat{X}(t)$ is called a weak solution to the SDE

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dB(t), \qquad (2.5.17)$$

Definition

Weak solution is called unique if whenever $X(t)$ and $X'(t)$ are two solutions (perhaps on different probability spaces) such that the distributions of $X(0)$ and $X'(0)$ are the same, then all finite dimensional distributions of $X(t)$ and $X'(t)$ are the same.

Example

(Tanaka's SDE)

$$dX(t) = \operatorname{sign}\big(X(t)\big)dB(t), \tag{2.5.18}$$

where

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x \le 0. \end{cases}.$$

It can be shown that a strong solution to Tanaks's SDE does not exist, for example, one can refer to Gihman I. I. and Skorohod A. V. (1982), *Stochastic Differential Equations*, Springer-Verlag.

We show that Brownian motion is the unique weak solution of Tanaka's SDE. Let X_0 be the initial value and $\overline{B}(t)$ be some standard Brownian motion defined on the same probability space. Consider the Processes

$$\begin{aligned} X(t) =& X_0 + \overline{B}(t), \\ \widehat{B}(t) =& \int_0^t \frac{1}{\operatorname{sign}\big(X(s)\big)} d\overline{B}(s) = \int_0^t \operatorname{sign}\big(X(s)\big) d\overline{B}(s). \end{aligned}$$

Then $\widehat{B}(t)$ is a continuous martingale with

$$[\widehat{B}, \widehat{B}](t) = \int_0^t \left[\mathsf{sign}\big(X(s)\big)\right]^2 ds = t.$$

So by Levy's theorem $\widehat{B}(t)$ is a Brownian motion, different to the original one $\overline{B}(t)$. Then

$$dX(t) = d\overline{B}(t) = \mathsf{sign}\big(X(t)\big)d\widehat{B}(t), \quad X(0) = X_0.$$

Levy's Theorem implies also any weak solution is a Brownian motion.

Existence and uniqueness of weak solution

Theorem

If for each $t > 0$, functions $\mu(x, t)$ and $\sigma(x, t)$ are bounded and continuous then the SDE (2.5.17) has at least one weak solution starting at time s and point x, for all s and x.

If in addition their partial derivatives with respect to x up to order two are also bounded and continuous, then the SDE (2.5.17)has unique weak solution starting at time s and point x, for all s and x.

Theorem

If $\sigma(x,t)$ is positive and continuous and for any $T>0$ there is K_T such that for all $x \in \mathscr{R}$ and $0 \le t \le T$,

$$\big|\mu(x,t)\big| + \big|\sigma(x,t)\big| \le K_T(1+|x|),$$

then the SDE (2.5.17) has unique weak solution starting at time s and point x, for all $s \ge 0$ and x.

The proof of the above two theorems can be found in D. Stroock and S.R.S. Varadhan (1979), *Multidimensional Diffusion Processes*, Springer-Verlag.

2.5.7 Martingale Problem and heat equation

Let $X(t)$ solve the stochastic differential equation

$$dX(t) = \mu\big(X(t), t\big)dt + \sigma\big(X(t), t\big)dB(t), \quad t \geq 0. \quad (2.5.19)$$

Then $X(t)$ is a Markov process.

2.5.7 Martingale Problem and heat equation

Let $X(t)$ solve the stochastic differential equation

$$dX(t) = \mu\big(X(t), t\big)dt + \sigma\big(X(t), t\big)dB(t), \quad t \geq 0. \qquad (2.5.19)$$

Then $X(t)$ is a Markov process.

Suppose $f(x,t) \in C^{2,1}$. Then

$$\begin{aligned} df\big(X(t),t\big) = & \frac{\partial f\big(X(t),t\big)}{\partial x} dX(t) + \frac{\partial f\big(X(t),t\big)}{\partial t} dt \\ & + \frac{1}{2}\frac{\partial^2 f\big(X(t),t\big)}{\partial x^2}\big(dX(t)\big)^2 \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{2}\sigma^2\big(X(t),t\big)\frac{\partial^2 f\big(X(t),t\big)}{\partial x^2} + \mu\big(X(t),t\big)\frac{\partial f\big(X(t),t\big)}{\partial x} \right.\\
&\qquad \left. + \frac{\partial f\big(X(t),t\big)}{\partial t} \right) dt + \sigma\big(X(t),t\big)\frac{\partial f\big(X(t),t\big)}{\partial x} dB(t) \\
&= \left(\mathscr{L}_t f\big(X(t),t\big) + \frac{\partial f}{\partial t}\big(X(t),t\big) \right) dt + \sigma\big(X(t),t\big)\frac{\partial f\big(X(t),t\big)}{\partial x} dB(t),
\end{aligned}
$$

where

$$
\mathscr{L}_t f(x,t) = \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f(x,t)}{\partial x^2} + \mu(x,t)\frac{\partial f(x,t)}{\partial x}.
$$

So,

$$
\begin{aligned}
& f\big(X(t), t\big) - f\big(X(0), 0\big) \\
= & \int_0^t \left(\mathscr{L}_u f + \frac{\partial f}{\partial t}\right)\big(X(u), u\big) du \\
& + \int_0^t \sigma\big(X(u), u\big) \frac{\partial f\big(X(u), u\big)}{\partial x} dB(u).
\end{aligned}
$$

So,

$$\begin{aligned} & f\big(X(t),t\big)-f\big(X(0),0\big) \\ = & \int_0^t \left(\mathscr{L}_u f+\frac{\partial f}{\partial t}\right)\big(X(u),u\big)du \\ & +\int_0^t \sigma\big(X(u),u\big)\frac{\partial f\big(X(u),u\big)}{\partial x}dB(u). \end{aligned}$$

Example

If $dX(t)=\alpha X(t)dt+\sigma X(t)dB(t)$, then

$$\mathscr{L}_t f(x,t)=\frac{1}{2}\sigma^2x^2\frac{\partial^2 f}{\partial x^2}(x,t)+\alpha x\frac{\partial f}{\partial x}(x,t).$$

Let

$$M_f(t) = f\big(X(t), t\big) - \int_0^t \left(\mathscr{L}_u f + \frac{\partial f}{\partial t}\right)\big(X(u), u\big) du.$$

Then

$$M_f(t) = f\big(X(0), 0\big) + \int_0^t \sigma\big(X(u), u\big) \frac{\partial f\big(X(u), u\big)}{\partial x} dB(u)$$

is a local martingale.

If f solves the partial differential equation

$$\mathscr{L}_t f(x,t) + \frac{\partial f}{\partial t}(x,t) = 0,$$

Then $f\big(X(t),t\big)$ is a local martingale. If

$$\mathsf{E}\left[\int_0^T \sigma^2(X(u),u)\left(\frac{\partial f\big(X(u),u\big)}{\partial x}\right)^2 du\right] < \infty,$$

then $f\big(X(t),t\big)$ is a martingale. In such case,

$$f\big(X(t),t\big) = \mathsf{E}\left[f\big(X(T),T\big)\big|\mathscr{F}_t\right] = \mathsf{E}\left[f\big(X(T),T\big)\big|X(t)\right],$$

$$f\big(x,t\big) = \mathsf{E}\left[f\big(X(T),T\big)\big|X(t)=x\right].$$

Theorem

Let $f(x,t)$ solve the backward equation

$$\mathscr{L}_t f(x,t) + \frac{\partial f}{\partial t}(x,t) = 0, \quad \text{with } f(x,T) = g(x).$$

Assume that $\frac{\partial f}{\partial x}(x,t)$ is bounded. Then

$$f(x,t) = \mathsf{E}\left[g\big(X(T)\big)\big|X(t) = x\right],$$

if the SDE

$$dX(t) = \mu\big(X(t),t\big)dt + \sigma\big(X(t),t\big)dB(t)$$

satisfies the following conditions

Theorem

1. *Coefficients are locally Lipschitz in x uniformly in t, that is for every N, there is a constant K depending only on T and N such that for all $|x|, |y| \le N$ and all $0 \le t \le T$,*

$$|\mu(x,t) - \mu(y,t)| + |\sigma(x,t) - \sigma(y,t)| \le K|x-y|,$$

2. *Coefficients satisfy the linear growth condition*

$$|\mu(x,t)| + |\sigma(x,t)| \le K(1 + |x|),$$

3. *$X(0)$ is independent of $(B(t), 0 \le t \le T)$, and $\mathsf{E}X^2(0) < \infty$.*

Proof.

$$\mathsf{E}\left[\int_0^T \sigma^2(X(u),u)\left(\frac{\partial f(X(u),u)}{\partial x}\right)^2 du\right]$$

$$\leq C\mathsf{E}\left[\int_0^T \sigma^2(X(u),u)du\right] \leq C\left[1+\int_0^T \mathsf{E}[X^2(u)]du\right]$$

$$\leq C\big(1+T\mathsf{E}[X^2(0)]\big) < \infty.$$

So, $f(X(t),t)$ is a martingale.

Theorem

(Feynman-Kac Formula) Let $X(t)$ be a solution of the SDE (2.5.19) Let $C(x,t)$ denote

$$C(x,t) = \mathsf{E}\left[e^{-\int_t^T r\left(X(u),u\right)du} g\big(X(T)\big)\Big| X(t) = x\right], \quad (2.5.20)$$

for given bounded functions $r(x,t)$ and $g(x)$. Assume that there is a solution to

$$\mathscr{L}_t f(x,t) + \frac{\partial f}{\partial t}(x,t) = r(x,t)f(x,t), \quad \text{with } f(x,T) = g(x). \quad (2.5.21)$$

Then the solution is unique and $C(x,t)$ is that solution.

(2.5.20) is equivalent to

$$e^{-\int_0^t r\left(X(u),u\right)du} C(x,t) = \mathsf{E}\left[e^{-\int_0^T r\left(X(u),u\right)du} g\big(X(T)\big) \Big| \mathscr{F}_t \right].$$

(2.5.20) is equivalent to

$$e^{-\int_0^t r\left(X(u),u\right)du} C(x,t) = \mathsf{E}\left[e^{-\int_0^T r\left(X(u),u\right)du} g\left(X(T)\right) \Big| \mathscr{F}_t\right].$$

Sketch of the proof. By Itô's formula,

$$\begin{aligned} df\left(X(t),t\right) = & \left(\mathscr{L}_t f\left(X(t),t\right) + \frac{\partial f}{\partial t}\left(X(t),t\right)\right) dt \\ & + \sigma\left(X(t),t\right) \frac{\partial f\left(X(t),t\right)}{\partial x} dB(t) \\ = & r\left(X(t),t\right) f\left(X(t),t\right) dt + \sigma\left(X(t),t\right) \frac{\partial f\left(X(t),t\right)}{\partial x} dB(t). \end{aligned}$$

Write $h_t = e^{-\int_0^t r\left(X(u),u\right)du}$ and $r_t = r\left(X(t),t\right)$.

Then

$$
\begin{aligned}
& d\left[h_t f\big(X(t),t\big)\right] \\
=& h_t d\left[f\big(X(t),t\big)\right] + f\big(X(t),t\big) dh_t + d[h_t] d\left[f\big(X(t),t\big)\right] \\
=& h_t r\big(X(t),t\big) f\big(X(t),t\big) dt + h_t \sigma\big(X(t),t\big) \frac{\partial f\big(X(t),t\big)}{\partial x} dB(t) \\
& + f\big(X(t),t\big) h_t\big(-r\big(X(t),t\big)\big) dt \\
=& h_t \sigma\big(X(t),t\big) \frac{\partial f\big(X(t),t\big)}{\partial x} dB(t).
\end{aligned}
$$

So, $h_t f\big(X(t), t\big)$ is a continuous martingale. That is, there is an increasing sequence of stopping times $\sigma_n \nearrow \infty$ a.s. such that $h_{t\wedge\sigma_n} f\big(X(t\wedge\sigma_n), t\wedge\sigma_n\big)$, $t \geq 0$, is a martingale.

So, $h_t f\big(X(t), t\big)$ is a continuous martingale. That is, there is an increasing sequence of stopping times $\sigma_n \nearrow \infty$ a.s. such that $h_{t\wedge\sigma_n} f\big(X(t\wedge\sigma_n), t\wedge\sigma_n\big)$, $t \geq 0$, is a martingale.

It follows that

$$h_{t\wedge\sigma_n} f\big(X(t\wedge\sigma_n), t\wedge\sigma_n\big) = \mathsf{E}\left[h_{T\wedge\sigma_n} f\big(X(T\wedge\sigma_n), T\wedge\sigma_n\big)\big|\mathscr{F}_t\right].$$

So, $h_t f\big(X(t),t\big)$ is a continuous martingale. That is, there is an increasing sequence of stopping times $\sigma_n \nearrow \infty$ a.s. such that $h_{t\wedge\sigma_n} f\big(X(t\wedge\sigma_n), t\wedge\sigma_n\big)$, $t \ge 0$, is a martingale.

It follows that

$$h_{t\wedge\sigma_n} f\big(X(t\wedge\sigma_n), t\wedge\sigma_n\big) = \mathsf{E}\left[h_{T\wedge\sigma_n} f\big(X(T\wedge\sigma_n), T\wedge\sigma_n\big)\big|\mathscr{F}_t\right].$$

Letting $n\to\infty$ yields

$$h_t f\big(X(t),t\big) = \mathsf{E}\left[h_T f\big(X(T),T\big)\big|\mathscr{F}_t\right].$$

So

$$
\begin{aligned}
f\left(X(t),t\right) =& \mathsf{E}\left[h_t^{-1}h_T f\left(X(T),T\right)\Big|\mathscr{F}_t\right] \\
=& \mathsf{E}\left[e^{-\int_t^T r\left(X(u),u\right)du} g\left(X(T)\right)\Big|\mathscr{F}_t\right] \\
=& \mathsf{E}\left[e^{-\int_t^T r\left(X(u),u\right)du} g\left(X(T)\right)\Big|X(t)\right],
\end{aligned}
$$

by the Markov property.

So

$$\begin{aligned} f\big(X(t),t\big) =& \mathsf{E}\left[h_t^{-1}h_T f\big(X(T),T\big)\Big|\mathscr{F}_t\right] \\ =& \mathsf{E}\left[e^{-\int_t^T r\left(X(u),u\right)du} g\big(X(T)\big)\Big|\mathscr{F}_t\right] \\ =& \mathsf{E}\left[e^{-\int_t^T r\left(X(u),u\right)du} g\big(X(T)\big)\Big|X(t)\right], \end{aligned}$$

by the Markov property. That is

$$f(x,t) = \mathsf{E}\left[e^{-\int_t^T r\left(X(u),u\right)du} g\big(X(T)\big)\Big|X(t)=x\right]. \quad \square$$

Example

If

$$dX(t) = \alpha X(t)dt + \sigma X(t)dB(t) = X(t)(\alpha dt + \sigma dB(t)),$$

then

$$\mathscr{L}_t f(x,t) = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(x,t) + \alpha x \frac{\partial f}{\partial x}(x,t).$$

So,

$$C(x,t) = \mathsf{E}\left[e^{-r(T-t)}g\big(X(T)\big)\Big|X(t)=x\right],$$

is a solution to

$$\frac{\partial f}{\partial t} + \alpha x\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 f}{\partial x^2} = rf, \quad \text{with } f(x,T)=g(x).$$

$C(x,t)$ satisfies the *Black-Scholes-Merton* partial differential equation