2.4 Itô formula

2.4.1 Itô formula for Brownian motion

Suppose f is a real function. We want calculate $df(B(t))$.

Recall the Taylor's formula,

$$\Delta f(x) = f'(x)\Delta x + \frac{1}{2} f''(x)(\Delta x)^2 + \frac{1}{3!} f'''(x)(\Delta x)^3 + \cdots .$$

That is

$$df(x) = f'(x)dx + \frac{1}{2} f''(x)(dx)^2 + \frac{1}{3!} f'''(x)(dx)^3 + \cdots .$$

If B is a real variable or a real deferential function x, we know that

$$df(x) = f'(x)dx,$$

because $(dx)^2$, $(dx)^3$ etc are zeros.

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$$\begin{aligned} df(B(t)) =& f'(B(t))dB(t) + \frac{1}{2}f''(B(t))\big(dB(t)\big)^2 \\ &+ \frac{1}{3!}f'''(B(t))\big(dB(t)\big)^3 + \cdots \end{aligned}$$

Notice $\big(dB(t)\big)^2 = dt$,
$\big(dB(t)\big)^3 = \big(dB(t)\big)^2 dB(t) = dt dB(t) = 0$, $\big(dB(t)\big)^k = 0$, $k = 3, 4, \ldots$.

It follows that

$$df(B(t)) = f'(B(t))dB(t) + \frac{1}{2}f''(B(t))dt$$

or in integral notations

$$f(B(t)) - f(B(0)) = \int_0^t f'(B(u))dB(u) + \frac{1}{2}\int_0^t f''(B(u))du.$$

This is the Itô formula. The first integral is the stochastic integral and the second is the usual integral.

Theorem

(Itô formula) *If $f(x)$ is twice continuous differential function, then for any t,*

$$f(B(t)) = f(B(0)) + \int_0^t f'(B(u))dB(u) + \frac{1}{2}\int_0^t f''(B(u))du.$$

The Itô formula can also be written in the stochastic differential form:

$$df(B(t)) = f'(B(t))dB(t) + \frac{1}{2}f''(B(t))dt.$$

Proof. Let $\Pi = \{t_i^n\}$ be a partition of $[0, t]$. Clearly,

$$f(B(t)) = f(B(0)) + \sum_{i=0}^{n-1} \Big(f\big(B(t_{i+1}^n)\big) - f\big(B(t_i^n)\big)\Big).$$

Apply now Taylor's formula to $f\big(B(t_{i+1}^n)\big) - f\big(B(t_i^n)\big)$ to obtain

$$\begin{aligned} & f\big(B(t_{i+1}^n)\big) - f\big(B(t_i^n)\big) \\ =& f'\big(B(t_i^n)\big)\big(B(t_{i+1}^n) - B(t_i^n)\big) + \frac{1}{2} f''(\theta_i^n)\big(B(t_{i+1}^n) - B(t_i^n)\big)^2, \end{aligned}$$

where $\theta_i^n \in \big(B(t_{i+1}^n), B(t_i^n)\big)$.

Thus,

$$
\begin{aligned}
f(B(t)) =& f(B(0)) + \sum_{i=0}^{n-1} f'\big(B(t_i^n)\big)\big(B(t_{i+1}^n) - B(t_i^n)\big) \\
&+ \frac{1}{2}\sum_{i=0}^{n-1} f''(\theta_i^n)\big(B(t_{i+1}^n) - B(t_i^n)\big)^2.
\end{aligned}
$$

Taking limits as $\|\Pi\| = \max_i(t_{i+1}^n - t_i^n) \to 0$, the first sum converges to the Itô integral $\int_0^t f'\big(B(u)\big)dB(u)$; the second converges to $\int_0^t f''\big(B(u)\big)du$ by the following lemma.

Lemma

If g is a continuous function and $\Pi = \{t_i\}$ represents partitions of $[0,t]$, then for any $\theta_i^n \in \big(B(t_i^n), B(t_{i+1}^n)\big)$,

$$\lim_{\|\Pi\|\to 0} \sum_{i=0}^{n-1} g(\theta_i^n)\big(B(t_{i+1}^n) - B(t_i^n)\big)^2 = \int_0^t g\big(B(s)\big)du \ \text{ in probability.}$$

Proof. Notice

$$\left|\sum_{i=0}^{n-1}\Big(g(\theta_i^n)-g(B(t_i^n))\Big)\big(B(t_{i+1}^n)-B(t_i^n)\big)^2\right|$$

$$\leq \max_i \left|g(\theta_i^n)-g(B(t_i^n))\right|\cdot\sum_{i=0}^{n-1}\big(B(t_{i+1}^n)-B(t_i^n)\big)^2$$

$$\to 0 \quad \text{in probability },$$

due to the fact that the first term converges to zero almost surely by the continuity of g and B, and the second converges in probability to the quadratic variation Brownian motion t.

Now, we show that

$$\sum_{i=0}^{n-1} g(B(t_i^n))\big(B(t_{i+1}^n)-B(t_i^n)\big)^2 \to \int_0^t g(B(s))ds \text{ in probability.} \tag{2.4.1}$$

By continuity of $g(B(t))$ and the definition of the integral, it follows that

$$\sum_{i=0}^{n-1} g(B(t_i^n))\big(t_{i+1}^n - t_i^n\big) \to \int_0^t g(B(s))ds \ \ a.s.$$

Next we show that the difference between sums converges to zero in probability,

$$\sum_{i=0}^{n-1} g(B(t_i^n)) \left[\left(B(t_{i+1}^n) - B(t_i^n) \right)^2 - \left(t_{i+1}^n - t_i^n \right) \right] \to 0 \text{ in probability.} \tag{2.4.2}$$

Also, for every given D,

$$\Big|\sum_{i=0}^{n-1} g(B(t_i^n))$$

$$\cdot I\{|g(B(t_i^n))| > D\}\left[\big(B(t_{i+1}^n) - B(t_i^n)\big)^2 - (t_{i+1}^n - t_i^n)\right]\Big|$$

$$\leq \max_{s\leq t} |g(B(s))I\{|g(B(s))| > D\}| \cdot \left[\sum_{i=0}^{n-1} \big(B(t_{i+1}^n) - B(t_i^n)\big)^2 + t\right]$$

$\to 0$ in probability as $n \to \infty$ and $D \to \infty$,

because, the first term converges to zero as $D \to \infty$ almost surely by continuity of g and B, and the sum in the second term converges to t in probability.

So in showing (2.4.2), without loss of generality, we may assume that g is bounded such that $|g(x)| \le D$. Write $\Delta B_i = B(t_{i+1}^n) - B(t_i^n)$, $\Delta t_i = t_{i+1}^n - t_i^n$ and $g_i = g(B(t_i^n))$. It is easily seen that $\{g_i\big((\Delta B_i)^2 - \Delta t_i\big), \mathscr{F}_{t_{i+1}}, i = 0, \ldots, n-1\}$ is a sequence of martingale differences with

$$\mathsf{E}\left[\left(g_i\big((\Delta B_i)^2 - \Delta t_i\big)\right)^2 \middle| \mathscr{F}_{t_i}\right]$$

$$= g_i^2 \mathsf{Var}\big[(\Delta B_i)^2\big] = 2g_i^2(\Delta t_i)^2 \le 2D^2(\Delta t_i)^2.$$

So

$$\mathsf{E}\left[\Big(g_i\big((\Delta B_i)^2-\Delta t_i\big)\Big)^2\right]\le 2D^2(\Delta t_i)^2.$$

It follows that

$$\begin{aligned}&\mathsf{E}\left[\Big(\sum_{i=0}^{n-1} g_i\big((\Delta B_i)^2-\Delta t_i\big)\Big)^2\right]\\ =&\sum_{i=0}^{n-1}\mathsf{E}\left[\Big(g_i\big((\Delta B_i)^2-\Delta t_i\big)\Big)^2\right]\\ \le&2D^2\sum_{i=0}^{n-1}(\Delta t_i)^2\le 2D^2t\|\Pi\|\to 0.\end{aligned}$$

(2.4.2) is now proved.

Remark: *The main step of proving the Itô formula is*

$$
\begin{aligned}
&\sum_{i=0}^{n-1} f''(\theta_i^n)\big(B(t_{i+1}^n) - B(t_i^n)\big)^2 \\
\approx &\sum_{i=0}^{n-1} f''\big(B(t_i^n)\big)\big(B(t_{i+1}^n) - B(t_i^n)\big)^2 \\
\approx &\sum_{i=0}^{n-1} f''\big(B(t_i^n)\big)\big(t_{i+1}^n - t_i^n\big)^2 \approx \int_0^t f''\big(B(u)\big)du,
\end{aligned}
$$

where in the second approximation we used the important property that the quadratic variation Brownian motion is t, that is,

$$\big(B(t_{i+1}^n) - B(t_i^n)\big)^2 \approx [B,B](t_{i+1}^n) - [B,B](t_i^n) = t_{i+1}^n - t_i^n.$$

Example

Take $f(x) = x^2$, we have

$$B^2(t) = 2\int_0^t B(u)dB(u) + \int_0^t du = 2\int_0^t B(u)dB(u) + t.$$

In general, take $f(x) = x^m$, $m \geq 2$, we have

$$B^m(t) = m\int_0^t B^{m-1}(u)dB(u) + \frac{m(m-1)}{2}\int_0^t B^{m-2}(u)du.$$

Example

Find $de^{B(t)}$.

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Find $de^{B(t)}$.

By using Itô formula with $f(x) = e^x$, we have $f'(x) = e^x$, $f''(x) = e^x$ and

$$\begin{aligned} de^{B(t)} =& df(B(t)) = f'\big(B(t)\big)dB(t) + \frac{1}{2}f''\big(B(t)\big)dt \\ =& e^{B(t)}dB(t) + \frac{1}{2}e^{B(t)}dt. \end{aligned}$$

Thus $X(t) = e^{B(t)}$ has stochastic differential

$$dX(t) = X(t)dB(t) + \frac{1}{2}X(t)dt.$$

Definition

Process $Y(t)$ is called an Itô process if it can be represented as

$$Y(t) = Y(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dB(s), \quad 0 \le t \le T, \quad (2.4.3)$$

or equivalently, it has a stochastic differential as

$$dY(t) = \mu(t)dt + \sigma(t)dB(t),$$

where processes $\mu(t)$ and $\sigma(t)$ satisfy conditions:

1. $\mu(t)$ is adapted and $\int_0^T |\mu(t)|dt < \infty$ a.s.
2. $\sigma(t)$ is predictable and $\int_0^T \sigma^2(s)ds < \infty$ a.s.

Function μ is often called the drift coefficient and function σ the diffusion coefficient. Notice that μ and σ can depend (and often do) on $Y(t)$ and $B(t)$.

Function μ is often called the drift coefficient and function σ the diffusion coefficient. Notice that μ and σ can depend (and often do) on $Y(t)$ and $B(t)$.

A important case is when dependence of μ and σ on t only through $Y(t)$:

$$dY(t) = \mu(Y(t))dt + \sigma(Y(t))dB(t), \quad 0 \le t \le T.$$

Quadratic variation

Recall that the quadratic variation of a stochastic process Y is defined as

$$[Y,Y](t)=[Y,Y]([0,t])=\lim_{\|\Pi\|\to 0}\sum_{i=0}^{n-1}\big(Y(t_{i+1}^n)-Y(t_i^n)\big)^2$$

in probability,

and the quadratic covariation of two stochastic processes X and Y is defined as

$$[X,Y](t) = \frac{1}{2}\Big([X+Y,X+Y](t) - [X,X](t) - [Y,Y](t)\Big)$$
$$= \lim_{\|\Pi\|\to 0} \sum_{i=0}^{n-1} \big(X(t_{i+1}^n) - X(t_i^n)\big)\big(Y(t_{i+1}^n) - Y(t_i^n)\big)$$

in probability,

where the limits are taken over all partitions $\Pi = \{t_i^n\}$ of $[0,t]$ with $\|\Pi\| = \max_i(t_{i+1}^n - t_i^n) \to 0$.

Let Y be a Itô process as

$$Y(t) = Y(0) + \int_0^t \mu(u)du + \int_0^t \sigma(u)dB(u).$$

Notice the process of the first is a adapted process of finite variation, and the second is an Itô integral process.

It follows that

$$
\begin{aligned}
[Y,Y](t) = & \left[\int_0^s \mu(u)du, \int_0^s \mu(u)du\right](t) \\
& + 2\left[\int_0^s \mu(u)du, \int_0^s \sigma(u)dB(u)\right](t) \\
& + \left[\int_0^s \sigma(u)dB(u), \int_0^s \sigma(u)dB(u)\right](t) \\
= & 0 + 0 + \int_0^t \sigma^2(s)ds = \int_0^t \sigma^2(s)ds.
\end{aligned}
$$

And similarly, if $X(t) = X(0) + \int_0^t \overline{\mu}(u)du + \int_0^t \overline{\sigma}(u)dB(u)$ is another Itô process, then

$$[X, Y](t) = [X, Y]([0, t]) = \int_0^t \overline{\sigma}(s)\sigma(s)ds.$$

Introduce a convention

$dX(t)dY(t) = d[X,Y](t)$ and in particular $(dY(t))^2 = d[Y,Y](t)$.

The following rules follow

$$(dt)^2 = d[s,s](t) = 0, \quad dB(t)dt = d[B(s),s](t) = 0,$$
$$\big(dB(t)\big)^2 = d[B,B](t) = dt.$$

Then in the stochastic differential notations,

$$
\begin{aligned}
&d[X,Y](t) = dX(t)dY(t) \\
=&\big(\overline{\mu}(t)dt + \overline{\sigma}(t)dB(t)\big)\big(\mu(t)dt + \sigma(t)dB(t)\big) \\
=&\overline{\mu}(t)\mu(t)(dt)^2 + \Big(\overline{\sigma}(t)\mu(t) + \overline{\mu}(t)\sigma(t)\Big)dB(t)dt \\
&+ \overline{\sigma}(t)\sigma(t)\big(dB(t)\big)^2 \\
=&0 + 0 + \overline{\sigma}(t)\sigma(t)
\end{aligned}
$$

2.4.2 Itô formula for Itô processes

Integrals with respect to stochastic differential

Suppose Y has a stochastic differential with respect to B,

$$dX(t) = \mu(t)dt + \sigma(t)dB(t),$$

and $H(t)$ is predictable and satisfies

$$\int_0^t H(s)^2\sigma(s)^2 ds < \infty, \quad \int_0^t |H(s)\mu(s)| ds < \infty,$$

then both $\int_0^t H(s)\mu(s)ds$ and $\int_0^t H(s)\sigma(s)dB(s)$ are well defined.

The stochastic integral $Z(t) = \int_0^t H(s)dX(s)$ is defined as

$$Z(t) = \int_0^t H(s)dX(s) := \int_0^t H(s)\mu(s)ds + \int_0^t H(s)\sigma(s)dB(s)$$

or

$$dZ(t) = H(t)dY(t) = H(t)\mu(t)dt + H(t)\sigma(t)dB(t).$$

Itô formula for $f(X(t))$ Let $X(t)$ have a stochastic differential

$$X(t) = \mu(t)dt + \sigma(t)dB(t).$$

Then $(dX(t))^2 = d[X,X](t) = \sigma^2(t)dt$, $(dX(t))^3 = d[X,X]dtdX(t) = \sigma^2(t)dt\mu(t)dt + \sigma^2(t)dt\sigma^2(t)dB(t) = 0$, $(dX(t))^k = 0$, $k = 1, 2, \ldots$.

So

$$
\begin{aligned}
df(X(t)) =& f'(X(t))dX(t) + \frac{1}{2}f''(X(t))(dX(t))^2 \\
& + \frac{1}{3!}f'''(X(t))(dX(t))^3 + \ldots \\
=& f'(X(t))dX(t) + \frac{1}{2}f''(X(t))(dX(t))^2 \\
=& f'(X(t))dX(t) + \frac{1}{2}f''(X(t))d[X,X](t) \\
=& f'(X(t))\mu(t)dt + f'(X(t))\sigma(t)dB(t) \\
& + \frac{1}{2}f''(X(t))\sigma^2(t)dt.
\end{aligned}
$$

Theorem

Let $X(t)$ have a stochastic differential

$$dX(t) = \mu(t)dt + \sigma(t)dB(t).$$

If $f(x)$ is twice continuously differentiable, then the stochastic differential of the process $Y(t) = f(X(t))$ exists and is given by

$$\begin{aligned} df(X(t)) =& f'(X(t))dX(t) + \frac{1}{2}f''(X(t))d[X,X](t) \\ =& \left(f'(X(t))\mu(t) + \frac{1}{2}f''(X(t))\sigma^2(t) \right) dt \\ &+ f'(X(t))\sigma(t)dB(t). \end{aligned}$$

In integral notations

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))dX(s) + \frac{1}{2} f''(X(s))\sigma^2(s)ds.$$

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$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))dX(s) + \frac{1}{2}f''(X(s))\sigma^2(s)ds.$$

Example

Find a process X having the stochastic differential

$$dX(t) = X(t)dB(t) + \frac{1}{2}X(t)dt. \tag{2.4.4}$$

Solution. Let's look for a positive process X. Let $f(x) = \log x$. Then $f'(x) = 1/x$ and $f''(x) = -1/x^2$. So

$$\begin{aligned} d\log X(t) =& \frac{1}{X(t)} dX(t) - \frac{1}{2}\frac{1}{X^2(t)}(dX(t))^2 \\ =& dB(t) + \frac{1}{2}dt - \frac{1}{2}\frac{1}{X^2(t)}X^2(t)dt \\ =& dB(t). \end{aligned}$$

So that $\log X(t) = \log X(0) + B(t)$, and we find that

$$X(t) = X(0)e^{B(t)}.$$

Using the Itô formula for $X(t) = X(0)e^{B(t)}$ we verify that this X indeed satisfied (2.4.4).

2.4.4 Itô formula for functions of two-variables

If $X(t)$ and $Y(t))$ have stochastic differentials,

$$dX(t) = \mu_X(t)dt + \sigma_X(t)dB(t).$$

$$dY(t) = \mu_Y(t)dt + \sigma_Y(t)dB(t).$$

Then

$$dX(t)dY(t) = d[X, Y](t) = \sigma_X(t)\sigma_Y(t)dt.$$

It follows that

$$
\begin{aligned}
d\big(X(t)Y(t)\big) =& X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t) \\
=& X(t)dY(t) + Y(t)dX(t) + \sigma_X(t)\sigma_Y(t)dt.
\end{aligned}
$$

So

$$
\begin{aligned}
& X(t)Y(t) - X(0)Y(0) \\
= & \int_0^t X(s)dY(s) + \int_0^t Y(s)dX(s) + \int_0^t d[X,Y](s) \\
= & \int_0^t X(s)dY(s) + \int_0^t Y(s)dX(s) + \int_0^t \sigma_X(s)\sigma_Y(s)ds.
\end{aligned}
$$

This is the formula for integration by parts.

Example

Let f and g be C^2 functions and $B(t)$ the Brownian motion. Find $df(B)g(B)$.

Example

Let f and g be C^2 functions and $B(t)$ the Brownian motion. Find $df(B)g(B)$.

Solution Using Itô formula

$$
\begin{aligned}
df(B) =& f'(B)dB + \frac{1}{2}f''(B)dt, \\
dg(B) =& g'(B)dB + \frac{1}{2}g''(B)dt.,
\end{aligned}
$$

Example

Let f and g be C^2 functions and $B(t)$ the Brownian motion. Find $df(B)g(B)$.

Solution Using Itô formula

$$\begin{aligned} df(B) =& f'(B)dB + \frac{1}{2}f''(B)dt, \\ dg(B) =& g'(B)dB + \frac{1}{2}g''(B)dt., \end{aligned}$$

So

$$df(B)dg(B) = f'(B)g'(B)(dB)^2 = f'(B)g'(B)dt.$$

$$
\begin{aligned}
d(f(B)g(B)) &= f(B)dg(B) + g(B)df(B) + df(B)dg(B) \\
&= \left[f(B)g'(B) + f'(B)g(B)\right] dB \\
&\quad + \frac{1}{2}\left[f''(B) + 2f'(B)g'(B) + g''(B)\right] dt
\end{aligned}
$$

In general, if $f(x,y)$ has continuous partial derivatives up to order two. Then

$$df(x,y) = \frac{\partial}{\partial x} f(x,y)dx + \frac{\partial}{\partial y} f(x,y)dy + \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} f(x,y)(dx)^2 + \frac{\partial^2}{\partial y^2} f(x,y)(dy)^2 + 2\frac{\partial^2}{\partial x \partial y} f(x,y)dxdy\right).$$

Now,

$$(dX(t))^2 = d[X,X](t) = \sigma_X^2(t)dt,$$

$$(dY(t))^2 = d[Y,Y](t) = \sigma_Y^2(t)dt,$$

$$dX(t)dY(t) = d[X,Y](t) = \sigma_X(t)\sigma_Y dt.$$

Theorem

Let $f(x,y)$ have continuous partial derivatives up to order two (a C^2 function) and X, Y be Itô process, then

$$
\begin{aligned}
& df(X(t),Y(t)) \\
=& \frac{\partial}{\partial x} f(X(t),Y(t))dX(t) + \frac{\partial}{\partial y} f(X(t),Y(t))dY(t) \\
& + \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} f(X(t),Y(t))d[X,X](t) \right. \\
& \quad + \frac{\partial^2}{\partial y^2} f((X(t),Y(t))d[Y,Y](t) \\
& \quad \left. +2\frac{\partial^2}{\partial x \partial y} f(X(t),Y(t))d[X,Y](t) \right).
\end{aligned}
$$

Theorem

So

$$
\begin{aligned}
& df(X(t), Y(t)) \\
=& \frac{\partial}{\partial x} f(X(t), Y(t)) dX(t) + \frac{\partial}{\partial y} f(X(t), Y(t)) dY(t) \\
& + \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} f(X(t), Y(t)) \sigma_X^2(t) + \frac{\partial^2}{\partial y^2} f((X(t), Y(t)) \sigma_Y^2(t) \right. \\
& \left. \quad + 2 \frac{\partial^2}{\partial x \partial y} f(X(t), Y(t)) \sigma_X(t) \sigma_Y(t) \right) dt.
\end{aligned}
$$

Example

Let $X(t) = e^{B(t)-t/2}$. Find $dX(t)$.

Example

Let $X(t) = e^{B(t)-t/2}$. Find $dX(t)$.

Solution. $f(x,t) = e^{x-t/2}$. Here $Y(t) = t$. Notice $(dt)^2 = 0$ and $dB(t)dt = 0$. We obtain

$$\begin{aligned} df(B(t),t) =& \frac{\partial f}{\partial x} dB + \frac{\partial f}{\partial t} dt \\ & + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dB)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial^2 f}{\partial x \partial t} dB dt \\ =& e^{B-t/2} dB - \frac{1}{2} e^{B-t/2} dt + \frac{1}{2} e^{B-t/2} dt \\ =& e^{B(t)-t/2} dB(t) = X(t) dB(t). \end{aligned}$$

So that $dX(t) = X(t)dB(t)$.