1.5 Markov Property of Brownian motion

1.5 Markov Property of Brownian motion

1.5.1 Markov Property

Definition

Let $X(t), t \geq 0$ be a stochastic process on filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathsf{P})$. The process is called a Markov process if for any t and $s > 0$, the conditional distribution of $X(t+s)$ given $\mathscr{F}_t$ is the same as the conditional distribution of $X(t+s)$ given $X(t)$,

Definition

that is,

$$\mathsf{P}\big(X(t+s) \le y|\mathscr{F}_t\big) = \mathsf{P}\big(X(t+s) \le y|X(t)\big);$$

or equivalently, if for any t and $s > 0$ and every nonnegative Borel-measurable function f, there is another Borel-measurable function g such that

$$\mathsf{E}\left[f\big(X(t+s)\big)|\mathscr{F}_t\right] = g\big(X(t)\big).$$

Theorem

Brownian motion $B(t)$ process Markov property.

Theorem

Brownian motion $B(t)$ process Markov property.

Proof. It is easy to see by using the moment generating function that the conditional distribution of $B(t+s)$ given $\mathscr{F}_t$ is the same as that given $B(t)$.

Indeed,

$$
\begin{aligned}
& \mathsf{E}\left[e^{uB(t+s)}|\mathscr{F}_t\right] \\
=& e^{uB(t)}\mathsf{E}\left[e^{u\{B(t+s)-B(t)\}}|\mathscr{F}_t\right] \\
=& e^{uB(t)}\mathsf{E}\left[e^{u\{B(t+s)-B(t)\}}\right] \\
& \quad \text{since } e^{u\{B(t+s)-B(t)\}} \text{ is independent of } \mathscr{F}_t \\
=& e^{uB(t)}\mathsf{E}\left[e^{u\{B(t+s)-B(t)\}}|B(t)\right] \\
& \quad \text{since } e^{u\{B(t+s)-B(t)\}} \text{ is independent of } B(t) \\
=& \mathsf{E}\left[e^{uB(t+s)}|B(t)\right].
\end{aligned}
$$

Remark: With the conditional characteristic function taking the place of conditional moment generating function, one can show that every stochastic process with independent increments is a Markov process.

Transition probability function of a Markov process X is defined as

$$P(y, t; x, s) = P(X(t) \leq y | X(s) = x)$$

the conditional distribution function of the process at time t, given that it is at point x at time s.

In the case of Brownian motion the transition probability function is given by the distribution of the normal $N(x, t-s)$ distribution

$$P(y,t;x,s) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi(t-s)}} \exp\left\{-\frac{(u-x)^2}{2(t-s)}\right\} du$$
$$= \int_{-\infty}^{y} p_{t-s}(x,u) du.$$

It is easy seen that, for Brownian motion $B(t)$, $P(y,t;x,s) = P(y,t-s;x,0)$. In other words,

$$\mathsf{P}(B(t) \leq y | B(s) = x) = P(B(t-s) \leq y | B(0) = x).$$

The above property states that Brownian motion is time-homogeneous, that is, its distributions do not change with a shift in time.

Now, we calculate $\mathsf{E}[f(B(t))|\mathscr{F}_s]$.

$$\mathsf{E}[f(B(t))|B(s)=x] = \int f(y)P(dy,t;x,s) = \int f(y)p_{t-s}(x,y)dy.$$

So,

$$\mathsf{E}[f(B(t))|\mathscr{F}_s] = \mathsf{E}[f(B(t))|B(s)] = \int f(y)p_{t-s}(B(s),y)dy.$$

1.5.2 Stopping time

Definition

A random time T is called a stopping for $B(t)$, $t \geq 0$, if for any t it is possible to decide whether T has occurred or not by observing $B(s), 0 \leq s \leq t$. More rigorously, for any t the sets $\{T \leq t\} \in \mathscr{F}_t = \sigma(B(s), 0 \leq s \leq t)$, the σ-field generated by $B(s), 0 \leq s \leq t$.

If T is a stopping time, events observed before or at time T are described by σ-field $\mathscr{F}_T$, defined as the collection of sets

$$\{A \in \mathscr{F} : \text{ for any } t,\ A \cap \{T \leq t\} \in \mathscr{F}_t\}.$$

Example

1. Any nonrandom time T is a stopping time. Formally, $\{T \leq t\}$ is either the $\emptyset$ or Ω, which are members of $\mathscr{F}_t$ for any t.

Example

2. The first passage time of level a, $T_a = \inf\{t > 0 : B(t) = a\}$ is a stoping time. Clearly, if we know $B(s)$ for all $s \leq t$ then we know whether the Brownian motion took the value a before or at t or not. Thus we know that $\{T_a \leq t\}$ has occurred or not just by observing the past of the process prior to t. Formally,

$$\{T_a \leq t\} = \{\max_{u \leq t} B(u) \geq a\} \in \mathscr{F}_t.$$

Example

3. Let T be the time when Brownian motion reaches its maximum on the interval $[0,1]$. Then clearly, to decide whether $\{T \le t\}$ has occurred or not, it is not enough to know the values of the process prior to time t, one needs to know all the values on the interval $[0,1]$. So that T is not a stopping time.

1.5.3 Strong Markov property

Strong Markov property is similar to the Markov property, except that in the definition a fixed time t is replaced by a stopping time.

Theorem

Brownian motion $B(t)$ has the strong Markov property: for a finite stopping time T the regular conditional distribution of $B(T+t), t \geq 0$ given $\mathscr{F}_T$ is $P_{B(T)}$, that is,

$$P\big(B(T+t) \leq y | \mathscr{F}_T\big) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{(u - B(T))^2}{2t}\right\} du.$$

Theorem

Let T be a finite stopping time. Define the new process in $t \geq 0$ by

$$\widehat{B}(t) = B(T+t) - B(T).$$

Then $\widehat{B}(t)$ is a Brownian motion started at zero and independent of $\mathscr{F}_T$.

Proof. Fix $0 = t_0 \le t_1 \le \cdots \le t_n$, and reals $\theta_1, \cdots, \theta_n$ and take some bounded $\mathscr{F}_T$ measurable random variables ξ. Let $s_j = t_j - t_{j-1}$, $Z_j = \widehat{B}(t_j) - \widehat{B}(t_{j-1})$, $j = 1, \cdots, n$. We want to show that

$$\mathsf{E}\left[\xi \exp\left\{\sum_{j=1}^n \theta_j Z_j\right\}\right] = \mathsf{E}[\xi] \exp\left\{\frac{1}{2}\sum_{j=1}^n \theta_j^2 s_j\right\}.$$

Note

$$M(t) = \exp\left\{\theta B(t) - \frac{1}{2}\theta^2 t\right\}$$

is a martingale. So

$$\mathsf{E}[M(s+t)|\mathscr{F}_s] = M(s).$$

Note

$$M(t) = \exp\left\{\theta B(t) - \frac{1}{2}\theta^2 t\right\}$$

is a martingale. So

$$\mathsf{E}[M(s+t)|\mathscr{F}_s] = M(s).$$

Let S be a stopping time. We will show that

$$\mathsf{E}[M(S+t)|\mathscr{F}_S] = M(S).$$

We need a stopping time theorem.

Lemma

(i) (Stopped martingale) If $M(t)$ is a martingale with filtration $\{\mathscr{F}_t\}$ and τ is a stopping time, then the stopped process $M(t\wedge\tau)$ is a martingale, in particular for any t, $\mathsf{E}M(t\wedge\tau)=\mathsf{E}M(0)$.

(ii) (Optional stopping time theorem) If $M(t)$ is a right-continuous martingale with filtration $\{\mathscr{F}_t\}$ and τ,σ are bounded stopping times such that $P(\tau\le\sigma)=1$. Then

$$\mathsf{E}[M(\sigma)|\mathscr{F}_\tau]=M(\tau).$$

The proof is omitted.

By applying the optional stopping time theorem to the stopping time $S \wedge N$, we obtain

$$\mathsf{E}\left[\exp\left\{\theta B(S\wedge N+t)-\frac{1}{2}\theta^2(S\wedge N+t)\right\}\Big|\mathscr{F}_{S\wedge N}\right]$$

$$\left(\mathsf{E}\left[M(S\wedge N+t)\big|\mathscr{F}_{S\wedge N}\right]\right)$$

$$=\exp\left\{\theta B(S\wedge N)-\frac{1}{2}\theta^2(S\wedge N)\right\}.$$

$$\left(M(S\wedge N)\right)$$

Letting $N \nearrow \infty$ yields

$$\mathsf{E}\left[\exp\left\{\theta B(S+t)-\frac{1}{2}\theta^2(S+t)\right\}\Big|\mathscr{F}_S\right]=\exp\left\{\theta B(S)-\frac{1}{2}\theta^2 S\right\},$$

i.e.,

$$\mathsf{E}\left[\exp\left\{\theta\big(B(S+t)-B(S)\big)-\frac{1}{2}\theta^2 t\right\}\Big|\mathscr{F}_S\right]=1.$$

Taking $S = T + t_{n-1}$ and $t = s_n$ yields

$$
\begin{aligned}
& \mathsf{E}\left[\xi \exp\left\{\sum_{j=1}^{n}\left[\theta_j Z_j - \frac{1}{2}\theta_j^2 s_j\right]\right\}\right] \\
=& \mathsf{E}\left[\xi \exp\left\{\sum_{j=1}^{n-1}\left[\theta_j Z_j - \frac{1}{2}\theta_j^2 s_j\right]\right\}\right. \\
& \quad \left.\mathsf{E}\left[\exp\left\{\theta_n\big(B(S+s_n) - B(S)\big) - \frac{1}{2}\theta_n^2 s_n\right\}\Big|\mathscr{F}_{T+t_{n-1}}\right]\right] \\
=& \mathsf{E}\left[\xi \exp\left\{\sum_{j=1}^{n-1}\left[\theta_j Z_j - \frac{1}{2}\theta_j^2 s_j\right]\right\}\right].
\end{aligned}
$$

Conditioning successively on $\mathscr{F}_{T+t_k}$ $(k = n-2, \cdots, 0)$ gives

$$\mathsf{E}\left[\xi \exp\left\{\sum_{j=1}^{n}\left[\theta_j Z_j - \frac{1}{2}\theta_j^2 s_j\right]\right\}\right] = \mathsf{E}[\xi].$$

The proof is completed. □

1.6 Functions of Brownian motion

1.6 Functions of Brownian motion

1.6.1 The first passage time

Let x be a real number, the first passage time of Brownian motion $B(t)$ is

$$T_x = \inf\{t > 0 : B(t) = x\}$$

where $\inf \emptyset = \infty$.

Theorem

$P_a(T_b < \infty) = 1$ *for all* a *and* b.

To prove this theorem, we need a property of martingale, the proof is omitted.

Lemma

(i) (Stopped martingale) If $M(t)$ is a martingale with filtration $\{\mathscr{F}_t\}$ and τ is a stopping time, then the stopped process $M(t\wedge\tau)$ is a martingale, in particular for any t, $\mathsf{E}M(t\wedge\tau)=\mathsf{E}M(0)$.

(ii) (Optional stopping theorem) If $M(t)$ is a right-continuous martingale with filtration $\{\mathscr{F}_t\}$ and τ,σ are bounded stopping times such that $P(\tau\le\sigma)=1$. Then

$$\mathsf{E}[M(\sigma)|\mathscr{F}_\tau]=M(\tau).$$

Proof of the Theorem. Notice

$$
\begin{aligned}
& P_a(T_b < \infty) \\
=& \mathsf{P}(\inf\{t > 0 : B(t) = b\} < \infty | B(0) = a\} \\
=& \mathsf{P}(\inf\{t > 0 : B(t) - B(0) = b - a\} < \infty | B(0) = a\} \\
=& P_0(T_{b-a} < \infty)
\end{aligned}
$$

and

$$
\begin{aligned}
& P_0(T_{b-a} < \infty) \\
=& \mathsf{P}(\inf\{t > 0 : B(t) = b - a\} < \infty | B(0) = 0\} \\
=& \mathsf{P}(\inf\{t > 0 : -B(t) = a - b\} < \infty | -B(0) = 0\} \\
=& P_0(T_{a-b} < \infty)
\end{aligned}
$$

by the symmetry of Brownian motion. So, without loss of generality we assume $a = 0$, $b \geq 0$ and that the Brownian motion starts at 0.

For $u > 0$, let

$$Z(t) = \exp\left\{uB(t) - \frac{u^2}{2}t\right\}.$$

Then $Z(t)$ is a martingale. By the theorem for stopped martingale, $\mathsf{E}Z(0) = \mathsf{E}Z(t \wedge T_b)$, that is

$$1 = \mathsf{E}\exp\left\{uB(t \wedge T_b) - \frac{u^2}{2}(t \wedge T_b)\right\}. \tag{1.6.1}$$

We will take $t \to \infty$ in (1.6.1).

First, notice that the Brownian motion is always at or below level b for $t \leq T_b$ and so

$$0 \leq \exp\left\{uB(t \wedge T_b) - \frac{u^2}{2}(t \wedge T_b)\right\} \leq \exp\{uB(t \wedge T_b)\} \leq e^{ub},$$

that is, the random variables in (1.6.1) is bounded by e^{ub}.

Next, on the event $\{T_b = \infty\}$,

$$\begin{aligned} & \exp\left\{uB(t\wedge T_b) - \frac{u^2}{2}(t\wedge T_b)\right\} \\ \le & e^{ub}\exp\left\{-\frac{u^2}{2}t\right\} \to 0, \quad t\to\infty, \end{aligned}$$

and on the event $\{T_b < \infty\}$,

$$\begin{aligned} & \exp\left\{uB(t\wedge T_b) - \frac{u^2}{2}(t\wedge T_b)\right\} \\ = & \exp\left\{ub - \frac{u^2}{2}(t\wedge T_b)\right\} \\ & \to \exp\left\{ub - \frac{u^2}{2}T_b\right\}, \quad t\to\infty. \end{aligned}$$

Taking the limits in (1.6.1) yields

$$\mathsf{E}\left[\mathbb{I}\{T_b < \infty\} \exp\left\{ub - \frac{u^2}{2} T_b\right\}\right] = 1 \tag{1.6.2}$$

by the Dominated Convergence Theorem.

Taking the limits in (1.6.1) yields

$$\mathsf{E}\left[\mathbb{I}\{T_b < \infty\}\exp\left\{ub - \frac{u^2}{2}T_b\right\}\right] = 1 \tag{1.6.2}$$

by the Dominated Convergence Theorem.

Again, random variables in (1.6.2) are bounded by 1. Taking $\mu \to 0$ yields

$$\mathsf{P}(T_b < \infty) = \mathsf{E}\left[\mathbb{I}\{T_b < \infty\}\right] = 1.$$

The proof is now completed.

And also from (1.6.2), it also holds that

$$\mathsf{E}\left[\exp\left\{-\frac{u^2}{2}T_b\right\}\right] = e^{-ub}.$$

Replacing $u^2/2$ by α yields

$$\mathsf{E}e^{-\alpha T_b} = e^{-b\sqrt{2\alpha}}.$$

This is the Laplace transform of T_b.

Theorem

For real number b, let the first passage time of Brownian motion $B(t)$ be T_b. Then the Laplace transform of the distribution of T_b is given by

$$\mathsf{E}e^{-\alpha T_b} = e^{-|b|\sqrt{2\alpha}} \quad \text{for all } \alpha > 0, \tag{1.6.3}$$

and the density of T_b is

$$f_{T_b}(t) = \frac{|b|}{\sqrt{2\pi}} t^{-3/2} \exp\left\{-\frac{b^2}{2t}\right\}, \quad t \geq 0 \tag{1.6.4}$$

for $b \neq 0$.

Proof. For non-negative level b, (1.6.3) is proved. If b is negative, then because Brownian motion is symmetric, the first passage time T_b and $T_{|b|}$ have the same distribution.

Proof. For non-negative level b, (1.6.3) is proved. If b is negative, then because Brownian motion is symmetric, the first passage time T_b and $T_{|b|}$ have the same distribution. Equation (1.6.3) for negative b follows. (1.6.4) follows because

$$\int_0^\infty e^{-\alpha t} f_{T_b}(t)dt = e^{-|b|\sqrt{2\alpha}} \quad \text{for all } \alpha > 0.$$

Checking this equality is another hard work. We omit it.

Differentiation of (1.6.3) with respect to α results in

$$\mathsf{E}\left[T_b e^{-\alpha T_b}\right] = \frac{|b|}{\sqrt{2\alpha}} e^{-|b|\sqrt{2\alpha}} \quad \text{for all } \alpha > 0.$$

Letting $\alpha \downarrow 0$, we obtain $\mathsf{E}T_b = \infty$ so long as $b \neq 0$.

Differentiation of (1.6.3) with respect to α results in

$$\mathsf{E}\left[T_b e^{-\alpha T_b}\right] = \frac{|b|}{\sqrt{2\alpha}} e^{-|b|\sqrt{2\alpha}} \quad \text{for all } \alpha > 0.$$

Letting $\alpha \downarrow 0$, we obtain $\mathsf{E}T_b = \infty$ so long as $b \neq 0$.

Corollary

$ET_b = \infty$ for any $b \neq 0$.

The next result looks at the first passage time T_x as a process in x.

Theorem

The process of the first passage times $\{T_x, x > 0\}$*, has increments independent of the past, that is, for any* $0 < a < b$, $T_b - T_a$ *is independent of* $B(t), t \le T_a$*, and the distribution of the increment* $T_b - T_a$ *is the same as that of* T_{b-a}.

Proof. By the continuity, the Brownian motion must pass a before it passes b. So, $T_a \le T_b$. By the strong Markov property $\widehat{B}(t) = B(T_a + t) - B(T_a)$ is Bownian motion started at zero, and independent of the past $B(t), t \le T_a$.

$$T_b - T_a = \inf\{t \ge 0 : B(t) = b\} - T_a$$

$$= \inf\{t \ge T_a : B(t) = b\} - T_a$$

$$= \inf\{t \ge T_a : B(t - T_a + T_a) - B(T_a) = b - B(T_a)\} - T_a$$

$$= \inf\{t \ge 0 : \widehat{B}(t) = b - a\}.$$

Hence $T_b - T_a$ is the same as first hitting time of $b - a$ by $\widehat{B}$.

The conclusion follows.

1.6.2 Maximum and Minimum

Let $B(t)$ be Brownian motion (which starts at zero). Define

$$M(t) = \max_{0\le s\le t} B(s) \quad \text{and } m(t) = \min_{0\le s\le t} B(s).$$

Theorem

For any $x > 0$,

$$P(M(t) \ge x) = 2P(B(t) \ge x) = 2\left(1 - \Phi(\frac{x}{\sqrt{t}})\right),$$

where $\Phi(x)$ *stands for the standard normal distribution function.*

Proof. The second equation is obvious. For the first one, notice

$$\mathsf{P}(M(t) \geq x) = \mathsf{P}(T_x \leq t)$$
$$= \int_0^t \frac{x}{\sqrt{2\pi}} u^{-3/2} \exp\left\{-\frac{x^2}{2u}\right\} du$$
$$\left(\text{letting } \frac{x^2}{u} = \frac{y^2}{t}, y > 0, \text{ then } u^{-3/2} du = -2du^{-1/2} = -\frac{1}{xt^{1/2}} dy\right)$$
$$= \int_x^\infty \sqrt{\frac{2}{\pi t}} \exp\left\{-\frac{y^2}{2t}\right\} dy.$$

The proof is now completed.

Another proof. Observe that the event $\{M(t) \geq x\}$ and $\{T_x \leq t\}$ are the same. Since

$$\{B(t) \geq x\} \subset \{T_x \leq t\},$$

$$\mathsf{P}(B(t) \geq x) = \mathsf{P}(B(t) \geq x, T_x \leq t).$$

As $B(T_x) = x$,

$$\mathsf{P}(B(t) \geq x) = \mathsf{P}(T_x \leq t, B(T_x + (t - T_x)) - B(T_x) \geq 0).$$

Since T_x is a finite stopping time. By the strong Markov property, $\widehat{B}(s) = B(T_x + s) - B(T_x)$ is a Brownian motion which is independent of $\mathscr{F}_{T_x}$. So when $t \ge T_x$,

$$\mathsf{P}\left(\widehat{B}(t - T_x) \ge 0 | \mathscr{F}_{T_x}\right) = \frac{1}{2}.$$

So,

$$\begin{aligned}
\mathsf{P}(B(t) \ge x) =& \mathsf{E}\left[\mathbb{I}\{T_x \le t\} \mathsf{P}\left(\widehat{B}(t - T_x) \ge 0 | \mathscr{F}_{T_x}\right)\right] \\
=& \mathsf{E}\left[\mathbb{I}\{T_x \le t\} \frac{1}{2}\right] \\
=& \frac{1}{2}\mathsf{P}(T_x \le t) = \frac{1}{2}\mathsf{P}(M(t) \ge x).
\end{aligned}$$

To find the distribution of the minimum of Brownian motion $m(t) = \min_{0\le s\le t} B(s)$ we use the symmetry argument, and that

$$-\min_{0\le s\le t} B(s) = \max_{0\le s\le t}(-B(s)).$$

Notice $-B(t)$ is also a Brrownian motion which has the property as $B(t)$. It follows that for $x < 0$,

$$\begin{aligned} \mathsf{P}(\min_{0\le s\le t} B(s) \le x) =& \mathsf{P}(\max_{0\le s\le t}(-B(s)) \ge -x) \\ =& 2\mathsf{P}(-B(t) \ge -x) = 2\mathsf{P}(B(t) \le x). \end{aligned}$$

Theorem

For any $x < 0$,

$$P(\min_{0\le s\le t} B(s) \le x) = 2P(B(t) \le x) = 2P(B(t) \ge -x).$$

1.6.3 Reflection principle and joint distribution

Theorem

(Reflection principle) Let T be a stopping time. Define $\widehat{B}(t) = B(t)$ for $t \le T$, and $\widehat{B}(t) = 2B(T) - B(t)$ for $t \ge T$. Then $\widehat{B}$ is also Brownian motion.

1.6.3 Reflection principle and joint distribution

Theorem

(Reflection principle) Let T be a stopping time. Define $\widehat{B}(t) = B(t)$ for $t \le T$, and $\widehat{B}(t) = 2B(T) - B(t)$ for $t \ge T$. Then $\widehat{B}$ is also Brownian motion.

Note that $\widehat{B}$ defined above coincides with $B(t)$ for $t \le T$, and then for $t \ge T$ it is the reflected path about the horizontal line passing though $(T, B(T))$, that

$$\widehat{B}(t) - B(T) = -(B(t) - B(T)),$$

which gives the name to the result.

Proof. Consider the process

$$Y(t) =: B(t)\ (0 \le t \le T), \quad Z(t) = B(t+T) - B(T)\ (t \ge 0).$$

By the strong Markov property, Z is a Brownian motion independent of Y and T.

Proof. Consider the process

$$Y(t) =: B(t) \ (0 \le t \le T), \quad Z(t) = B(t+T) - B(T) \ (t \ge 0).$$

By the strong Markov property, Z is a Brownian motion independent of Y and T. So, $-Z$ is also a Brownian motion, also independent of Y and T. Thus $(Y, T, Z) \stackrel{d}{=} (Y, T, -Z)$.

Proof. Consider the process

$$Y(t) =: B(t)\ (0 \le t \le T),\ \ Z(t) = B(t+T) - B(T)\ (t \ge 0).$$

By the strong Markov property, Z is a Brownian motion independent of Y and T. So, $-Z$ is also a Brownian motion, also independent of Y and T. Thus $(Y, T, Z) \overset{d}{=} (Y, T, -Z)$. The map

$$\varphi : (Y, T, Z) \to \Big(Y(t)I\{t \le T\} + (Y(T) + Z(t-T))I\{t > T\}\Big)_{t \ge 0}$$

produces a continuous process, which will therefore have the same law as $\varphi(Y, T, -Z)$. But $\varphi(Y, T, Z) = B$, $\varphi(Y, T, -Z) = \widehat{B}$. □

Corollary

For any $y > 0$, let $\widehat{B}(t)$ be $B(t)$ reflected at T_y, that is, $\widehat{B}(t)$ equals $B(t)$ before $B(t)$ hits y, and is the reflection of $B(t)$ after the first hitting time. Then $\widehat{B}(t)$ is also a Brownian motion.

Theorem

The joint distribution of $(B(t), M(t))$ *has the density*

$$f_{B,M}(x,y) = \sqrt{\frac{2}{\pi}} \frac{2y-x}{t^{3/2}} \exp\left\{-\frac{(2y-x)^2}{2t}\right\}, \quad \textit{for } y \geq 0, x \leq y.$$

Proof. Let $y > 0$ and $y > x$. Let $\widehat{B}(t)$ be $B(t)$ reflected T_y. Then

$$\mathsf{P}(B(t) \le x, M(t) \ge y) = \mathsf{P}(B(t) \le x, T_y \le t)$$

$$= \mathsf{P}(T_y \le t, \widehat{B}(t) \ge 2y - x) \quad (\text{on } \{T_y \le t\}, \widehat{B}(t) = 2y - B(t)\ )$$

$$= \mathsf{P}(\widehat{T}_y \le t, \widehat{B}(t) \ge 2y - x) = \mathsf{P}(T_y \le t, B(t) \ge 2y - x)$$

(since T_y is the same for $\widehat{B}$ and B)

$$= \mathsf{P}(B(t) \ge 2y - x)$$

(since $y - x > 0$, and $\{B(t) \ge 2y - x\} \subset \{T_y \le t\}$)

$$= 1 - \Phi\left(\frac{2y - x}{\sqrt{t}}\right).$$

That is

$$\int_{-\infty}^{x}\int_{y}^{\infty} f_{B,M}(u,v)dudv = 1 - \Phi(\frac{2y-x}{\sqrt{t}}).$$

The density is obtained by differentiation.

Corollary

The conditional distribution of $M(t)$ given $B(t) = x$ is

$$f_{M|B}(y|x) = \frac{2(2y-x)}{t} \exp\left\{-\frac{2y(y-x)}{t}\right\}, \quad y > 0, x \le y.$$

Proof. The density of $B(t)$ is

$$f_B(x)=\frac{1}{\sqrt{2\pi t}}\exp\left\{-\frac{x^2}{2t}\right\}.$$

So for $y>0$ and $x\le y$,

$$\begin{aligned}
f_{M|B}(y|x)=&\frac{f_{B,M}(x,y)}{f_B(x)}\\
=&\sqrt{\frac{2}{\pi}}\frac{2y-x}{t^{3/2}}\cdot\sqrt{2\pi t}\exp\left\{-\frac{(2y-x)^2}{2t}+\frac{x^2}{2t}\right\}\\
=&\frac{2(2y-x)}{t}\exp\left\{-\frac{2y(y-x)}{t}\right\}.
\end{aligned}$$

1.6.4 Zeros of Brownian motion. Arcsine law

A time point τ is called a zero of Brownian motion if $B(\tau)=0$. Let $\{B^x(t)\}$ denotes Brownian motion started at x.

Theorem

For any $x\neq 0$, the probability that $\{B^x(t)\}$ has at least one zero in the interval $(0,t)$, is give by

$$\frac{|x|}{\sqrt{2\pi}}\int_0^t u^{-3/2}\exp\left\{-\frac{x^2}{2u}\right\}du=\sqrt{\frac{2}{\pi t}}\int_{|x|}^{\infty}\exp\left\{-\frac{y^2}{2t}\right\}dy,$$

that is the same probability of $P_0(T_{|x|}\leq t)$.

Proof. If $x<0$, then due to continuity of $B^x(t)$, the events $\{B^x$ has at least one zero between 0 and t $\}$ and $\{\max_{0\le s\le t} B^x(t)\ge 0\}$ are the same. Since $B^x(t)=B(t)+x$. So

$$\begin{aligned}
&\mathsf{P}(B^x \text{ has at least a zero between 0 and } t)=\mathsf{P}(\max_{0\le s\le t} B^x(t)\ge 0)\\
=&P_0(\max_{0\le s\le t} B(t)+x\ge 0)=P_0(\max_{0\le s\le t} B(t)\ge -x)\\
=&2P_0(B(t)\ge -x)=P_0(T_{|x|}\le t).
\end{aligned}$$

If $x>0$, then $-B^x(t)$ is a Brownian motion started at $-x$ by the symmetry of Brownian motion. The result follows.

Theorem

The probability that Brownian motion $B(t)$ has no zeros in the time interval (a, b) is given by

$$\frac{2}{\pi}\arcsin\sqrt{\frac{a}{b}}.$$

Proof. Denote by

$$h(x) = \mathsf{P}(B \text{ has at least one zero in } (a,b) | B_a = x).$$

By the Markov property

$\mathsf{P}(B$ has at least one zero in $(a,b)|B_a = x)$ is the same as

$\mathsf{P}(B^x$ has at least one zero in $(0, b-a))$. So

$$h(x) = \frac{|x|}{\sqrt{2\pi}} \int_0^{b-a} u^{-3/2} \exp\left\{-\frac{x^2}{2u}\right\} du.$$

By conditioning

$$
\begin{aligned}
&\mathsf{P}(B \text{ has at least one zero in } (a,b)) \\
=&\int_{-\infty}^{\infty} \mathsf{P}(B \text{ has at least one zero in } (a,b) | B_a = x)\mathsf{P}(B_a \in dx) \\
=&\int_{-\infty}^{\infty} h(x)\mathsf{P}(B_a \in dx) \\
=&\sqrt{\frac{2}{\pi a}}\int_0^{\infty} h(x)\exp\left\{-\frac{x^2}{2a}\right\}dx \\
=&\sqrt{\frac{2}{\pi a}}\int_0^{\infty}\left[\frac{|x|}{\sqrt{2\pi}}\int_0^{b-a} u^{-3/2}\exp\left\{-\frac{x^2}{2u}\right\}du\right]\exp\left\{-\frac{x^2}{2a}\right\}dx.
\end{aligned}
$$

By conditioning

$$
\begin{aligned}
&\mathsf{P}(B \text{ has at least one zero in } (a,b)) \\
&= \int_{-\infty}^{\infty} \mathsf{P}(B \text{ has at least one zero in } (a,b) | B_a = x)\mathsf{P}(B_a \in dx) \\
&= \int_{-\infty}^{\infty} h(x)\mathsf{P}(B_a \in dx) \\
&= \sqrt{\frac{2}{\pi a}} \int_0^{\infty} h(x) \exp\left\{-\frac{x^2}{2a}\right\} dx \\
&= \sqrt{\frac{2}{\pi a}} \int_0^{\infty} \left[\frac{|x|}{\sqrt{2\pi}} \int_0^{b-a} u^{-3/2} \exp\left\{-\frac{x^2}{2u}\right\} du\right] \exp\left\{-\frac{x^2}{2a}\right\} dx.
\end{aligned}
$$

Performing the necessary calculations we obtain

$$
\mathsf{P}(B \text{ has at least one zero in } (a,b)) = \frac{2}{\pi} \arccos\sqrt{\frac{a}{b}}.
$$

Let

$$\gamma_t = \sup\{s \le t : B(s) = 0\} = \text{ last zero before } t.$$

$$\beta_t = \inf\{s \ge t : B(s) = 0\} = \text{ first zero after } t.$$

Note that β_t is a stopping time but γ_t is not.

Let

$$\gamma_t = \sup\{s \le t : B(s) = 0\} = \text{ last zero before } t.$$

$$\beta_t = \inf\{s \ge t : B(s) = 0\} = \text{ first zero after } t.$$

Note that β_t is a stopping time but γ_t is not.

Corollary

$$P(\gamma_t \le x) = \frac{2}{\pi} \arcsin \sqrt{\frac{x}{t}}, \quad x \le t,$$

$$P(\beta_t \ge y) = \frac{2}{\pi} \arcsin \sqrt{\frac{t}{y}}, \quad y \ge t,$$

$$P(\gamma_t \le x, \beta_t \ge y) = \frac{2}{\pi} \arcsin \sqrt{\frac{x}{y}}, \quad x \le t \le y.$$

Proof.

$$\{\gamma_t \le x\} = \{B \text{ has no zeros in } (x,t)\},$$

$$\{\beta_t \ge y\} = \{B \text{ has no zeros in } (t,y)\},$$

$$\{\gamma_t \le x, \beta_t \ge y\} = \{B \text{ has no zeros in } (x,y)\}.$$

Theorem

The set of zeros of a Brownian motion is a random uncountable closed set without isolated points and has Lebesgue measure zero.