

## 1.3 Properties of Brownian motion paths

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### 1.3.1 Continuity and differentiability

Almost every sample path  $B(t)$ ,  $0 \leq t \leq T$

- ① is a continuous function of  $t$ ,
- ② is not monotone in any interval, no matter how small the interval is;
- ③ is not differentiable at any point.

## 1.3 Properties of Brownian motion paths

## 1.3.1 Continuity and differentiability

Property 1 is known. Property 2 follows from Property 3. For Property 3, one can show that

$$\lim_{h \rightarrow 0} \inf_{0 \leq t \leq T} \max_{0 < s \leq h} \frac{|B(t+s) - B(t)|}{h} = \infty \text{ a.s.}$$

We will not prove this result. Instead, we prove a weaker one.

## Theorem

For every  $t_0$ ,

$$\limsup_{t \rightarrow t_0} \left| \frac{B(t) - B(t_0)}{t - t_0} \right| = \infty \text{ a.s.},$$

which implies that for any  $t_0$ , almost every sample  $B(t)$  is not differentiable at this point.

**Proof.** Without loss of generality, we assume  $t_0 = 0$ . If one considers the event

$$A(h, \omega) = \left\{ \sup_{0 < s \leq h} \left| \frac{B(s)}{s} \right| > D \right\},$$

then for any sequence  $\{h_n\}$  decreasing to 0, we have

$$A(h_n, \omega) \supset A(h_{n+1}, \omega),$$

and

$$A(h_n, \omega) \supset \left\{ \left| \frac{B(h_n)}{h_n} \right| > D \right\}.$$

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## 1.3.1 Continuity and differentiability

So,

$$\begin{aligned}P(A(h_n)) &\geq P\left(|B(h_n)/\sqrt{h_n}| > D\sqrt{h_n}\right) \\ &= P(|B(1)| > D\sqrt{h_n}) \rightarrow 1 \quad n \rightarrow \infty.\end{aligned}$$

Hence,

$$P\left(\bigcap_{n=1}^{\infty} A(h_n)\right) = \lim P(A(h_n)) = 1.$$

It follows that

$$\sup_{0 < s \leq h_n} \left| \frac{B(s)}{s} \right| \geq D \quad a.s. \text{ for all } n \text{ and } D > 0.$$

Hence

$$\lim_{h \rightarrow 0} \sup_{0 < s \leq h} \left| \frac{B(s)}{s} \right| = \infty \quad a.s.$$

## 1.3.2 Variation and quadratic variation

If  $g$  is a function real variable, define its quadratic variation over the interval  $[0, t]$  as limit (when it exists)

$$[g, g](t) = [g, g]([0, t]) = \lim \sum_{i=1}^n (g(t_i^n) - g(t_{i-1}^n))^2, \quad (1.3.1)$$

where limit is taken over partitions:

$$0 = t_0^n < t_1^n < \dots < t_n^n = t,$$

with  $\delta = \max_{1 \leq i \leq n} (t_i^n - t_{i-1}^n) \rightarrow 0$ .

The variation of  $g$  on  $[0, t]$  is

$$V_g(t) = \sup \sum_{i=1}^n |g(t_i^n) - g(t_{i-1}^n)|, \quad (1.3.2)$$

where supremum is taken over partitions:

$$0 = t_0^n < t_1^n < \dots < t_n^n = t.$$

Notice that the summation in (1.3.2) is non-decreasing on the partitions, that is, the thinner is the partition, the larger is the summation. So

$$V_g(t) = \lim \sum_{i=1}^n |g(t_i^n) - g(t_{i-1}^n)|,$$

where limit is taken over partitions:

$$0 = t_0^n < t_1^n < \dots < t_n^n = t,$$

with  $\delta = \max_{1 \leq i \leq n} (t_i^n - t_{i-1}^n) \rightarrow 0$ . However, the quadratic variation has not such property.

From the definition of the quadratic variation, the following theorem follows.

### Theorem

- ① *Let  $a \leq c \leq b$ . Then the quadratic variation of  $g$  over the interval  $[a, b]$  exists if and only if its quadratic variations on both the intervals  $[a, c]$  and  $[c, b]$  exist.*

*Further*

$$[g, g]([a, b]) = [g, g]([a, c]) + [g, g]([c, b]), \quad a \leq c \leq b.$$

## Theorem

② If  $g(t) = A + B(f(t) - C)$  for  $t \in [a, b]$ , then

$$[g, g]([a, b]) = B^2 \cdot [f, f]([a, b]).$$

## Theorem

*If  $g$  is continuous and of finite variation then its quadratic variation is zero.*

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### Proof.

$$\begin{aligned} [g, g](t) &= \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} (g(t_{i+1}^n) - g(t_i^n))^2 \\ &\leq \lim_{n \rightarrow \infty} \max_{1 \leq i \leq n-1} |g(t_{i+1}^n) - g(t_i^n)| \cdot \sum_{i=0}^{n-1} |g(t_{i+1}^n) - g(t_i^n)| \\ &\leq \lim_{n \rightarrow \infty} \max_{1 \leq i \leq n-1} |g(t_{i+1}^n) - g(t_i^n)| \cdot V_g(t) = 0. \end{aligned}$$

For Brownian motion  $B$  (or other stochastic process), we can similarly define its variation and quadratic variation. In such case, the variation and quadratic variation are both random variables. Also, in the definition of quadratic variation, the limit in (1.3.1) must be specified in which sense, convergence in probability, almost sure convergence or others, because the summation is now a random variables. We take the weakest one, limit in probability.

## Definition

The quadratic variation of Brownian motion  $B(t)$  is defined as

$$[B, B](t) = [B, B]([0, t]) = \lim \sum_{i=1}^n |B(t_i^n) - B(t_{i-1}^n)|^2, \quad (1.3.3)$$

where for each  $n$ ,  $\{t_i^n, 0 \leq i \leq n\}$  is a partition of  $[0, t]$ , and the limit is taken over all partitions with

$\delta_n = \max_i(t_{i+1}^n - t_i^n) \rightarrow 0$  as  $n \rightarrow \infty$ , and in the sense of convergence in probability.

## Theorem

*Quadratic variation of a Brownian motion over  $[0, t]$  is  $t$ , that is, the Brownian motion accumulates quadratic variation at rate one per unit time.*

**Proof.** Let  $T_n = \sum_{i=1}^n |B(t_i^n) - B(t_{i-1}^n)|^2$ . Then

$$\mathbb{E}T_n = \sum_{i=1}^n \mathbb{E} |B(t_i^n) - B(t_{i-1}^n)|^2 = \sum_{i=1}^n (t_i^n - t_{i-1}^n) = t.$$

We want to show that  $T_n - \mathbb{E}T_n \rightarrow 0$  in probability.

Notice for a standard normal random variable  $N$ ,

$$EN^4 = 3, \quad EN^2 = 1.$$

So,

$$\text{Var}(N^2) = EN^4 - (EN^2)^2 = 3 - 1 = 2.$$

It follows that for a normal  $N(0, \sigma^2)$  random variable  $X$ ,

$$\text{Var}(X^2) = \sigma^4 \text{Var}((X/\sigma)^2) = \sigma^4 \cdot 2(\text{Var}(X/\sigma))^2 = 2(\text{Var}X)^2.$$

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Hence

$$\text{Var}((B(t) - B(s))^2) = 2(\text{Var}(B(t) - B(s)))^2 = 2(t - s)^2.$$

The variance of  $T_n$  is

$$\begin{aligned}\text{Var}(T_n) &= \sum_{i=1}^n \text{Var} \left( |B(t_i^n) - B(t_{i-1}^n)|^2 \right) \\ &= \sum_{i=1}^n 2(t_i^n - t_{i-1}^n)^2 \\ &\leq 2 \max_i (t_i^n - t_{i-1}^n) \sum_{i=1}^n (t_i^n - t_{i-1}^n) \leq 2\delta_n t.\end{aligned}$$

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So

$$T_n - \mathbb{E}T_n \rightarrow 0 \quad \text{in probability.}$$

The proof is completed.

Actually, we have shown that  $T_n \rightarrow t$  in  $L_2$ , that is

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If one uses the following inequality for independent mean-zero random variables  $\{X_i\}$  as

$$\mathbb{E} \left| \sum_{i=1}^n X_i \right|^p \leq C_p \left\{ \left( \sum_{i=1}^n \text{Var} X_i \right)^{p/2} + \sum_{i=1}^n \mathbb{E}|X_i|^p \right\} \text{ for } p \geq 2,$$

it can be showed that

$$\mathbb{E}|T_n - \mathbb{E}T_n|^p \leq C_p \left\{ (\delta_n t)^{p/2} + \delta_n^{p-1} t \right\} \rightarrow 0 \text{ for all } p.$$

So, for Brownian motion, the limit in the definition of quadratic variation can be taken in sense of any  $L_p$  convergence. However, in the sense of almost sure convergence, the limit does not exist unless additional condition on  $\delta_n$  is assumed for example,  $\delta_n = o(1/(\log n)^{1/2})$ .

Combing Theorem 4 and Theorem 6 yields the following corollary.

### Corollary

*Almost every path  $B(t)$  of Brownian motion has infinite variation on any interval, no matter how small it is.*

The quadratic variation property of the Brownian motion is important for considering the stochastic calculus. It means that

$$(B(t_{j+1}) - B(t_j))^2 \approx t_{j+1} - t_j.$$

Although this approximating is not true locally, because

$$\frac{(B(t_{j+1}) - B(t_j))^2}{t_{j+1} - t_j}$$

is a normal  $N(0, 1)$  random variable and so not near 1, no matter how small we make  $t_{j+1} - t_j$ ,

the approximating makes sense if we consider the summation over a partition on a interval since the summation of the left hand can be approximated by the summation of the right hand:

$$\sum_j (B(t_{j+1}) - B(t_j))^2 \approx \sum_j (t_{j+1} - t_j).$$

We may write the approximation informally in the differential notations

$$dB(t)dB(t) = (dB(t))^2 = dt.$$

Recall that for a real differential function  $f(x)$ , we have

$$f(t_{j+1}) - f(t_j) \approx f'(t_j)(t_{j+1} - t_j),$$

that is

$$df(t) = f'(t)dt.$$

And so

$$df(t)df(t) = (df(t))^2 = (f'(t))^2(dt)^2 = 0.$$

This difference will be the main different issue between the stochastic calculus and usual real calculus.

## 1.3 Properties of Brownian motion paths

## 1.3.3 Law of large numbers and law of the iterated logarithm

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## Theorem

*(Law of Large Numbers)*

$$\lim_{t \rightarrow \infty} \frac{B(t)}{t} = 0 \quad a.s.$$

## Theorem

*(Law of the Iterated Logarithm)*

$$\limsup_{t \rightarrow \infty} \frac{B(t)}{\sqrt{2t \log \log t}} = 1, \quad a.s.$$

$$\liminf_{t \rightarrow \infty} \frac{B(t)}{\sqrt{2t \log \log t}} = -1 \quad a.s.$$

## 1.3 Properties of Brownian motion paths

## 1.3.3 Law of large numbers and law of the iterated logarithm

Notice  $\widehat{B}(t) = t(B(\frac{1}{t}))$  is also a Brownian motion.

### Theorem

*(Law of the Iterated Logarithm)*

$$\limsup_{t \rightarrow 0} \frac{B(t)}{\sqrt{2t \log \log(1/t)}} = 1, \quad a.s.$$

$$\liminf_{t \rightarrow 0} \frac{B(t)}{\sqrt{2t \log \log(1/t)}} = -1 \quad a.s.$$

## 1.4 Martingale property for Brownian motion

### Definition

A stochastic process  $\{X(t), t \geq 0\}$  is a martingale if for any  $t$  it is integrable,  $E|X(t)| < \infty$ , and for any  $s > 0$

$$E[X(t+s)|\mathcal{G}_t] = X(t) \text{ a.s.}, \quad (1.4.1)$$

where  $\mathcal{G}_t$  is the information about the process up to time  $t$ , that is,  $\{\mathcal{G}_t\}$  is a collection of  $\sigma$ -algebras such that

- 1  $\mathcal{G}_u \subset \mathcal{G}_t$ , if  $u \leq t$ ;
- 2  $X(t)$  is  $\mathcal{G}_t$  measurable.

## Theorem

Let  $B(t)$  be a Brownian motion. Then

- 1  $B(t)$  is a martingale.
- 2  $B^2(t) - t$  is a martingale.
- 3 For any  $u$ ,  $e^{uB(t) - \frac{u^2}{2}t}$  is a martingale.

**Proof.** Let  $\mathcal{F}_t$  is a filtration for  $B(t)$ .

1. By definition,  $B(t) \sim N(0, t)$ , so that  $B(t)$  is integrable with  $EB(t) = 0$ . Then, for  $s < t$ ,

$$\begin{aligned} E[B(t)|\mathcal{F}_s] &= E[B(s) + (B(t) - B(s))|\mathcal{F}_s] \\ &= E[B(s)|\mathcal{F}_s] + E[B(t) - B(s)|\mathcal{F}_s] \\ &= B(s) + E[B(t) - B(s)] = B(s). \end{aligned}$$

So,  $B(t)$  is a martingale.

2. Notice  $\mathbb{E}B^2(t) = t < \infty$ . Therefore  $B^2(t) - t$  is integrable.

Since,

$$\begin{aligned} & \mathbb{E}[B^2(t) - t | \mathcal{F}_s] \\ &= \mathbb{E} \left[ B(s)^2 + (B(t) - B(s))^2 + 2B(s)(B(t) - B(s)) | \mathcal{F}_s \right] - t \\ &= B(s)^2 + \mathbb{E}(B(t) - B(s))^2 + 2B(s)\mathbb{E}[B(t) - B(s)] - t \\ &= B(s)^2 + (t - s) + 0 - t = B^2(s) - s. \end{aligned}$$

$B^2(t) - t$  is a martingale.

3. The moment generating function of the  $N(0, t)$  variable  $B(t)$  is

$$\mathbb{E}e^{uB(t)} = e^{tu^2/2} < \infty.$$

Therefore  $e^{uB(t)-tu^2/2}$  is integrable with

$$\mathbb{E}e^{uB(t)-tu^2/2} = 1.$$

Further for  $s < t$

$$\begin{aligned}\mathbb{E} \left[ e^{uB(t)} \mid \mathcal{F}_s \right] &= \mathbb{E} \left[ e^{uB(s)} e^{u(B(t)-B(s))} \mid \mathcal{F}_s \right] \\ &= e^{uB(s)} \mathbb{E} \left[ e^{u(B(t)-B(s))} \mid \mathcal{F}_s \right] \\ &= e^{uB(s)} \mathbb{E} \left[ e^{u(B(t)-B(s))} \right] \\ &= e^{uB(s)} e^{(t-s)u^2/2}.\end{aligned}$$

It follows that

$$\mathbb{E} \left[ e^{uB(t)-tu^2/2} \mid \mathcal{F}_s \right] = e^{uB(s)-su^2/2},$$

and then  $e^{uB(t)-tu^2/2}$  is a martingale.

All three martingales have a central place in the theory.

- The martingale  $B^2(t) - t$  provides a characterization (Levy's characterization) of Brownian motion. It can be shown that if a process  $X(t)$  is a continuous martingale such that  $X^2(t) - t$  is also a martingale, then  $X(t)$  is Brownian motion.
- The martingale  $e^{uB(t) - tu^2/2}$  is known as the exponential martingale. Actually, a continuous process  $B(t)$  with property 3 and  $B(0) = 0$  must be a Brownian motion.

### Theorem

Let  $X(t)$  be a continuous process such that for any  $u$ ,  $e^{uX(t)-tu^2/2}$  is a martingale. Then  $X(t)$  is a Brownian motion.

**Proof.** Since  $e^{uX(t)-tu^2/2}$  is a martingale,

$$\mathbb{E}[e^{uX(t)-tu^2/2} | \mathcal{F}_s] = e^{uX(s)-su^2/2}.$$

It follows that

$$\mathbb{E}[e^{u\{X(t)-X(s)\}} | \mathcal{F}_s] = e^{(t-s)u^2/2}.$$

**Proof.** Since  $e^{uX(t)-tu^2/2}$  is a martingale,

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It follows that

$$\mathbb{E}[e^{u\{X(t)-X(s)\}} | \mathcal{F}_s] = e^{(t-s)u^2/2}.$$

Taking expectation yields the moment generating function of  $X(t) - X(s)$  is

$$\mathbb{E}[e^{u\{X(t)-X(s)\}}] = e^{(t-s)u^2/2}.$$

So,  $X(t) - X(s) \sim N(0, t - s)$ . The above conditional expectation also tells us that the conditional moment generating function given  $\mathcal{F}_s$  is a non-random function, which implies that  $X(t) - X(s)$  is independent of  $\mathcal{F}_s$  and so independent of all  $X(u)$ ,  $u \leq s$ . So, by definition,  $X(t)$  is a Brownian motion.