Brownian motion and stochastic calculus

These materials are adapted from related chapters of the following two books as

- Fima C Klebaner, Introduction to Stochastic Calculus with Applications, Imperial College Press, 1998.
- Steven E. Shreve, Stochastic Calculus for Finance II: Continuous-Time Models, Springer, 2004.

Chapter 1. Brownian motion

1.1 Basic concepts on stochastic processes

A stochastic process X is an umbrella term for any collection of random variables $\{X(t,\omega)\}$ depending on time t, which is defined on the same probability space $(\Omega, \mathscr{F}, \mathsf{P})$. Time can be discrete, for example, $t = 0, 1, 2, \ldots$, or continuous, $t \geq 0$.

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For fixed time t, the observation is described by a random variable which we denote by X_t or $X(t)$.

For fixed $\omega \in \Omega$, $X(t)$ is a single realization (single path) of this process. Any single path is a function of time t, $x_t = x(t)$, $t \geq 0$.

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A stochastic process is determined by all its finite dimensional distributions, that is, probabilities of the form

$$\mathsf{P}\big(X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots, X(t_n) \leq x_n\big), \qquad (1.1.1)$$

for any choice of time points $0 \leq t_1 < t_2 < \ldots < t_n$, any $n \geq 1$ with $x_1, \ldots, x_n \in \mathbb{R}$.

1.1.1. Gaussian process

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Because, a multivariate normal distribution is determined by its mean and covariance matrix, a Gaussian process is determined by it mean function $m(t) = \mathsf{E}X(t)$ and covariance function $\gamma(t, s) = \mathsf{Cov}\{X(t), X(s)\}$.

1.2 Brownian motion

1.2.1 Definition of Brownian motion

Definition

Brownian motion $\{B(t)\}$ is a stochastic process with the following three properties.

1. (Independence of increments) For all $0 = t_0 < t_1 < \ldots < t_m$, the increments

$$B(t_1) - B(t_0), \quad B(t_2) - B(t_1), \quad \ldots, \quad B(t_m) - B(t_{m-1})$$

are independent.

Definition

2. (Stationary normal increments) $B(t) - B(s)$ has normal distribution with mean zero and variance $t - s$.
3. (Continuity of paths) $B(t)$,$t \geq 0$ are continuous functions of t.

If the process is started at x, then $B(t)$ has the $N(x,t)$ distribution. This can be written as

$$P_x\big(B(t)\in(a,b)\big)=\int_a^b \frac{1}{\sqrt{2\pi t}}e^{-\frac{(y-x)^2}{2t}}\,dy.$$

P_x denotes the probability of events when the process starts at x. The function under the above integral is called the transition probability density of Brownian motion,

$$p_t(x,y)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{(y-x)^2}{2t}}.$$

If $B^x(t)$ denotes a Brownian motion started at x, then $B^x(t) - x$ is a Brownian motion started at 0, and $B^0(t) + x$ is Brownian motion started at$x$, in other words

$$B^x(t) = x + B^0(t).$$

So, usually we also assume $B(0) = 0$ if not specified, that is, the process is started at 0.

Example

Calculate $P(B(1) \leq 0, B(2) \leq 0)$.

Solution.

$$
\begin{aligned}
& P(B(1) \leq 0, B(2) \leq 0) \\
=& P(B(1) \leq 0, B(1) + B(2) - B(1) \leq 0) \\
=& \int_{-\infty}^{\infty} P(B(1) \leq 0, B(1) + B(2) - B(1) \leq 0 | B(1) = y_1) \frac{1}{\sqrt{2\pi}} e^{-\frac{y_1^2}{2}} dy_1 \\
=& \int_{-\infty}^{0} P(B(1) + B(2) - B(1) \leq 0 | B(1) = y_1) \frac{1}{\sqrt{2\pi}} e^{-\frac{y_1^2}{2}} dy_1 \\
=& \int_{-\infty}^{0} P(B(2) - B(1) \leq -y_1) \frac{1}{\sqrt{2\pi}} e^{-\frac{y_1^2}{2}} dy_1 \\
=& \int_{-\infty}^{0} \Phi(-y_1) d\Phi(y_1) = \int_{-\infty}^{0} [1 - \Phi(y_1)] \, d\Phi(y_1) = \frac{1}{2} - \int_{0}^{1/2} y dy = \frac{3}{8}.
\end{aligned}
$$

1.2.2 Distribution of Brownian motion

Brownian motion is a Gaussian process

Because the increments

$$B(t_1) = B(t_1) - B(0), \quad B(t_2) - B(t_1), \quad \ldots, \quad B(t_m) - B(t_{m-1})$$

are independent and normal distributed, as their linear transform, the random variables $B(t_1), B(t_2), \ldots, B(t_m)$ are jointly normally distributed, that is, the finite dimensional of Brownian motion is multivariate normal.

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are independent and normal distributed, as their linear transform, the random variables $B(t_1), B(t_2), \dots, B(t_m)$ are jointly normally distributed, that is, the finite dimensional of Brownian motion is multivariate normal. So Brownian motion is a Gaussian process with mean 0 and covariance function

$$\gamma(t, s) = \mathsf{Cov}\{B(t), B(s)\} = \mathsf{E}B(t)B(s).$$

If $t < s$, then $B(s) = B(t) + B(s) - B(t)$, and

$$\mathsf{E}B(t)B(s) = \mathsf{E}B^2(t) + \mathsf{E}B(t)(B(s) - B(t)) = \mathsf{E}B^2(t) = t.$$

Similarly if $t > s$, $\mathsf{E}B(t)B(s) = s$. Therefore

$$\gamma(t, s) = \min(t, s).$$

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$$\gamma(t, s) = \min(t, s).$$

On the other hand, a continuous mean zero Gaussian process with covariance function $\gamma(t, s) = \min(t, s)$ is a Brownian motion.

Example

1. For any $T>0$, $\{T^{-1/2}B(Tt)\}$ is Brownian motion.
2. The process

$$\xi_0 \frac{t}{\sqrt{\pi}} + \frac{2}{\pi}\sum_{j=1}^{\infty}\frac{\sin(jt)}{j}\xi_j,$$

where $\xi_j's$, $, = 0, 1, \ldots,$ are independent standard normal random variables, is Brownian motion on $[0,\pi]$.

Example

3. $\{-B(t), t \geq 0\}$ is also a Brownian motion.
4. $\{tB(\frac{1}{t}), t > 0\}$ is also a Brownian motion.
5. If $B(t)$ is a Brownian motion on $[0, 1]$, then $(t+1)B(\frac{1}{t+1}) - B(1)$ is a Brownian motion on $[0, \infty)$.

The first statement is the self-similarity property of the Brownian motion.

The second is the random series representation of Brownian motion.

The third is the symmetry of Brownian motion.

The fourth allows to transfer results on the behavior of the paths of Brownian motion for large t to that of small t.

The second and the the last show the existence of Brownian motion.

Each of the above can be shown by checking the mean and covariance function.

The finite dimensional distribution of Brownian motion

Notice, the jointly density function of increments

$$B(t_1) = B(t_1) - B(0), \quad B(t_2) - B(t_1), \quad \ldots, \quad B(t_m) - B(t_{m-1})$$

is

$$p_{t_1}(0, x_1) p_{t_2 - t_1}(0, x_2) \cdots p_{t_m - t_{m-1}}(0, x_m).$$

The jointly density function of

$$
\begin{aligned}
B(t_1) =& B(t_1), \\
B(t_2) =& B(t_1) + [B(t_2) - B(t_1)], \\
& \cdots\cdots \\
B(t_m) =& B(t_1) + [B(t_2) - B(t_1)] + \ldots + [B(t_m) - B(t_{m-1})],
\end{aligned}
$$

is

$$p_{t_1}(0, y_1) p_{t_2-t_1}(y_1, y_2) \cdots p_{t_m-t_{m-1}}(y_{m-1}, y_m).$$

So,

$$
\begin{aligned}
&\mathsf{P}\Big(B(t_1) \le x_1, B(t_2) \le x_2, \ldots, B(t_m) \le x_m)\Big) \\
&= \int_{-\infty}^{x_1} p_{t_1}(0, y_1) dy_1 \int_{-\infty}^{x_2} p_{t_2-t_1}(y_1, y_2) dy_2 \\
&\qquad \cdots \int_{-\infty}^{x_m} p_{t_m-t_{m-1}}(y_{m-1}, y_m) dy_m.
\end{aligned}
$$

So,

$$\mathsf{P}\Big(B(t_1)\le x_1, B(t_2)\le x_2, \ldots, B(t_m)\le x_m\Big)$$
$$=\int_{-\infty}^{x_1} p_{t_1}(0,y_1)dy_1\int_{-\infty}^{x_2} p_{t_2-t_1}(y_1,y_2)dy_2$$
$$\cdots\int_{-\infty}^{x_m} p_{t_m-t_{m-1}}(y_{m-1},y_m)dy_m.$$

In general, if the process starts at x, the

$$P_x\Big(B(t_1)\le x_1, B(t_2)\le x_2, \ldots, B(t_m)\le x_m\Big)$$
$$=\int_{-\infty}^{x_1} p_{t_1}(x,y_1)dy_1\int_{-\infty}^{x_2} p_{t_2-t_1}(y_1,y_2)dy_2$$
$$\cdots\int_{-\infty}^{x_m} p_{t_m-t_{m-1}}(y_{m-1},y_m)dy_m.$$

Example

We know that

$$\frac{1}{\sqrt{n}} S_{[nt]} \xrightarrow{D} B(t).$$

Then

$$n^{-3/2} \sum_{m=1}^{n-1} S_m = \int_0^1 \frac{1}{\sqrt{n}} S_{[nt]} dt \xrightarrow{D} \int_0^1 B(t) dt.$$

Next, we want to find the distribution of $\int_0^1 B(t)dt$.

Notice,

$$\int_0^1 B(t)dt = \lim \sum B(t_i)(t_{i+1} - t_i),$$

and $\sum B(t_i)(t_{i+1} - t_i)$ are normal random variables with mean zeros. So $\int_0^1 B(t)dt$ is a normal random variable with mean zero.

On the other hand,

$$
\begin{aligned}
\mathsf{Var}\left\{\int_0^1 B(t)dt\right\} =& \mathsf{E}\left[\int_0^1 B(t)dt \int_0^1 B(s)ds\right] \\
=& \mathsf{E}\left[\int_0^1 \int_0^1 B(t)B(s)dtds\right] \\
=& \int_0^1 \int_0^1 \mathsf{E}[B(t)B(s)]dtds \\
=& \int_0^1 \int_0^1 \min(t,s)dtds = 1/3.
\end{aligned}
$$

Exchanging the integrals and expectation is justified by Fubini's theorem since

$$\int_0^1\int_0^1 E|B(t)B(s)|dtds \le \int_0^1\int_0^1 \sqrt{\mathsf{E}B^2(t)\mathsf{E}B^2(s)}dsdt < 1.$$

Thus $\int_0^1 B(t)dt$ has $N(0,1/3)$ distribution.

1.2.3 Filtration for Brownian motion

In addition to the Brownian motion itself, we will need some notation for the amount of information available at each time, We do that with a filtration.

Definition

Let $(\Omega, \mathscr{F}, \mathsf{P})$ be a probability space on which is defined a Brownian motion $B(t)$, $t \geq 0$. A filtration for the Brownian motion is a collection of σ-algebras $\mathscr{F}_t$,$t \geq 0$, satisfying

1. (Information accumulates) For $0 \leq s < t$, every $\mathscr{F}_s$ is also in $\mathscr{F}_t$. In other words, there is at least as much information available at the later time $\mathscr{F}_t$ as there is at the earlier time $\mathscr{F}_s$.

Definition

2. (Adaptivity) For each $t \geq 0$, the Brownian motion $B(t)$ at time t is $\mathscr{F}_t$-measurable. In other words, the information available at time t is sufficient to evaluate the Bownian motion $B(t)$ at that time.
3. (Independence of future increments) For $0 \leq t < u$, the increment $B(u) - B(t)$ is independent of $\mathscr{F}_t$. In other words, any increments of the Brownian motion after time t is independent of the information available at time t.

Properties (1) and (2) in the definition above guarantee that the information available at each time t is at least as much as one would learn from observing the Brownian motion up to time t. Property (3) says that this information is of no use in predicting future movements of the Brownian motion.

If $\mathscr{F}_t = \sigma(B(u) : u \leq t)$, then $\mathscr{F}_t$ is called the nature filtration of the Brownian motion.

Here, for a stochastic process $X(t)$,$t \geq 0$, $\sigma(X(u), u \leq t)$ is the smallest σ-field that contains sets of the form $\{a \leq X(u) \leq b\}$ for all $0 \leq u \leq t$, $a, b \in \mathbb{R}$. It is the information available to an observer of X up to time t.

In general,

Definition

A family $\mathbb{F} = \{\mathscr{F}_t\}$ of increasing sub σ-fields on $(\Omega, \mathscr{F})$ is called a filtration. Usually, $\mathscr{F}_0$ is defined to be $\{\emptyset, \Omega\}$. $(\Omega, \mathscr{F}, \mathbb{F}, \mathsf{P})$ is called the filtered probability space.

A stochastic process $X(t)$, $t \geq 0$, is called adapted if for all t, $X(t)$ is $\mathscr{F}_t$-measurable, that is, if for any t, $\mathscr{F}_t$ contains all the information about $X(t)$ (as well as all $X(u)$, $u \leq t$) but it may contain some extra information as well.

For the Brownian motion, the extra information contained in $\mathscr{F}_t$ is not allowed to give clues about the future increments of B because of Property (3).