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Computer Aided Geometric Design 23 (2006) 243–260
COMPUTER AIDED GEOMETRIC DESIGN
www.elsevier.com/locate/cagd

Normal based subdivision scheme for curve design

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Received 7 November 2004; received in revised form 22 June 2005; accepted 21 October 2005

Available online 28 November 2005

Abstract

In this paper we propose a new kind of nonlinear and geometry driven subdivision scheme for curve interpolation. Instead of using linear combination of old vertexes, displacement vector for every new vertex is given by normal vectors at old vertexes. The normal vectors are computed adaptively for each time of subdivision, and the limit curve is G^1 smooth with wide ranges of free parameters. With this new scheme, normal vectors at selected vertexes can be interpolated efficiently. A shape preserving subdivision scheme with explicit choices of all free parameters is also presented.

Keywords: Curve interpolation; Normal based subdivision; Nonlinear subdivision; Shape preserving

1. Introduction

Curve interpolation by repeated subdivision is an efficient shape design method in the field of computer aided geometric design and many curve subdivision schemes serve as the foundations for surface subdivision. For interpolating curve subdivision, new vertexes will be computed and added to the old polygons for each time of subdivision and the limit curve will pass through all the vertexes of the original control polygon. In this paper we present a new scheme for curve interpolation by subdivision.

A well known work for interpolatory subdivision scheme is four point subdivision scheme proposed by Dyn et al. (1987). Four point subdivision scheme is a stationary linear subdivision scheme and polynomials of order up to three can be reproduced by this scheme. Recently, several new schemes are proposed as the extensions of four point subdivision scheme (Hassan et al., 2002; Marinov et al., 2004). To obtain a fair subdivision curve, Kobbelt (1996) proposed a non-stationary subdivision scheme for curve interpolation. Aspert et al. (2003) proposed a nonlinear interpolatory subdivision scheme based on spherical coordinates transformation. Besides interpolating the original control vertexes, derivatives at the initial data can also be set ahead and interpolated using Hermite subdivision schemes (Jüttler and Schwanecke, 2002).

Besides convergence and continuity, another important property for geometric design by subdivision is shape preserving property. Because shape preservation is often dealt as a nonlinear and geometric problem, several shape preserving subdivision schemes were only concerned with univariate functional data or convex polygonal curve in the literature. LeMéhauté and Utreras (1994) proposed a convexity preserving subdivision scheme that generates C^1

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doi:10.1016/j.cagd.2005.10.001

smooth curves interpolating functional data. When applied for univariate data, four point subdivision scheme can also be used for convexity preserving purpose (Kuijt and van Damme, 1998; Dyn et al., 1999). For general convex data, a geometric subdivision scheme was proposed by Dyn et al. (1992).

In this paper we present a new nonlinear and geometric dependent subdivision scheme for curve interpolation. With the observation that every intermediate polygon is a piecewise linear approximation to a final interpolating curve, and then every new added vertex can be estimated from the intermediate polygon geometrically. From a geometric point of view, a new vertex can be obtained by adding a displacement vector from a selected point on the old polygon. When we choose a point on an edge we just split the edge into two sub-edges. Then the displacement vector from the split point can be computed as a combination of the projection of the sub-edges on the normals at the end points of the old edge. It can be shown that the limit curve is G^1 smooth with adaptive computation of normal vectors for each intermediate polygon. Moreover, we can easily obtain shape preserving subdivision scheme with explicit choices of the subdivision parameters, and some other properties such as normal interpolation at selected points, straight line or circular arc generation can be easily achieved too.

The organization of the paper is as follows. In Section 2 we will introduce general idea of normal based subdivision scheme, and we present the smoothness analysis of this scheme in Section 3. In Section 4 we will present a shape preserving subdivision scheme. The experimental examples are presented in Section 5. Section 6 is devoted to the conclusion of the paper.

2. The subdivision scheme

Let $\{p_i^0\}_i$ be a sequence of control points, we define the normal based subdivision scheme as

$$p_{2i}^{k+1} = p_i^k, \tag{1}$$

$$p_{2i-1}^{k+1} = \left(1 - s_i^k\right) p_{i-1}^k + s_i^k p_i^k + v_i^k, \tag{2}$$

where $0 < \varepsilon_a \leqslant s_i^k \leqslant 1 - \varepsilon_b < 1$ be the subdivision parameter and v_i^k be the displacement vector corresponding to the edge $p_{i-1}^k p_i^k$.

Before defining the displacement vector v_i^k we define the unit normal at each vertex firstly. Except for fixed normals at selected vertexes, normal vectors at other vertexes will be computed adaptively after each time of subdivision. Let p_{i-1}^k, p_i^k and p_{i+1}^k be three consecutive vertexes with different positions, we define the normal vector n_i^k at p_i^k as paralleling the bisector of the angle $\angle p_{i-1}^k p_i^k p_{i+1}^k$. Let T_i^k be the unit tangent vector at p_i^k, we have

$$T_i^k = \frac{T_i^- + T_i^+}{\|T_i^- + T_i^+\|},$$

where $T_i^- = \frac{p_i^k - p_{i-1}^k}{\|p_i^k - p_{i-1}^k\|}$ and $T_i^+ = \frac{p_{i+1}^k - p_i^k}{\|p_{i+1}^k - p_i^k\|}$. Assume that $T_i^k = (T_i^k.x, T_i^k.y)$, then the unit normal vector n_i^k at p_i^k can be computed as $n_i^k = (-T_i^k.y, T_i^k.x)$. For fixed normals, they can be either computed from the initial control polygon or given ahead by users, but they will be kept unchanged during the subdivision.

With the normals at all vertexes defined, local shapes corresponding to individual edges will be determined efficiently by the vertex positions as well as the normals. As in the following definition, we can classify all edges into three types, convex edges, inflection edges and straight edges (see Fig. 1).

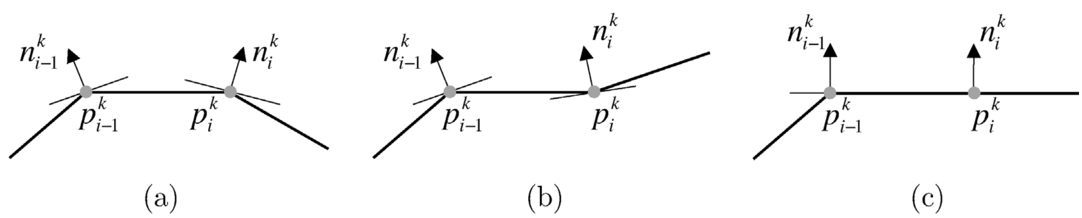


Fig. 1. (a) Convex edge; (b) inflection edge; (c) straight edge.

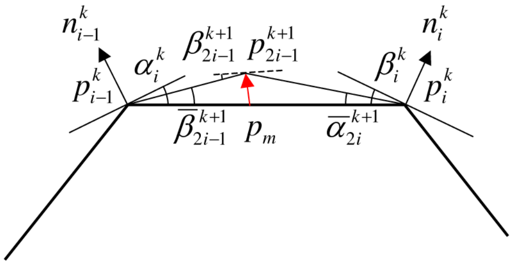


Fig. 2. Compute a new vertex by adding a displacement vector.

Definition. Let $p_{i-1}^k p_i^k$ be an arbitrary edge with normal vectors n_{i-1}^k and n_i^k at p_{i-1}^k and p_i^k, respectively, the projections of the edge onto the normals at the end points are computed as $l_i^k = (p_{i-1}^k - p_i^k)n_{i-1}^k$ and $r_i^k = (p_i^k - p_{i-1}^k)n_i^k$. We define the edge $p_{i-1}^k p_i^k$ a convex edge if $l_i^k r_i^k > 0$. The edge will be defined an inflection edge when $l_i^k r_i^k < 0$ or one of these two projections vanishes. If both l_i^k and r_i^k vanish, the edge $p_{i-1}^k p_i^k$ is defined a straight edge.

Now, we define the displacement vector for an edge $p_{i-1}^k p_i^k$ (see Fig. 2). Let α_i^k and β_i^k be the unsigned angles between the chord $p_{i-1}^k p_i^k$ with the tangent line T_{i-1}^k at p_{i-1}^k or with the tangent line T_i^k at p_i^k, respectively, we have $0 \leqslant \alpha_i^k < \frac{\pi}{2}$ and $0 \leqslant \beta_i^k < \frac{\pi}{2}$. Let $p_m = (1 - s_i^k)p_{i-1}^k + s_i^k p_i^k$, the displacement vector for a convex edge can be defined as

$$v_i^k = w\left(\lambda_i^k n_{i-1}^k + \mu_i^k n_i^k\right), \tag{3}$$

where $\lambda_i^k = (p_{i-1}^k - p_m)n_{i-1}^k$ and $\mu_i^k = (p_i^k - p_m)n_i^k$. The tension parameter w here is a positive number. If $p_{i-1}^k p_i^k$ is an inflection edge or a straight edge, we choose $s_i^k = \frac{1}{2}$ and the displacement vector is defined as

$$v_i^k = \begin{cases} w(\lambda_i^k n_{i-1}^k + \mu_i^k n_i^k), & \text{if } \alpha_i^k + \beta_i^k \leqslant \frac{\pi}{2}, \\ w(2\sigma_i^k e_i^k - \lambda_i^k n_{i-1}^k - \mu_i^k n_i^k), & \text{otherwise}, \end{cases} \tag{4}$$

where $e_i^k = \frac{p_i^k - p_{i-1}^k}{\|p_i^k - p_{i-1}^k\|}$ and $\sigma_i^k = (\lambda_i^k n_{i-1}^k + \mu_i^k n_i^k)e_i^k$. The definition of v_i^k under $\alpha_i^k + \beta_i^k > \frac{\pi}{2}$ is a symmetric vector of $w(\lambda_i^k n_{i-1}^k + \mu_i^k n_i^k)$ with respect to the edge $p_{i-1}^k p_i^k$.

In the following text we will show that limit curves by this subdivision scheme exist and are tangent smooth within wide ranges of free parameters. Moreover, the scheme can be modified for shape preserving curve interpolation just by choosing some proper subdivision parameters.

3. Smoothness analysis

With the definition of chord tangent angles, the absolute values of λ_i^k and μ_i^k can be obtained as

$$\left|\lambda_i^k\right| = s_i^k \left\| p_{i-1}^k - p_i^k \right\| \sin\alpha_i^k, \tag{5}$$

$$\left|\mu_i^k\right| = \left(1 - s_i^k\right) \left\| p_i^k - p_{i-1}^k \right\| \sin\beta_i^k. \tag{6}$$

From Eqs. (5) and (6) we have

$$\sin\alpha_i^k = \frac{|\lambda_i^k|}{s_i^k \|p_{i-1}^k - p_i^k\|},$$

$$\sin\beta_i^k = \frac{|\mu_i^k|}{(1 - s_i^k)\|p_i^k - p_{i-1}^k\|}.$$

If $p_{i-1}^k p_i^k$ is a convex edge we can assume that the normal n_{i-1}^k and n_i^k are both pointing toward the convex side of the polygon, then we have $\lambda_i^k > 0$ and $\mu_i^k > 0$. It can be easily verified that the displacement vector v_i^k lies at

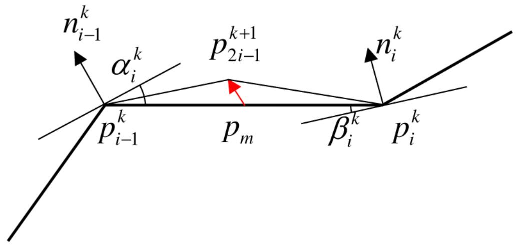


Fig. 3. Displacement vector for an inflection edge.

the convex side of the edge $p_{i-1}^k p_i^k$ too. For a straight edge it is clear that the displacement vector vanishes and a new added vertex lies on the edge itself. As to the problem of which side will the displacement vector v_i^k lie for an inflection edge, we have the following theorem.

Theorem 1. *Let $p_{i-1}^k p_i^k$ be an inflection edge, the chord tangent angles at p_{i-1}^k and p_i^k are α_i^k and β_i^k, respectively, then the displacement vector v_i^k as defined in Eq.* (4) *lies on the side with larger chord tangent angle.*

Proof. It is clear that the theorem holds when $\alpha_i^k = 0$ or $\beta_i^k = 0$. We then need to prove the theorem under the condition that $\alpha_i^k > 0$ and $\beta_i^k > 0$. Without loss of generality, we can assume that the normal n_{i-1}^k is pointing toward the convex side of the polygon at vertex p_{i-1}^k (see Fig. 3), then we have $\lambda_i^k > 0$ and $\mu_i^k < 0$. To judge which side does the displacement vector v_i^k lie to the edge $p_{i-1}^k p_i^k$, we should then determine the sign of the projection of v_i^k onto the normal vector of the edge. By deleting a positive coefficient, the perpendicular part of the vector v_i^k to the edge $p_{i-1}^k p_i^k$ is $\lambda_i^k \cos\alpha_i^k + \mu_i^k \cos\beta_i^k$ when $\alpha_i^k + \beta_i^k < \frac{\pi}{2}$ or $-\lambda_i^k \cos\alpha_i^k - \mu_i^k \cos\beta_i^k$ when $\alpha_i^k + \beta_i^k > \frac{\pi}{2}$. By substituting Eqs. (5) and (6), we have

$$\begin{aligned}\lambda_i^k \cos\alpha_i^k + \mu_i^k \cos\beta_i^k &= \frac{1}{2}\left\| p_{i-1}^k - p_i^k \right\| \sin\alpha_i^k \cos\alpha_i^k - \frac{1}{2}\left\| p_i^k - p_{i-1}^k \right\| \sin\beta_i^k \cos\beta_i^k \\ &= \frac{1}{4}\left\| p_i^k - p_{i-1}^k \right\| \left(\sin 2\alpha_i^k - \sin 2\beta_i^k\right) \\ &= \frac{1}{2}\left\| p_i^k - p_{i-1}^k \right\| \sin\left(\alpha_i^k - \beta_i^k\right) \cos\left(\alpha_i^k + \beta_i^k\right).\end{aligned}$$

From the above equality we can see that the sign of $\lambda_i^k \cos\alpha_i^k + \mu_i^k \cos\beta_i^k$ is same as the sign of $\alpha_i^k - \beta_i^k$ when $\alpha_i^k + \beta_i^k < \frac{\pi}{2}$. If $\alpha_i^k + \beta_i^k > \frac{\pi}{2}$, the sign of $-\lambda_i^k \cos\alpha_i^k - \mu_i^k \cos\beta_i^k$ agrees with the sign of $\alpha_i^k - \beta_i^k$ too. The displacement vector v_i^k for an inflection edge $p_{i-1}^k p_i^k$ lies at the side with the angle α_i^k when $\alpha_i^k > \beta_i^k$ and lies at the side with β_i^k otherwise. The theorem is proven. □

Before discussing the convergence and smoothness of normal based subdivision scheme, we give a few formulae about trigonometric functions and the bounds of the subdivision parameters. These formulae can be easily verified and we leave the proof to interested readers.

(a) Let $0 < \theta < \theta_k \leqslant \frac{\pi}{2}$, then

$$\frac{\sin\theta_k}{\theta_k}\theta \leqslant \sin\theta \leqslant \theta. \tag{7}$$

(b) Assume that $0 < \phi_1, \phi_2 < \frac{\pi}{2}$, if $\sin\phi_1 < r\sin\phi_2$ with $r \leqslant 1$, then we have

$$\phi_1 < r\phi_2. \tag{8}$$

(c) With the assumption that $0 < \varepsilon_a \leqslant s_i^k \leqslant 1 - \varepsilon_b < 1$, we have

$$\frac{1 - s_i^k}{s_i^k} \geqslant \frac{\varepsilon_b}{1 - \varepsilon_b} \geqslant s_a, \qquad \frac{s_i^k}{1 - s_i^k} \geqslant \frac{\varepsilon_a}{1 - \varepsilon_a} \geqslant s_a,$$

where $s_a = \min\{1, \frac{\varepsilon_a}{1-\varepsilon_a}, \frac{\varepsilon_b}{1-\varepsilon_b}\}$. With this definition we have $0 < s_a \leqslant 1$.

The normal based subdivision scheme permits subdivision parameter s_i^k for each edge and tension parameter w for all displacement vectors. Let $w_s = \min\{2\varepsilon_a(1-\varepsilon_a), 2\varepsilon_b(1-\varepsilon_b)\}$, it can be verified that $w_s \leqslant 0.5$, and $2s_i^k(1-s_i^k) > w_s$ when $\varepsilon_a < s_i^k < 1-\varepsilon_b$. Let $\theta_k = \max_i\{\alpha_i^k, \beta_i^k\}$, we have the following theorem.

Theorem 2. *For the normal based subdivision scheme defined by Eqs.* (1)–(4), *if we choose* s_i^k *and* w *satisfying* $\varepsilon_a < s_i^k < 1-\varepsilon_b$ *and* $0 < w < w_s \cos\frac{\pi}{4}$, *then we have* $\lim_{k\to\infty}\theta_k = 0$.

Proof. The essence of the proof of this theorem is the estimation of new chord tangent angles after each subdivision. The proof is consisting of three main steps, chord tangent angles estimation at new inserted vertexes, chord tangent angles estimation at old vertexes, convergence analysis of the angles.
(1) Chord tangent angles at new inserted vertexes

Let p_{2i-1}^{k+1} be the new inserted vertex corresponding to the edge $p_{i-1}^k p_i^k$ (see Fig. 2), we estimate new chord tangent angles β_{2i-1}^{k+1} and α_{2i}^{k+1} at p_{2i-1}^{k+1}. Let $\bar\beta_{2i-1}^{k+1} = \angle p_{2i-1}^{k+1} p_{i-1}^k p_m$ and $\bar\alpha_{2i}^{k+1} = \angle p_{2i-1}^{k+1} p_i^k p_m$, and denote the angle $\angle p_{i-1}^k p_{2i-1}^{k+1} p_m$ as γ_{2i-1}, we have

$$\sin\bar\beta_{2i-1}^{k+1} = \sin\gamma_{2i-1}\frac{\|v_i^k\|}{\|p_m - p_{i-1}^k\|} \leqslant \frac{w\|\lambda_i^k n_{i-1}^k + \mu_i^k n_i^k\|}{\|p_m - p_{i-1}^k\|}$$
$$\leqslant w\sin\alpha_i^k + \frac{1-s_i^k}{s_i^k} w\sin\beta_i^k \leqslant \frac{w}{s_i^k}\max\{\sin\alpha_i^k, \sin\beta_i^k\} \leqslant \frac{w}{s_i^k}\sin\theta_k.$$

In the same way we can define $\angle p_i^k p_{2i-1}^{k+1} p_m$ as γ_{2i} and we have

$$\sin\bar\alpha_{2i}^{k+1} = \sin\gamma_{2i}\frac{\|v_i^k\|}{\|p_m - p_i^k\|} \leqslant \frac{w}{1-s_i^k}\max\{\sin\alpha_i^k, \sin\beta_i^k\} \leqslant \frac{w}{1-s_i^k}\sin\theta_k.$$

With simple calculation, we can see that the projection of p_{2i-1}^{k+1} onto the edge $p_{i-1}^k p_i^k$ lies between p_{i-1}^k and p_i^k, then we have $0 < \bar\beta_{2i-1}^{k+1} < \frac{\pi}{2}$ and $0 < \bar\alpha_{2i}^{k+1} < \frac{\pi}{2}$. Moreover, we can derive upper bounds for $\bar\beta_{2i-1}^{k+1}$ and $\bar\alpha_{2i}^{k+1}$ in terms of θ_k explicitly. Because $0 < \bar\beta_{2i-1}^{k+1} < \frac{\pi}{2}$, we have

$$\sin\bar\beta_{2i-1}^{k+1} = 2^m \sin\frac{\bar\beta_{2i-1}^{k+1}}{2^m}\cos\frac{\bar\beta_{2i-1}^{k+1}}{2}\cdots\cos\frac{\bar\beta_{2i-1}^{k+1}}{2^m} > 2^m\sin\frac{\bar\beta_{2i-1}^{k+1}}{2^m}\cos^m\frac{\pi}{4} = 2^{\frac{m}{2}}\sin\frac{\bar\beta_{2i-1}^{k+1}}{2^m}.$$

Then we have

$$\sin\frac{\bar\beta_{2i-1}^{k+1}}{2^m} < \frac{w}{s_i^k}\frac{1}{2^{m/2}}\sin\theta_k.$$

So, with proper choice of m, we have $\frac{w}{s_i^k}\frac{1}{2^{m/2}} < 1$ for $0 < w < 0.5$ and $\varepsilon_a < s_i^k < 1-\varepsilon_b$. From Eq. (8) we have

$$\bar\beta_{2i-1}^{k+1} < \frac{w}{s_i^k}2^{m/2}\theta_k \leqslant 2K\theta_k,$$

where $K = \frac{1}{2}\frac{0.5}{\min\{\varepsilon_a,\varepsilon_b\}}2^{m/2}$. Similarly, we have $\bar\alpha_{2i}^{k+1} < \min\{2K\theta_k, \frac{\pi}{2}\}$. With the bounds of $\bar\beta_{2i-1}^{k+1}$ and $\bar\alpha_{2i}^{k+1}$ obtained, we have

$$\cos\frac{1}{2}\big(\bar\beta_{2i-1}^{k+1} - \bar\alpha_{2i}^{k+1}\big) \geqslant \max\left\{\cos K\theta_k, \cos\frac{\pi}{4}\right\}.$$

When we compute the new normal at vertex p_{2i-1}^{k+1} as paralleling the bisector of the angle $\angle p_{i-1}^k p_{2i-1}^{k+1} p_i^k$, the chord tangent angles β_{2i-1}^{k+1} and α_{2i}^{k+1} are equal. Then we have

$$\beta_{2i-1}^{k+1} = \alpha_{2i}^{k+1} = \frac{1}{2}\big(\bar\beta_{2i-1}^{k+1} + \bar\alpha_{2i}^{k+1}\big).$$

Now, we compute the bounds of sine value of these angles as follows.

$$2\sin\beta_{2i-1}^{k+1}\cos\frac{\pi}{4}\leqslant 2\sin\frac{1}{2}\left(\bar\beta_{2i-1}^{k+1}+\bar\alpha_{2i}^{k+1}\right)\cos\frac{1}{2}\left(\bar\beta_{2i-1}^{k+1}-\bar\alpha_{2i}^{k+1}\right)=\sin\bar\beta_{2i-1}^{k+1}+\sin\bar\alpha_{2i}^{k+1}$$
$$\leqslant\frac{w}{s_i^k(1-s_i^k)}\max\left\{\sin\alpha_i^k,\sin\beta_i^k\right\}.$$

From this inequality we have

$$\sin\beta_{2i-1}^{k+1}<\frac{w}{w_s\cos\frac{\pi}{4}}\max\left\{\sin\alpha_i^k,\sin\beta_i^k\right\}.$$

From Eq. (8) we have $\beta_{2i-1}^{k+1}<\frac{w}{w_s\cos\frac{\pi}{4}}\max\{\alpha_i^k,\beta_i^k\}$ when $w<w_s\cos\frac{\pi}{4}$.

As illustrated in the rest of this proof, θ_k will approach zero along with the subdivision. We have $\cos K\theta_k\geqslant\cos\frac{\pi}{4}$ when k is large enough. We have

$$\sin\beta_{2i-1}^{k+1}<r_k^1\max\left\{\sin\alpha_i^k,\sin\beta_i^k\right\},$$

where $r_k^1=\frac{w}{w_s\cos K\theta_k}<1$. Furthermore, we have

$$\beta_{2i-1}^{k+1}=\alpha_{2i}^{k+1}<r_k^1\max\left\{\alpha_i^k,\beta_i^k\right\}.$$

(2) New chord tangent angles at old vertexes

To compute new chord tangent angles β_{2i}^{k+1} and α_{2i+1}^{k+1} at vertex p_i^k, we first estimate the lower bounds of angles $\bar\beta_{2i-1}^{k+1}$ at p_{i-1}^k and $\bar\alpha_{2i}^{k+1}$ at p_i^k according to whether the edge $p_{i-1}^k p_i^k$ is a convex edge or some other type of edge. After that, we will obtain the upper bound of α_{2i+1}^{k+1} and β_{2i}^{k+1}.

(2.1) $\lambda_i^k\mu_i^k>0$

In this case, the edge $p_{i-1}^k p_i^k$ is a convex edge and the length of the subvector of v_i^k perpendicular to edge $p_{i-1}^k p_i^k$ is $w|\lambda_i^k\cos\alpha_i^k+\mu_i^k\cos\beta_i^k|$. Then, we have

$$\frac{\|v_i^k\|}{\|p_{i-1}^k-p_m\|}=\frac{w\|\lambda_i^k n_{i-1}^k+\mu_i^k n_i^k\|}{s_i^k\|p_i^k-p_{i-1}^k\|}\geqslant\frac{w|\lambda_i^k\cos\alpha_i^k+\mu_i^k\cos\beta_i^k|}{s_i^k\|p_i^k-p_{i-1}^k\|}$$
$$\geqslant w\max\left\{\frac{|\lambda_i^k|\cos\alpha_i^k}{s_i^k\|p_i^k-p_{i-1}^k\|},\frac{|\mu_i^k|\cos\beta_i^k}{s_i^k\|p_i^k-p_{i-1}^k\|}\right\}$$
$$=w\max\left\{\sin\alpha_i^k\cos\alpha_i^k,\frac{1-s_i^k}{s_i^k}\sin\beta_i^k\cos\beta_i^k\right\}$$
$$\geqslant w\min\left\{1,\frac{1-s_i^k}{s_i^k}\right\}\cos\theta_k\max\left\{\sin\alpha_i^k,\sin\beta_i^k\right\}$$
$$\geqslant w\min\left\{1,\frac{1-s_i^k}{s_i^k}\right\}\cos\theta_k\frac{\sin\theta_k}{\theta_k}\max\left\{\alpha_i^k,\beta_i^k\right\}.$$

When the lower bound of $\frac{\|v_i^k\|}{\|p_{i-1}^k-p_m\|}$ is obtained, we have

$$\bar\alpha_{2i-1}^{k+1}=\alpha_i^k-\bar\beta_{2i-1}^{k+1}\leqslant\alpha_i^k-\sin\bar\beta_{2i-1}^{k+1}=\alpha_i^k-\sin\gamma_{2i-1}\frac{\|v_i^k\|}{\|p_{i-1}^k-p_m\|}$$
$$\leqslant\left(1-w\sin\gamma_{2i-1}\cos\theta_k\frac{\sin\theta_k}{\theta_k}\min\left\{1,\frac{1-s_i^k}{s_i^k}\right\}\right)\max\left\{\alpha_i^k,\beta_i^k\right\}.$$

In a similar method we can obtain the upper bound of $\bar\beta_{2i}^{k+1}$ by estimating the lower bound of the angle $\bar\alpha_{2i}^{k+1}$.

$$\bar{\beta}_{2i}^{k+1} = \beta_i^k - \bar{\alpha}_{2i}^{k+1} \leqslant \beta_i^k - \sin\gamma_{2i}\frac{\|v_i^k\|}{\|p_i^k - p_m\|}$$
$$\leqslant \left(1 - w\sin\gamma_{2i}\cos\theta_k\frac{\sin\theta_k}{\theta_k}\min\left\{1, \frac{s_i^k}{1-s_i^k}\right\}\right)\max\{\alpha_i^k, \beta_i^k\}.$$

Let $\rho_k = \min_i\{\sin\gamma_{2i-1}, \sin\gamma_{2i}\}\cos\theta_k\frac{\sin\theta_k}{\theta_k}$, and it is clear that $0 < \rho_k \leqslant 1$. Then, the upper bounds for $\bar{\alpha}_{2i-1}^{k+1}$ and $\bar{\beta}_{2i}^{k+1}$ can be reformulated as

$$\bar{\alpha}_{2i-1}^{k+1} \leqslant (1 - w\rho_k s_a)\max\{\alpha_i^k, \beta_i^k\}$$

and

$$\bar{\beta}_{2i}^{k+1} \leqslant (1 - w\rho_k s_a)\max\{\alpha_i^k, \beta_i^k\}.$$

(2.2) $\lambda_i^k\mu_i^k \leqslant 0$

According to the subdivision rule, we choose $s_i^k = \frac{1}{2}$ in this case. We compute here the bound for $\bar{\alpha}_{2i-1}^{k+1}$ at p_{i-1}^k under the condition of $\lambda_i^k\mu_i^k < 0$. The bound for $\bar{\beta}_{2i}^{k+1}$ at p_i^k and the bounds under $\lambda_i^k\mu_i^k = 0$ will be obtained similarly. If $\lambda_i^k\mu_i^k < 0$, the edge $p_{i-1}^k p_i^k$ is an inflection edge and we will compute the bound for $\bar{\alpha}_{2i-1}^{k+1}$ according to $\alpha_i^k > \beta_i^k$ or $\alpha_i^k < \beta_i^k$, respectively.

Firstly, when the conditions $\lambda_i^k\mu_i^k < 0$ and $\alpha_i^k > \beta_i^k$ hold, we have

$$\sin\bar{\beta}_{2i-1}^{k+1} = \sin\gamma_{2i-1}\frac{\|v_i^k\|}{\frac{1}{2}\|p_i^k - p_{i-1}^k\|} \leqslant \frac{w\|\lambda_i^k n_{i-1}^k + \mu_i^k n_i^k\|}{\frac{1}{2}\|p_i^k - p_{i-1}^k\|} \leqslant w(\sin\alpha_i^k + \sin\beta_i^k) \leqslant 2w\sin\alpha_i^k.$$

Since $0 < w \leqslant 0.5$ always holds during the subdivision, we have $0 < \bar{\beta}_{2i-1}^{k+1} \leqslant \alpha_i^k$. From this inequality we obtain the bound for $\bar{\alpha}_{2i-1}^{k+1}$ as

$$0 < \bar{\alpha}_{2i-1}^{k+1} = \alpha_i^k - \bar{\beta}_{2i-1}^{k+1} < \alpha_i^k.$$

Secondly, when the conditions $\lambda_i^k\mu_i^k < 0$ and $\alpha_i^k < \beta_i^k$ hold, the displacement vector v_i^k lies on the side with the angle β_i^k. Moreover, the sign of $\lambda_i^k\sin\alpha_i^k + \mu_i^k\sin\beta_i^k$ is the same as that of μ_i^k which means that the angle $\angle p_{2i-1}^{k+1} p_m p_i^k$ is an acute angle (see Fig. 4). We denote the subvector of v_i^k perpendicular to the edge $p_{i-1}^k p_i^k$ as $p_m p_v$, and denote $\angle p_m p_{i-1}^k p_v = \eta$. It is clear that $\bar{\beta}_{2i-1}^{k+1} < \eta$. As for η we have

$$\tan\eta = \frac{w|\lambda_i^k\cos\alpha_i^k + \mu_i^k\cos\beta_i^k|}{\frac{1}{2}\|p_i^k - p_{i-1}^k\|} = w\left|\sin\alpha_i^k\cos\alpha_i^k - \sin\beta_i^k\cos\beta_i^k\right|$$
$$= w\sin(\beta_i^k - \alpha_i^k)\left|\cos(\beta_i^k + \alpha_i^k)\right| \leqslant w(\beta_i^k - \alpha_i^k).$$

Since $\eta < \tan\eta$, we have $\bar{\alpha}_{2i-1}^{k+1} = \alpha_i^k + \bar{\beta}_{2i-1}^{k+1} < \beta_i^k$.

So, whether $\alpha_i^k > \beta_i^k$ or $\alpha_i^k < \beta_i^k$ holds, we all have

$$\bar{\alpha}_{2i-1}^{k+1} \leqslant \max\{\alpha_i^k, \beta_i^k\}.$$

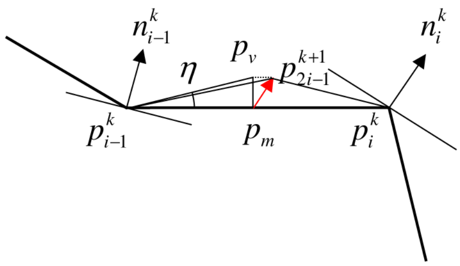


Fig. 4. Chord tangent angle at an inflection edge.

Similarly, we have $\bar{\beta}_{2i}^{k+1} \leqslant \max\{\alpha_i^k, \beta_i^k\}$ for the inflection edge $p_{i-1}^k p_i^k$. It is clear that the above bound for $\bar{\alpha}_{2i-1}^{k+1}$ and $\bar{\beta}_{2i}^{k+1}$ also holds when λ_i^k or μ_i^k vanishes.

(2.3) The bound of new chord tangent angles

If $p_{i-1}^k p_i^k$ and $p_i^k p_{i+1}^k$ are two inflection edges with one common inflexion after some subdivision, it can be easily verified that vertexes p_{2i-1}^{k+1}, p_i^k and p_{2i+1}^{k+1} are on a line. Then both β_{2i}^{k+1} and α_{2i+1}^{k+1} are zero.

Assume that p_i^k is not an inflexion, two abutting edges $p_{i-1}^k p_i^k$ and $p_i^k p_{i+1}^k$ are among the following three cases, convex and convex, convex and inflection, or convex and straight. If $p_{i-1}^k p_i^k$ and $p_i^k p_{i+1}^k$ are two convex edges we have

$$\beta_{2i}^{k+1} = \alpha_{2i+1}^{k+1} = \frac{1}{2}\left(\bar{\beta}_{2i}^{k+1} + \bar{\alpha}_{2i+1}^{k+1}\right) \leqslant (1 - w\rho_k s_a)\max\left\{\alpha_i^k, \beta_i^k, \alpha_{i+1}^k, \beta_{i+1}^k\right\} \leqslant (1 - w\rho_k s_a)\theta_k.$$

If one of these two edges is a convex edge and the other one is an inflection edge, we have

$$\beta_{2i}^{k+1} = \alpha_{2i+1}^{k+1} = \frac{1}{2}\left(\bar{\beta}_{2i}^{k+1} + \bar{\alpha}_{2i+1}^{k+1}\right) \leqslant \left(1 - \frac{1}{2} w\rho_k s_a\right)\theta_k.$$

If $p_{i-1}^k p_i^k$ or $p_i^k p_{i+1}^k$ is a straight edge, we have $\alpha_{2i+1}^{k+1} = \bar{\alpha}_{2i+1}^{k+1} \leqslant (1 - w\rho_k s_a)\theta_k$ or $\beta_{2i}^{k+1} = \bar{\beta}_{2i}^{k+1} \leqslant (1 - w\rho_k s_a)\theta_k$.

(3) Convergence of chord tangent angles

By summarizing steps (1) and (2), we can conclude that for arbitrary i, $\alpha_i^{k+1} < r_k\theta_k$ and $\beta_i^{k+1} < r_k\theta_k$, where $r_k = \max\{r_k^1, 1 - \frac{1}{2} w\rho_k s_a\}$. Then, we have

$$\theta_{k+1} < r_k\theta_k.$$

When we choose $0 < w < w_s \cos\frac{\pi}{4}$, we have $r_k^1 < 1$. Consequently, we have $r_k < 1$ and $\theta_{k+1} < \theta_k$. Since the sequence θ_k $(k = 0, 1, \ldots)$ is decreasing, we can conclude that $\cos\theta_k \frac{\sin\theta_k}{\theta_k} > \cos\theta_0 \frac{\sin\theta_0}{\theta_0}$ for $k > 0$. To find the lower bound of ρ_k, we should also compute a lower bound for $\sin\gamma_{2i-1}$ or $\sin\gamma_{2i}$ in advance. From the geometry of local displacement vector we have

$$\sin\bar{\beta}_{2i-1}^{k+1} = \sin\gamma_{2i-1}\frac{\|v_i^k\|}{\|p_m - p_{i-1}^k\|} = \sin\gamma_{2i-1}\frac{w\|\lambda_i^k n_{i-1}^k + \mu_i^k n_i^k\|}{\|p_m - p_{i-1}^k\|}$$
$$\leqslant \sin\gamma_{2i-1}\left(w\sin\alpha_i^k + \frac{1 - s_i^k}{s_i^k} w\sin\beta_i^k\right) \leqslant \sin\gamma_{2i-1}\frac{w}{s_i^k}\sin\theta_k.$$

As illustrated in step (1), $\sin\bar{\beta}_{2i-1}^{k+1} > 2^{\frac{m}{2}} \sin\frac{\bar{\beta}_{2i-1}^{k+1}}{2^m}$, then we have

$$\sin\frac{\bar{\beta}_{2i-1}^{k+1}}{2^m} < \sin\gamma_{2i-1}\frac{w}{s_i^k}\frac{\sin\theta_k}{2^{\frac{m}{2}}}.$$

Assume that m is a properly selected number such that $\frac{w}{s_i^k}\frac{\sin\theta_k}{2^{\frac{m}{2}}} < 1$ for $0 < w < w_s$. From Eq. (8) we have $\bar{\beta}_{2i-1}^{k+1} < 2^m\gamma_{2i-1}$. On another hand (see Fig. 2), $\bar{\beta}_{2i-1}^{k+1} + \gamma_{2i-1} = \angle p_{2i-1}^{k+1} p_m p_i^k > \frac{\pi}{2} - \beta_i^k > \frac{\pi}{2} - \theta_k$. Then we have $\gamma_{2i-1} > \frac{1}{2^m+1}(\frac{\pi}{2} - \theta_k) > \frac{1}{2^m+1}(\frac{\pi}{2} - \theta_0)$. In a similar way we have $\gamma_{2i} > \frac{1}{2^m+1}(\frac{\pi}{2} - \theta_0)$. Now, we obtain one lower bound of $\rho_k (k = 0, 1, \ldots)$ as

$$\rho_k \geqslant \underline{\rho} = \cos\theta_0 \sin\frac{\frac{\pi}{2} - \theta_0}{2^m + 1}\frac{\sin\theta_0}{\theta_0}.$$

With lower bound of ρ_k, we have upper bound of r_k as

$$r_k \leqslant \bar{r} = \max\left\{\frac{w}{w_s\cos\frac{\pi}{4}}, 1 - \frac{1}{2} w\underline{\rho} s_a\right\}.$$

Because $\bar{r} < 1$, then we have

$$\lim_{k\to\infty}\theta_k = 0.$$

To investigate the limit of r_k, we compute the limit of r_k^1 and ρ_k, respectively. Based on $\lim_{k\to\infty}\theta_k = 0$, we have $\lim_{k\to\infty} r_k^1 = \frac{w}{w_s}$. To compute the limit of ρ_k, we first reformulate $\gamma_{2i-1} = \frac{\pi}{2} - \bar{\beta}_{2i-1}^{k+1} - \varphi_i^k$, where $\varphi_i^k = \frac{\pi}{2} - \angle p_{2i-1}^{k+1} p_m p_i^k$. It is clear that $-\alpha_i^k < \varphi_i^k < \beta_i^k$, and then $\lim_{k\to\infty}\varphi_i^k = 0$. On another hand, $\sin\bar{\beta}_{2i-1}^{k+1} \leqslant \frac{w}{s_i^k}\sin\theta_k$ which implies that $\lim_{k\to\infty}\bar{\beta}_{2i-1}^{k+1} = 0$. With these two limits, we have $\lim_{k\to\infty}\gamma_{2i-1} = \frac{\pi}{2}$ and $\lim_{k\to\infty}\rho_k = 1$. Now, we have

$$\lim_{k\to\infty} r_k = \max\left\{\frac{w}{w_s}, 1 - \frac{1}{2}ws_a\right\}.$$

The theorem is proven. □

It should be pointed out that the proof of Theorem 2 is conservative. In fact, we can choose $w < w_s$ such that $\lim_{k\to\infty} r_k < 1$ for subdivision curve generation. For the purpose of clarity, we assume that $\theta_{k+1} < r\theta_k$ with $r < 1$ in the following text.

Theorem 3. *If $\varepsilon_a < s_i^k < 1 - \varepsilon_b$ and $0 < w < w_s\cos\frac{\pi}{4}$, then the normal based subdivision scheme defined by Eqs.* (1)–(4) *converges and the limit curve is G^1 smooth.*

Proof. To prove this theorem we address three main points: (a) any polygon sequence generated by normal based subdivision converges to a continuous limit curve; (b) tangent at each point on the limit curve exists; (c) the tangent line for the limit curve is continuous.

Let Γ_k be the polygon after kth subdivision, we compute the distance d_k between Γ_{k+1} and Γ_k. Let p_{2i-1}^{k+1} be a new added vertex corresponding to edge $p_{i-1}^k p_i^k$, then we have

$$\begin{aligned}\|p_{i-1}^k p_{2i-1}^{k+1}\| &\leqslant \|p_{i-1}^k p_m\| + \|v_i^k\| = \|p_{i-1}^k p_m\| + w\|\lambda_i^k n_{i-1}^k + \mu_i^k n_i^k\| \\ &\leqslant s_i^k\|p_i^k - p_{i-1}^k\| + w\|p_i^k - p_{i-1}^k\|\big[s_i^k\sin\alpha_i^k + \big(1 - s_i^k\big)\sin\beta_i^k\big] \\ &\leqslant \|p_i^k - p_{i-1}^k\|\big(s_i^k + w\sin\theta_k\big).\end{aligned}$$

From Theorem 2, we have $\lim_{k\to\infty}\theta_k = 0$, then the coefficient $s_i^k + w\sin\theta_k$ will be less than 1 after finite times of subdivision. Because $\varepsilon_a < s_i^k < 1 - \varepsilon_b$, we have $\lim_{k\to\infty}\|p_i^k - p_{i-1}^k\| = 0$. Consequently, we have $\|p_i^k - p_{i-1}^k\| \leqslant L$, where L is a positive constant. Now, the distance from p_{2i-1}^{k+1} to the edge $p_{i-1}^k p_i^k$ can be computed as

$$\begin{aligned} d_i^k &= w\big|\lambda_i^k\cos\alpha_i^k + \mu_i^k\cos\beta_i^k\big| \\ &\leqslant w\|p_i^k - p_{i-1}^k\|\big(s_i^k\sin\alpha_i^k\cos\alpha_i^k + \big(1 - s_i^k\big)\sin\beta_i^k\cos\beta_i^k\big) \\ &\leqslant w\|p_i^k - p_{i-1}^k\|\sin\frac{1}{2}\big(\alpha_i^k + \beta_i^k\big) \leqslant L\theta_k.\end{aligned}$$

Let $d_k = \max_i d_i^k$, we have $d_k \leqslant L\theta_k \leqslant r^k L\theta_0$. This means that the polygon sequence $\{\Gamma_k\}$ is a Cauchy sequence and this sequence of polygons converge uniformly. Since each polygon is a piecewise linear curve, the limit curve is continuous.

Because all p_i^ks are densely lying on the limit curve, we should just prove that the tangent at each p_i^k exists. To prove the existence of tangent line at p_i^k, we will prove that for any points approaching p_i^k on the limit curve, the lines connecting these points to p_i^k converge. Without loss of generality, we prove that the line connecting p_i^k and any point on the right side of p_i^k converges, and the left case can be proved in a similar way. Moreover, it can be easily verified that if two sets of lines connecting points on either side of p_i^k converge, they will converge to a same limit line. As to the convergence problem on the right side, we address it in two steps: firstly, we will show that a sequence of selected lines $p_i^k p_{2^l i+1}^{k+l}$ ($l \geqslant 0$) converge; secondly, we will prove that any line connecting p_i^k to other point on the limit curve converges to the same limit line.

Notice that $p_i^k = p_{2i}^{k+1} = p_{2^2 i}^{k+2} = \cdots$, it can be easily verified that $p_{2^{l+1}i+1}^{k+l+1}$ is a new vertex added corresponding to edge $p_i^k p_{2^l+1}^{k+l}$. Recalling proof of Theorem 2, the angle $\bar{\beta}_{2i+1}^{k+1} = \angle p_{2i+1}^{k+1} p_i^k p_{i+1}^k$ satisfies

$$\frac{2}{\pi}\bar{\beta}_{2i+1}^{k+1} < \sin\bar{\beta}_{2i+1}^{k+1} < \frac{w}{s_{i+1}^k}\sin\theta_k,$$

then we have $\bar{\beta}_{2i+1}^{k+1} < c\theta_k$, where $c = \frac{\pi}{2}\frac{w}{\min\{\varepsilon_a,\varepsilon_b\}}$. In the same way, let $\bar{\beta}_{2^{l+1}i+1}^{k+l+1} = \angle p_{2^{l+1}i+1}^{k+l+1} p_i^k p_{2^l i+1}^{k+l}$, then we have $\bar{\beta}_{2^{l+1}i+1}^{k+l+1} < c\theta_{k+l} < c\theta_k r^l$ $(l = 0, 1, \ldots)$. So lines $p_i^k p_{2^l+1}^{k+l}$ $(l = 0, 1, \ldots)$ form a Cauchy sequence and the limit line exists when l goes to infinity. Assume $\bar{T}_i^k$ be the limit line of the sequence, we then prove that any line connecting p_i^k and a point between $p_{2^{l+1}i+1}^{k+l+1}$ and $p_{2^l i+1}^{k+l}$ approaches $\bar{T}_i^k$ too when l goes to infinity. Assume $p_{k,l}^\infty$ be an arbitrary point lying between $p_{2^{l+1}i+1}^{k+l+1}$ and $p_{2^l i+1}^{k+l}$ on the limit curve, it may be reached or approximated with arbitrary closeness by repeated subdivision. Let $\psi_{k,l} = \angle p_{k,l}^\infty p_i^k p_{2^l i+1}^{k+l}$, then we have

$$\begin{aligned}\psi_{k,l} &\leqslant \angle p_{2^{l+1}i+1}^{k+l+1} p_i^k p_{2^l i+1}^{k+l} + \angle p_{2^{l+2}i+3}^{k+l+2} p_{2^{l+1}i+1}^{k+l+1} p_{2^l i+1}^{k+l} + \cdots = \bar{\beta}_{2^{l+1}i+1}^{k+l+1} + \bar{\beta}_{2^{l+2}i+1}^{k+l+2} + \cdots \\ &< c\theta_k(r^l + r^{l+1} + \cdots) = \frac{c\theta_k}{1-r} r^l.\end{aligned}$$

Consequently, we have $\lim_{l\to\infty}\psi_{k,l} = 0$. This implies that the line $p_i^k p_{k,l}^\infty$ approaches $\bar{T}_i^k$ too, and $\bar{T}_i^k$ is just the tangent line at p_i^k on the limit curve.

Let ϕ_i^k be the chord tangent angle between $\bar{T}_i^k$ and $p_i^k p_{i+1}^k$, we have

$$\begin{aligned}\phi_i^k &\leqslant \angle p_{2i+1}^{k+1} p_i^k p_{i+1}^k + \angle p_{2^2 i+1}^{k+2} p_i^k p_{2i+1}^{k+1} + \cdots + \angle p_{2^{l+1}i+1}^{k+l+1} p_i^k p_{2^l i+1}^{k+l} + \cdots \\ &= \bar{\beta}_{2i+1}^{k+1} + \bar{\beta}_{2^2 i+1}^{k+2} + \cdots + \bar{\beta}_{2^{l+1}i+1}^{k+l+1} + \cdots < c\theta_k(1 + r + \cdots + r^l + \cdots) = \frac{c\theta_k}{1-r}.\end{aligned}$$

Similarly, the angle between the tangent $\bar{T}_{i+1}^k$ at p_{i+1}^k with $p_i^k p_{i+1}^k$ is bounded by $\frac{c}{1-r}\theta_k$ too. Assume that $\bar{T}_i^k$ also denotes the unit tangent direction at p_i^k, then we have $\|\bar{T}_{i+1}^k - \bar{T}_i^k\| < \frac{2c}{1-r}\theta_k$. Let $\bar{T}_{k,l}^\infty$ be the unit tangent vector at $p_{k,l}^\infty$ which is lying between $p_{2^{l+1}i+1}^{k+l+1}$ and $p_{2^l i+1}^{k+l}$ on the limit curve, we compute the bound for $\|\bar{T}_i^k - \bar{T}_{k,l}^\infty\|$ by repeated subdivision again. Then we have

$$\begin{aligned}\|\bar{T}_i^k - \bar{T}_{k,l}^\infty\| &\leqslant \|\bar{T}_i^k - \bar{T}_{2^{l+1}i+1}^{k+l+1}\| + \|\bar{T}_{2^{l+1}i+1}^{k+l+1} - \bar{T}_{2^{l+2}i+3}^{k+l+2}\| + \cdots \\ &< \frac{2c}{1-r}(\theta_{k+l+1} + \theta_{k+l+2} + \cdots) < \frac{2c}{(1-r)^2}\theta_k r^{l+1}.\end{aligned}$$

From this inequality, we have $\lim_{l\to\infty}\|\bar{T}_i^k - \bar{T}_{k,l}^\infty\| = 0$. This means that for any point $p_{k,l}^\infty$ approaching p_i^k, the tangent line $\bar{T}_{k,l}^\infty$ approaches $\bar{T}_i^k$ too. So, the limit curve is G^1 smooth.

This proves the theorem. □

From Theorem 3 we see that any closed curve obtained by normal based subdivision scheme is G^1 smooth. For open polygons, we choose fixed normals at the ends for the subdivision and the limit curves are convergent and smooth at the end points too. From the proofs of Theorems 2 and 3 we can see that the fixed normals at selected vertexes will be interpolated.

Corollary. *Let p_l^0 be a vertex of the original control polygon and n_l^0 be the fixed normal at the vertex, then the normal vector n_l^0 as well as the position of p_l^0 will be interpolated by the normal based subdivision curve.*

4. Shape preserving subdivision

As analyzed in Section 3, normal based subdivision scheme is an efficient method for smooth curve generation. In this section we present subdivision scheme for shape preserving interpolation with explicit choices of the subdivision parameters.

By a shape preserving subdivision scheme, total number of inflexions defined by the initial polygon as well as the normals at initial vertexes will be kept and the straight edges along the initial control polygon will be preserved too. In this paper we define straight edges just when three consecutive vertexes are on a line on the initial control polygon. If two neighboring straight edges intersect at a vertex, this vertex will be dealt as a sharp corner and the initial control polygon can just be divided into two polygons at the corner for shape preserving interpolation. In the following text we assume that there is no sharp corner within the initial control polygon.

Explicit scheme for shape preserving subdivision depends on two aspects, normal computation and parameter selection. Except for those vertexes with fixed normals, we compute the vertex normals for the initial and all intermediate polygons using the normal formula introduced in Section 2. The normal for vertexes on a straight edge is computed as the normal of the line, and this normal will be dealt as a fixed normal during the following subdivision process.

With the normals at all vertexes properly defined, we should then choose the free parameters explicitly for shape preserving interpolation. Let α_i^k and β_i^k be the chord tangent angles of the edge $p_{i-1}^k p_i^k$ at p_{i-1}^k and p_i^k, respectively, we pick the subdivision parameter for the edge $p_{i-1}^k p_i^k$ as

$$s_i^k = \begin{cases} \frac{\sin\beta_i^k}{\sin\alpha_i^k+\sin\beta_i^k}, & \text{convex edge,} \\ \frac{1}{2}, & \text{otherwise.} \end{cases} \tag{9}$$

With the subdivision parameters chosen above, we have a shape preserving interpolation algorithm immediately.

Theorem 4. *For an initial control polygon without sharp corner and with all initial vertex normals properly defined, if we choose the subdivision parameter using Eq.* (9)*, then for* $0 < w < 0.5$ *the normal based subdivision curves are shape preserving and* G^1 *smooth.*

Proof. The proof of the smoothness of the limit curves for Theorem 4 is similar to that of Theorems 2 and 3. Because the choices of the subdivision parameters for convex edges within these theorems are different, we derive the range for w by computing the bounds for $\bar\beta_{2i-1}^{k+1}$ and $\bar\alpha_{2i}^{k+1}$ again.

In a similar way as the proof of Theorem 2 and from Eq. (9), we have

$$\sin\bar\beta_{2i-1}^{k+1} = \sin\gamma_{2i-1}\frac{\|v_i^k\|}{\|p_m - p_{i-1}^k\|} \leqslant w\sin\alpha_i^k + \frac{1-s_i^k}{s_i^k}w\sin\beta_i^k \leqslant 2w\sin\alpha_i^k,$$

and

$$\sin\bar\alpha_{2i}^{k+1} = \sin\gamma_{2i}\frac{\|v_i^k\|}{\|p_m - p_i^k\|} \leqslant 2w\sin\beta_i^k.$$

Then we have $\bar\beta_{2i-1}^{k+1} \leqslant 2w\alpha_i^k$ and $\bar\alpha_{2i}^{k+1} \leqslant 2w\beta_i^k$ when $0 < w < 0.5$. So, the upper bound for chord tangent angles at new vertex p_{2i-1}^{k+1} can be computed as

$$\beta_{2i-1}^{k+1} = \alpha_{2i}^{k+1} = \frac{1}{2}\left(\bar\beta_{2i-1}^{k+1} + \bar\alpha_{2i}^{k+1}\right) \leqslant 2w\max\left\{\alpha_i^k, \beta_i^k\right\}.$$

With this bound and in a same way as the proof of Theorem 3, we can conclude that the normal based subdivision curves are G^1 smooth for $0 < w < 0.5$.

Within the rest of this proof we address the shape preserving property of the subdivision scheme. To prove that the subdivision curve is shape preserving, we should then prove that straight edges will be preserved and no more inflexions will be introduced after each time of subdivision.

If $p_{i-1}^k p_i^k$ is a straight edge, the normal vectors n_{i-1}^k at p_{i-1}^k and n_i^k at p_i^k are both perpendicular to the edge $p_{i-1}^k p_i^k$. From Eq. (3), the displacement vector v_i^k vanishes. Then the new added vertex p_{2i-1}^{k+1} corresponding to $p_{i-1}^k p_i^k$ is just the midpoint of the edge. On another hand, the normal vectors corresponding to straight edges are kept unchanged, and the normal vector at new added vertex is perpendicular to the edge too. Then initial local straight lines will be preserved during the subdivision.

If $p_{i-1}^k p_i^k$ is a convex edge, we choose the subdivision parameter s_i^k as in Eq. (9). From Eqs. (5) and (6) we have $\lambda_i^k = \mu_i^k$. Then, the displacement vector v_i^k is parallel to the bisector of the angle formed by $\lambda_i^k n_{i-1}^k$ and $\mu_i^k n_i^k$ (see

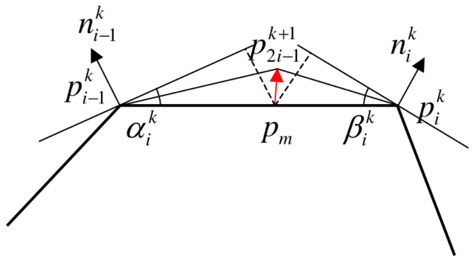


Fig. 5. Shape preserving subdivision for convex edges.

Fig. 5). For $0 < w < 0.5$, the new vertex p_{2i-1}^{k+1} lies within the triangle formed by the tangent line T_{i-1}^k at the vertex p_{i-1}^k, the tangent line T_i^k at p_i^k and the edge $p_{i-1}^k p_i^k$. In case $p_{i-1}^k p_i^k$ is an inflection edge, from Theorem 1, the new vertex p_{2i-1}^{k+1} lies within the angle between the edge $p_{i-1}^k p_i^k$ and the tangent line T_{i-1}^k at p_{i-1}^k when $\alpha_i^k > \beta_i^k \geqslant 0$ or within the angle between the edge $p_i^k p_{i-1}^k$ and the tangent line T_i^k at p_i^k for $0 \leqslant \alpha_i^k < \beta_i^k$ (see Fig. 3).

Assume that $p_{i-1}^k p_i^k$ is a convex edge and none of p_{i-1}^k or p_i^k is inflexion, then new added vertexes p_{2i-3}^{k+1}, p_{2i-1}^{k+1} and chords $p_{i-2}^k p_{i-1}^k$, $p_{i-1}^k p_i^k$ all lie at one side to the tangent line T_{i-1}^k at p_{i-1}^k. Vertexes p_{2i-1}^{k+1}, p_{2i+1}^{k+1} and chords $p_{i-1}^k p_i^k$, $p_i^k p_{i+1}^k$ lie at one side to the tangent line T_i^k at p_i^k too. Then, the local polygon $p_{2i-3}^{k+1} p_{2i-2}^{k+1} p_{2i-1}^{k+1} p_{2i}^{k+1} p_{2i+1}^{k+1}$ is convex. After the definition or redefinition of normal vectors at p_{2i-2}^{k+1}, p_{2i-1}^{k+1} and p_{2i}^{k+1}, respectively, the new edges $p_{2i-2}^{k+1} p_{2i-1}^{k+1}$ and $p_{2i-1}^{k+1} p_{2i}^{k+1}$ are two convex edges. Similarly, if $p_{i-1}^k p_i^k$ is an inflection edge with $\alpha_i^k \neq \beta_i^k$, then it will be replaced by one convex edge and one new inflection edge after a subdivision.

If $p_{i-1}^k p_i^k$ is an inflection edge with α_i^k equal to β_i^k, we have $p_{2i-1}^{k+1} = \frac{1}{2}(p_{i-1}^k + p_i^k)$ and then the edge will be replaced by two inflection edges with one common inflexion after a subdivision. In this case, the vertexes p_{i-1}^k, p_i^k and p_{2i-1}^{k+1} are on a line and the polygon is local symmetric with respect to the vertex p_i^k. With simple calculation, we can see that the inflexion will be preserved during the subdivision.

From the above analysis, the number of inflexions implied by the original control polygon does not increase after each time of subdivision. The theorem is proven. □

Though shape preserving subdivision curves can be generated by choosing adaptive subdivision parameters for all edges, due to the fixed tension parameter w for every subdivision step, the curves are sometimes not fair. To obtain a shape preserving as well as fair subdivision curve, we should determine the tension parameter for every edge adaptively. For practical computation, we scale the displacement vectors for convex edges adaptively, and the displacement vectors for inflection edges or straight edges are still defined as in Eq. (4).

Let v_i^k be the displacement vector defined by Eq. (3) for a convex edge, we compute the scaling factor for the displacement vector according to the criterion that local circular arc will be reproduced provided that the end vertexes and normals at the ends of the edge are sampled from an arc. For a given edge $p_{i-1}^k p_i^k$, we define

$$t_\lambda = \begin{cases} \frac{1}{1+\cos\alpha_i^k} \frac{|\lambda_i^k|}{\|v_i^k\|}, & \text{convex edge,} \\ 1, & \text{otherwise,} \end{cases}$$

and

$$t_\mu = \begin{cases} \frac{1}{1+\cos\beta_i^k} \frac{|\mu_i^k|}{\|v_i^k\|}, & \text{convex edge,} \\ 1, & \text{otherwise.} \end{cases}$$

Now, we choose the scaling factor for the displacement vector v_i^k as $t = \min\{t_\lambda, t_\mu\}$ and the new vertex corresponding to the edge $p_{i-1}^k p_i^k$ can be defined

$$p_{2i-1}^{k+1} = (1 - s_i^k) p_{i-1}^k + s_i^k p_i^k + t v_i^k. \tag{10}$$

From the definition of t_λ and t_μ, if $p_{i-1}^k p_i^k$ is a convex edge, we have

$$t \leqslant t_\lambda < \frac{|\lambda_i^k|}{\|v_i^k\|} < \frac{\lambda_i^k}{v_i^k n_{i-1}^k},$$

and

$$t \leqslant t_\mu < \frac{|\mu_i^k|}{\|v_i^k\|} < \frac{\mu_i^k}{v_i^k n_i^k}.$$

With these two inequalities, we have $|t v_i^k \cdot n_{i-1}^k| < |\lambda_i^k|$ and $|t v_i^k \cdot n_i^k| < |\mu_i^k|$ which imply that the scaled displacement vector $t v_i^k$ lies under the tangent lines T_{i-1}^k at p_{i-1}^k and T_i^k at p_i^k, simultaneously. Then, the subdivision scheme defined by Eqs. (9) and (10) is a shape preserving subdivision scheme for plane curve modelling. In the following text we refer this scheme as shape preserving subdivision scheme. Moreover, this scheme can be used for circular arc generation provided that the boundary data are properly defined.

Theorem 5. *Let $p_{i-1}^0 p_i^0$ be a convex edge on the original control polygon with fixed end normal n_{i-1}^0 and n_i^0 at p_{i-1}^0 and p_i^0, respectively, if α_i^0 is equal to β_i^0, the subdivision curve defined by Eqs.* (9) *and* (10) *is a circular arc.*

Proof. When $p_{i-1}^0 p_i^0$ is a convex edge and α_i^0 is equal to β_i^0, from Eqs (5), (6) and (9) we have $s_i^0 = 0.5$ and $\lambda_i^0 = \mu_i^0$. Furthermore, the displacement vector v_i^0 is perpendicular to the edge $p_{i-1}^0 p_i^0$ and $t_\lambda = t_\mu$ for the edge. Then, the scaling factor for the displacement vector v_i^0 is $t = \frac{1}{1+\cos\alpha_i^0}\frac{\lambda_i^0}{\|v_i^0\|}$. Let $\gamma = \angle p_{2i-1}^1 p_{i-1}^0 p_i^0$, then we have

$$\tan\gamma = \frac{t\|v_i^0\|}{\|p_i^0 - p_m\|} = \frac{1}{1+\cos\alpha_i^0}\frac{\lambda_i^0}{\|p_i^0 - p_m\|} = \frac{1}{1+\cos\alpha_i^0}\sin\alpha_i^0 = \tan\frac{\alpha_i^0}{2}.$$

Because $0 < \gamma < \frac{\pi}{2}$, we have $\gamma = \frac{\alpha_i^0}{2}$. If we fix the normal n_{i-1}^0 at p_{i-1}^0, n_i^0 at p_i^0 and compute the normal n_{2i-1}^1 at p_{2i-1}^1 as paralleling the bisector of $\angle p_{i-1}^0 p_{2i-1}^1 p_i^1$, then the chord tangent angles α_{2i-1}^1, β_{2i-1}^1, α_{2i}^1 and β_{2i}^1 are equal with each other after one time of subdivision. When new vertexes have been computed and added repeatedly, the limit curve is an interpolating circular arc. □

For general type of data, the shape preserving subdivision scheme can also be used for smooth curve generation. In a similar way as the proofs of Theorems 3 and 4, we have the following theorem.

Theorem 6. *The subdivision scheme defined by Eqs.* (1), (9) *and* (10) *is a shape preserving subdivision scheme, and for* $0 < w < 0.5$ *the limit curves generated by this scheme are G^1 smooth.*

From Eq. (10), we can see that a new added vertex p_{2i-1}^{k+1} will be independent of the parameter w when $p_{i-1}^k p_i^k$ is a convex edge. So, one can choose any $w > 0$ for an initial local convex control polygon. To control the shape of the subdivision curve, usually the first few steps of the subdivision, we can compute scale t for Eq. (10) as $t = \min\{1, t_\lambda, t_\mu\}$. We will illustrate the influence of this scale by the fourth example in next section.

Remark 7. Suppose p_i^k be a vertex added corresponding to an inflection edge, it is then always close to the edge $p_{i-1}^k p_{i+1}^k$. When we compute the normal n_i^k at p_i^k as paralleling the bisector of $\angle p_{i-1}^k p_i^k p_{i+1}^k$, the chord tangent angle β_i^k or α_{i+1}^k is much less than α_i^k or β_{i+1}^k. From Eq. (9), s_i^k and s_{i+1}^k will be close to zero or 1 for the edge $p_{i-1}^k p_i^k$ and $p_i^k p_{i+1}^k$, and then new vertexes p_{2i-1}^{k+1} and p_{2i+1}^{k+1} will be close to the initial inflection edge. Though the subdivision curve is smooth in the end, but it is always flat between p_{i-1}^k and p_{i+1}^k. To increase the fairness of the subdivision curve near the inflection edge we can modify the normal vector at p_i^k for the following subdivision.

Let

$$e_a = \frac{p_{i+1}^k - p_{i-1}^k}{\|p_{i+1}^k - p_{i-1}^k\|}$$

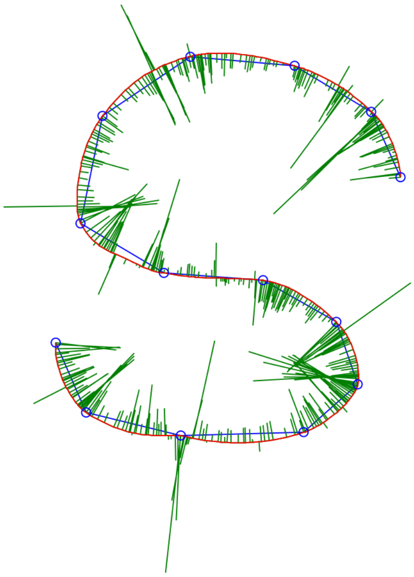
and

$$n_a = \frac{n_{i-1}^k + n_{i+1}^k}{\|n_{i-1}^k + n_{i+1}^k\|},$$

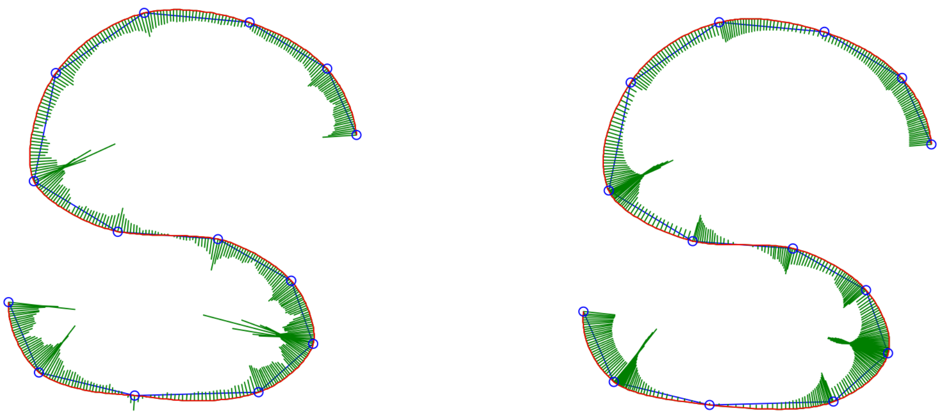
then we have the reflection vector of n_a with respect to a line perpendicular to e_a as

$$n_r = n_a - 2(n_a e_a) e_a.$$

Now, the modified normal vector at p_i^k can be defined as $\bar{n}_i^k = (n_i^k + n_r)/\|n_i^k + n_r\|$. With $\bar{n}_i^k$ as the new normal at p_i^k, either p_i^k will become an inflexion itself or one and only one of the edge $p_{i-1}^k p_i^k$ or $p_i^k p_{i+1}^k$ is an inflection edge. Then the subdivision is still shape preserving with this modification. On another hand, the upper bound for new chord tangent angles at p_i^k can be estimated as $((\alpha_i^k + 2\beta_i^k + \beta_{i+1}^k)/2 + 2\beta_i^k)/2 = \frac{1}{4}(\alpha_i^k + \beta_{i+1}^k) + \frac{3}{2}\beta_i^k$. Because β_i^k is always much less than α_i^k or β_{i+1}^k when p_i^k is an inflexion or a vertex added corresponding to an inflection edge, then we can



(a)



(b) (c)

Fig. 6. Curve interpolation by normal based subdivision scheme: (a) with random $s_i^k \in [0.25, 0.75]$; (b) with fixed $s_i^k = 0.5$; (c) shape preserving subdivision.

believe that the maximum chord tangent angle θ_k will not be influenced by the normal modification at inflexions. In fact, this modification has made subdivision more uniform and the final subdivision curve more fair than by a simple shape preserving scheme.

5. Examples

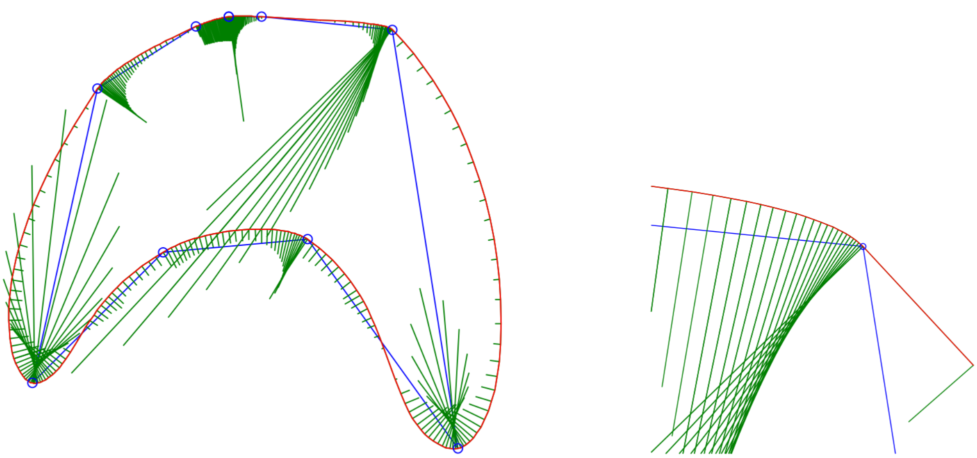
In this section we present several interesting examples for smooth plane curve design or shape preserving interpolation by normal based subdivision scheme. To illustrate the quality of a subdivision curve more clearly, we compute and plot the discrete curvatures along with the curve after finite times of subdivision.

In the first example we compute subdivision curves with an S-shaped control polygon. With normal vectors at initial vertexes properly defined, there exists one inflection edge within the control polygon. Three curves are defined interpolating all the initial control points. We construct the first subdivision curve by choosing the subdivision parameters s_i^k randomly within the interval $(0.25, 0.75)$ and setting $w = 0.25$ for all steps of subdivision (see Fig. 6(a)). When we choose s_i^k as a constant 0.5 and set $w = 0.25$ again we obtain another subdivision curve (Fig. 6(b)). From the figures we can also see that both of these two curves are smooth, however, neither of them is shape preserving. By using the subdivision scheme proposed in Section 4, we obtain a shape preserving subdivision curve with $w = 0.3$ (Fig. 6(c)).

The second example is concerned about shape preserving interpolation of a planar closed polygon with nonuniform control points. Both the lengths of edges and the turning angles of the original control polygon are non-uniform and changing rapidly (see Fig. 7). We choose $w = 0.4$ for this example, and obtain a shape preserving subdivision curve with natural shape after 5 times of subdivision. From the figure we can see that the subdivision is nonuniform too, but the limit curve seems piecewise fair.

In the third example we design a bottle like shape by shape preserving subdivision scheme. Three straight line segments are defined within the original control polygon (see Fig. 8(a)). As the normal vectors for the vertexes on straight edges are fixed during the subdivision, the curve segments connecting these three lines have fixed boundary normals. In fact, the connecting segments are circular arcs in this example. As a result, the interpolating subdivision curve is not only shape preserving, but also G^1 smooth along the whole curve with straight lines and circular arcs imbedded (see Fig. 8(b)).

In the fourth example we interpolate a star shape polygon by shape preserving subdivision curves with various tensions. The initial control polygon is local convex and there is no inflection edge within the polygon. As discussed in Section 4, the tension w can be picked in an even larger range. This example also shows that w can be used to adjust the fullness of the subdivision curve.



(a) (b)

Fig. 7. (a) Shape preserving interpolation of a closed polygon with nonuniform control points; (b) zoom in of the curve near the top right control vertex.

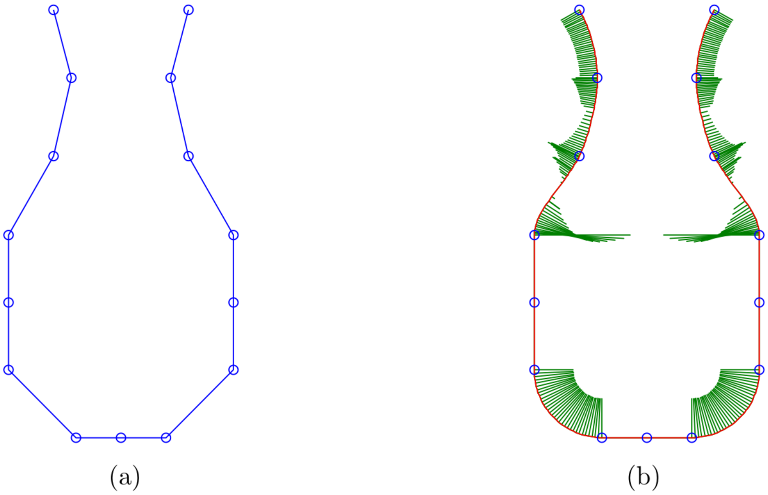


Fig. 8. Shape preserving interpolation with local straight lines and circular arcs generation: (a) the control polygon; (b) the subdivision curve.

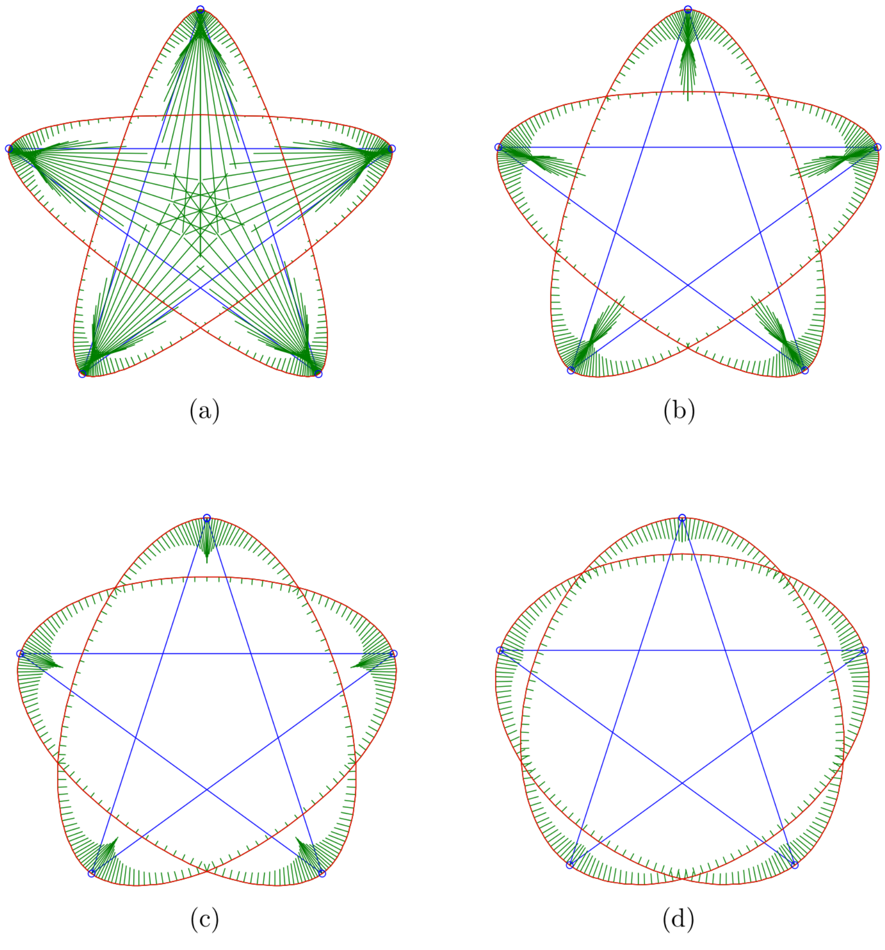


Fig. 9. Shape preserving interpolation of a local convex polygon: (a) $w = 0.3$; (b) $w = 0.5$; (c) $w = 0.7$; (d) $w = 0.9$.

Table 1
The maximum chord tangent angles and the convergence ratios after each subdivision

k	0	1	2	3	4	5
S(a)	0.615712	0.424489	0.265348	0.183638	0.124351	0.061576
		0.689383	0.625099	0.692067	0.677155	0.495177
S(b)	0.615712	0.417069	0.274952	0.174585	0.106985	0.063631
		0.677332	0.659249	0.634964	0.612796	0.594766
S(c)	0.615712	0.318427	0.160222	0.080186	0.040098	0.020049
		0.517136	0.503166	0.500468	0.500063	0.500008
close plyg	1.336695	1.063081	0.594761	0.306461	0.154072	0.077097
		0.795306	0.559469	0.515268	0.502744	0.500400
bottle	0.785398	0.427510	0.217122	0.108838	0.054438	0.027220
		0.544322	0.507876	0.501275	0.500178	0.500023
star(b)	1.256637	0.970793	0.537907	0.275958	0.138604	0.069347
		0.772532	0.554091	0.513021	0.502265	0.500326

Finally, we present the maximum chord tangent angles and the convergence ratios of the chord tangent angles after each subdivision for above mentioned examples (see Table 1). From the table, we can see that the maximum chord tangent angles θ_k decrees rapidly for normal based subdivision scheme. Except for the example with random subdivision parameter s_i^k, the convergence ratios r_k also converge monotonically.

6. Conclusions

A new subdivision scheme, normal based subdivision scheme, has been introduced for curve interpolation. The main feature of this scheme is that displacement vector for every new vertex is given as a linear combination of normal vectors at old vertexes. With proper choices of the subdivision parameters, the limit curves are G^1 smooth.

Because new vertexes depend on normal vectors at old vertexes, the normal vectors at selected vertexes can be interpolated by this new subdivision scheme. By this scheme, shape preserving interpolation of plane curves has been reduced to the computation of a set of proper subdivision parameters. Straight lines and circular arcs can be generated along with the subdivision under properly defined initial conditions. The experimental examples also show the efficiency of normal based subdivision scheme.

One interesting future topic about normal based subdivision is how to construct curvature continuous subdivision curves. The normal based subdivision scheme can be generalized for surface interpolation of meshes directly (Yang, 2005), but how to design shape preserving subdivision schemes for other types of data such as space curve interpolation or surface interpolation deserve further study in the future.

Acknowledgement

The author owes thanks to referees for their helpful comments and suggestions which have helped to improve the paper greatly. This work is supported by 973 program of China (2002CB312101) and NSFC (60303015, 60333010).

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