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High accuracy approximation of helices by quintic curves

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Abstract

In this paper we present methods for approximating a helix segment by quintic Bézier curves or quintic rational Bézier curves based on the geometric Hermite interpolation technique in space. The fitting curve interpolates the curvatures as well as the Frenet frames of the original helix at both ends. We achieve a high accuracy of the approximation by giving a proper parametrization of the curve, and the approximation order of the height function along the helix axis is 9 provided that the screw angle of the helix is fixed. Numerical examples are also presented to illustrate the efficiency of the new method.

Keywords: Helix; Bézier curves; Rational Bézier curves; Interpolation; Parametrization

1. Introduction

A helix segment is a kind of geometric curve with non-vanishing constant curvature and non-vanishing constant torsion (do Carmo, 1976), and it is a natural generalization of circular arc in 3D space. In the fields of computer aided design and computer graphics, helices can be used for the tool path description, the simulation of kinematic motion or the design of highways, etc. Though a helix segment can be represented accurately with a combination of trigonometric functions and polynomials, there is no exact representation for it by polynomials or rational polynomials. A high accuracy of approximation by NURBS will make the helix more convenient in use within most of current CAD/CAM systems.

Several authors have discussed the problem of approximating the helix by rational Bézier curves in the literature. Mick and Röschel (1990) have presented a direct approach to interpolate the helix by rational cubic Bézier curves. Juhász (1995) has studied the same problem and presented an improved algorithm with careful error estimation. Recently, Seemann (1997) has presented algorithms to approximate a helix segment with rational Bézier curves of order 4, 5 and 6. The main idea of Seemann is to increase the

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freedoms for the approximation by a generalized degree elevation technique. Though the fitting curve lies on the same cylinder surface as the helix, the principal normal directions at two ends are not always interpolated yet. For applications such as motion simulation in space, the accuracy of the principal normal or the binormal plays the same important roles as the accuracy of the positions.

In this paper we present an efficient new algorithm for approximating a segment of helix by quintic polynomial curves or quintic rational curves. Because the projection of a helix segment onto a plane orthogonal to the axis of the helix is a circular arc, the approximation of the helix can be formulated as the problem of approximating the circular arc on the plane and approximating the height function along the cylinder axis simultaneously. The approximation to a circular arc by polynomial curves has been studied extensively (de Boor et al., 1987; Goldapp, 1991; Fang, 1998). We show that the latter approximation problem is a Hermite interpolation problem of which the end derivatives are determined by the equation of the circular arc. Then, the fitting curve not only interpolates the end positions and end curvatures of the helix but also matches the tangent directions and the principal normal vectors of the helix at the ends. Even more, when the circular arc is approximated by a quintic Bézier curve with G^2 contact at the ends or represented as a rational Bézier curve of degree larger than or equal to four, the parametrization of the curve is flexible and we achieve a high accuracy of the approximation by selecting a proper parametrization of the plane curve.

The organization of the paper is as follows. In Section 2, we will formulate the problem and propose a general solution to the problem. In Section 3, we present an explicit algorithm for approximating a helix segment by a quintic polynomial curve. Since circular arcs can be represented accurately by rational curves, we discuss how to approximate a helix segment by a quintic rational Bézier curve in Section 4. Examples and comparison with some existing methods are presented in Section 5. In the last section we conclude the paper and discuss some potential problems.

2. Problem formulation

Without loss of generality we can assume that the helix segment is defined as follows:

$$x = R\cos\varphi,$$
$$y = R\sin\varphi,$$
$$z = b\varphi,$$

where $0 \leqslant \varphi \leqslant 2\theta$ is the center angle of the corresponding circular arc on xy-plane. Because a circular arc can be represented by rational curves (Farin, 1990) or approximated by polynomial curves with high accuracy (Fang, 1998), we then assume that the circular arc is represented or approximated as a parametric curve $Q(t) = (x(t), y(t))$ for $0 \leqslant t \leqslant 1$.

A polynomial curve or a rational curve cannot be with arc length parametrization unless it is linear (Farouki and Sakkalis, 1991), thus the arc length of the plane curve $Q(t)$ cannot be the parameter t. Nevertheless the arc length of the curve can still be represented as follows:

$$\Upsilon(t) = \int_0^t \| Q'(\xi) \| \, d\xi. \qquad (1)$$

Now the height function of the helix can be obtained as $h(t) = \mathrm{Y}(t)\frac{b}{R}$. From formula (1), we can derive the first and the second order derivatives of $\mathrm{Y}(t)$ as

$$\frac{\mathrm{dY}(t)}{\mathrm{d}t} = \|Q'(t)\| = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}, \tag{2}$$

$$\frac{\mathrm{d}^2\mathrm{Y}(t)}{\mathrm{d}t^2} = \frac{x'x'' + y'y''}{\|Q'(t)\|}. \tag{3}$$

When $Q(t)$ is a rational curve representing a circular arc or an approximating polynomial curve, the height function $h(t)$ is not a polynomial or a rational polynomial any more. Then it is necessary to approximate the original function $h(t)$ with a rational polynomial or a polynomial function of the same degree as that of the curve $Q(t)$. To approximate the circular arc and the helix with enough accuracy and with appropriate freedoms, we represent or approximate the arc segment with quintic rational Bézier curves or quintic Bézier curves. The approximation of the function $h(t)$ can then be formulated as a quintic Hermite interpolation problem.

The boundary value of the function $h(t)$ is the corresponding coordinate value at the ends of the helix, i.e., $h(0) = 0$ and $h(1) = 2\theta b$. The first and the second order derivatives of the function $h(t)$ at two ends are as follows:

$$h'(0) = \mathrm{Y}'(0)\frac{b}{R}, \tag{4}$$

$$h'(1) = \mathrm{Y}'(1)\frac{b}{R}, \tag{5}$$

$$h''(0) = \mathrm{Y}''(0)\frac{b}{R}, \tag{6}$$

$$h''(1) = \mathrm{Y}''(1)\frac{b}{R}. \tag{7}$$

With the boundary data defined above, a polynomial function or a rational function with prescribed weights can be computed from the data. Let $z(t)$ be the Hermite interpolation function to the original function $h(t)$, then the fitting error in z direction is $e(t) = z(t) - h(t)$. Because the interpolation function $z(t)$ is determined by the boundary conditions or the derivatives of the plane curve $Q(t)$, we can achieve a high accuracy of the approximation to the helix by constructing a plane curve $Q(t)$ with a proper parametrization.

When the interpolating function $z(t)$ is obtained, a space polynomial curve or a space rational curve $P(t)$ can be obtained as $P(t) = (x(t), y(t), z(t))$. In addition to interpolate two ends of the helix, the curve $P(t)$ also interpolates the midpoint of the helix and matches the Frenet frames of the helix at the two ends.

Theorem 1. *If the curve $Q(t)$ interpolates the end positions, end tangents and curvatures of the projected arc, the space curve $P(t)$ not only interpolates the end positions and end curvatures but also matches the Frenet frames at the ends of the helix.*

Proof. Let $H(s) = (x_c(s), y_c(s), h(s))$ be the helix segment defined over [0, 1], then the interpolating curve $Q(t)$ is with G^2 contact to the arc $C(s) = (x_c(s), y_c(s))$ at the ends. From differential geometry we know that there exists a parameter transformation for $Q(t)$, i.e., $t = \phi(s)$, such that $Q(\phi(s))$ or $Q(s)$

for simplicity, has the same derivatives $Q'(s)$ and $Q''(s)$ with $C(s)$ at $s = 0$ and $s = 1$. Consequently, $z'(s) = h'(s)$ and $z''(s) = h''(s)$ for $s = 0, 1$. Then, the curvatures and the Frenet frames of the helix $H(s)$ at two ends are the same as $P(s)$ and interpolated by $P(t)$. □

Theorem 2. *Let $f(t)$ be a differentiable function defined over interval $[0,1]$ and $f(t)$ satisfies $f'(t) = f'(1-t)$ for $t \in [0,1]$, then $f(\frac{1}{2}) = \frac{1}{2}[f(0) + f(1)]$.*

Proof. Because the function $f(t)$ is differentiable, we have, for arbitrary $t \in [0,1]$,

$$f(t) = f(0) + \int_0^t f'(\xi)\,\mathrm{d}\xi.$$

By substituting the equality $f'(\xi) = f'(1-\xi)$ into the above integral formula, we have

$$f(t) = f(0) + \int_0^t f'(1-\xi)\,\mathrm{d}\xi = f(0) + \int_{1-t}^1 f'(\xi)\,\mathrm{d}\xi.$$

Then we have $f(\frac{1}{2}) = f(0) + \int_{\frac{1}{2}}^1 f'(\xi)\,\mathrm{d}\xi$, from which we can derive the formula $f(\frac{1}{2}) = \frac{1}{2}[f(0) + f(1)]$. This proves the theorem. □

Corollary. *Let $f(t)$ be a function defined as in Theorem 2 and if it is twice differentiable, then we have $f''(t) = -f''(1-t)$.*

3. Approximating a helix by quintic Bézier curves

In this section we propose an algorithm for approximating the helix by a quintic polynomial curve. With the projection of the helix as a segment of a circular arc, the approximant to the helix should consist of two dependent parts: the approximation to the planar arc and a Hermite interpolation for the height function.

For the simplicity of description we can assume that the quintic curve $Q(t)$ is a Bézier curve $Q(t) = \sum_{i=0}^5 Q_i B_i^5(t)$, where $B_i^5(t) = \frac{5!}{i!(5-i)!} t^i (1-t)^{5-i}$ $(i = 0, 1, \ldots, 5)$ are Bernstein basis functions. The end control points of the curve can be obtained from the definition of the helix, i.e., $Q_0 = (R, 0)$ and $Q_5 = (R\cos 2\theta, R\sin 2\theta)$. The unit tangent vector of the arc at the end Q_0 is $V_0 = (0, 1)$ and the tangent direction at Q_5 is $V_1 = (-\sin 2\theta, \cos 2\theta)$. Let U_0 be the unit vector pointing from Q_0 to Q_5, and U_1 be the unit vector perpendicular to U_0 and lying on the same side of the chord as the arc segment (see Fig. 1), then the missing control points for the Bézier curve can be obtained based on the symmetry property of the curve:

$$\begin{aligned} Q_1 &= Q_0 + \lambda V_0, \\ Q_2 &= Q_0 + s_0 U_0 + s_1 U_1, \\ Q_3 &= Q_5 - s_0 U_0 + s_1 U_1, \\ Q_4 &= Q_5 - \lambda V_1. \end{aligned} \tag{8}$$

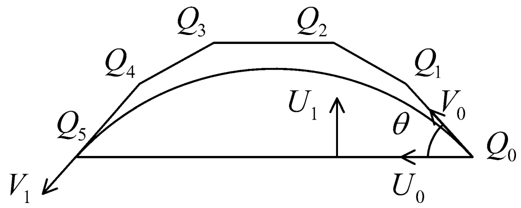

Fig. 1. Approximating the plane arc by a quintic Bézier curve $Q(t)$.

In the literature the methods for the determination of the control polygon of the Bézier curve are mainly based on the criteria that the fitting curve is with the optimal fitting accuracy to the arc or with higher continuity order at the ends (see (Floater, 1997; Fang, 1998)). In this paper, we compute the three unknowns s_0, s_1 and λ according to the criteria that the curve $Q(t)$ passes through the midpoint of the arc segment and interpolates the curvatures at the ends, as well as the interpolating function $z(t)$ approximates the height function $h(t)$ of the helix with a high accuracy.

It can be easily derived that the midpoint of the arc segment is $Q_m = \frac{1}{2}(Q_0 + Q_5) + R(1 - \cos\theta)U_1$, where R is the radius of the arc and θ is the angle between the tangent V_0 and the chord of the arc (see Fig. 1). With the interpolation requirement we have

$$Q\left(\frac{1}{2}\right) = Q_m. \tag{9}$$

Because the curve $Q(t)$ interpolates the curvatures of the arc segment at the ends, i.e., $\frac{\|Q'(0)\wedge Q''(0)\|}{\|Q'(0)\|^3} = k_0 = \frac{1}{R}$ or $\frac{\|Q'(1)\wedge Q''(1)\|}{\|Q'(1)\|^3} = k_0$, then we have

$$u_0 s_0 + u_1 s_1 = \frac{5}{4} k_0 \lambda^2, \tag{10}$$

where $u_0 = \sin\theta$ and $u_1 = -\cos\theta$. From Eq. (9) we have $s_1 = \frac{8}{5}R(1 - \cos\theta) - \frac{1}{4}l_1\lambda$, where $l_1 = 2\sin\theta$, and from Eq. (10) we have $s_0 = (\frac{5}{4}k_0\lambda^2 - u_1 s_1)/u_0$. As to the free variable λ, it may somewhat influence the approximation accuracy to the arc segment, but the derivative length of the plane curve or the parameter speed of $\Upsilon(t)$ are more sensitive to this variable. Hence we can choose values for λ with the aim that both the arc and the height function of the helix can be approximated with high accuracies.

When the curve $Q(t)$ or the function $\Upsilon(t)$ has been obtained, we can then compute the derivatives of the function $h(t)$ immediately. At first, we assume that the first and the second derivatives of the function $h(t)$ at the ends are denoted as h'_0, h'_1, h''_0 and h''_1, respectively. From Eq. (4) we have $h'_0 = 5\lambda\frac{b}{R}$, and from Eq. (6) we have $h''_0 = 20(v_0 s_0 + v_1 s_1 - 2\lambda)\frac{b}{R}$ where $v_0 = \cos\theta$ and $v_1 = \sin\theta$. Based on the symmetry property of the fitting curve $Q(t)$, we have $\|Q'(t)\| = \|Q'(1-t)\|$. Then the derivatives at the other end can be obtained as $h'_1 = h'_0$ and $h''_1 = -h''_0$.

With the boundary data stated above, a quintic polynomial $z(t)$ can be constructed by the Hermite interpolation method. Let the fitting function be

$$z(t) = \sum_{i=0}^{5} z_i B_i^5(t), \tag{11}$$

then the coefficients of the function are as follows:

$$
\begin{aligned}
z_0 &= 0,\\
z_5 &= 2\theta b,\\
z_1 &= z_0 + \frac{1}{5}h_0',\\
z_4 &= z_5 - \frac{1}{5}h_1',\\
z_2 &= \frac{1}{20}h_0'' + 2z_1 - z_0,\\
z_3 &= \frac{1}{20}h_1'' + 2z_4 - z_5.
\end{aligned}
$$

Based on the coefficient formula of the function $z(t)$ and from Eq. (11) we have

$$z'(0.5) = \frac{15}{4}\theta b - \frac{7}{8}h_0' - \frac{1}{16}h_0''. \tag{12}$$

Substituting the expressions of h_0' and h_0'', Eq. (12) can be changed into

$$z'(0.5) = \left[\frac{15}{4}\theta R - \frac{5}{4}(v_0 s_0 + v_1 s_1) - \frac{15}{8}\lambda\right]\frac{b}{R}. \tag{13}$$

On the other hand, the magnitude of the derivative at the midpoint of curve $Q(t)$ can be obtained as

$$\|Q'(0.5)\| = 5\left(\frac{3}{8}l_2 - \frac{1}{4}s_0 - \frac{3}{16}l_0\lambda\right), \tag{14}$$

where $l_0 = 2\cos\theta$ and $l_2 = \|Q_5 - Q_0\| = 2R\sin\theta$.

It can be easily verified that $z'(t) = z'(1-t)$, then from Theorem 2 we can conclude that $z(0.5) = (h_0 + h_1)/2$. This means that the quintic space curve not only interpolates two ends, but also interpolates the midpoint of the original helix. From the corollary of Theorem 2, we have $z''(0.5) = h''(0.5) = 0$. Then it can be easily verified that the principal normal of the Bézier curve is the same with that of the helix at the point. Even more, we can assume that $z'(0.5) = h'(0.5)$, which means that the approximating curve have the same tangent direction with the helix at the midpoint too, we have

$$\frac{z'(0.5)}{\|Q'(0.5)\|} = \frac{b}{R}. \tag{15}$$

By substituting Eqs. (13), (14) into (15), we have

$$a_0\lambda^2 + b_0\lambda + c_0 = 0, \tag{16}$$

where $a_0 = \frac{25}{16}\frac{k_0}{\sin\theta}(1-\cos\theta)$, $b_0 = -\frac{5}{4}(1-\cos\theta)$ and $c_0 = -\frac{2R}{\sin\theta}(1-\cos\theta)^2 + \frac{15}{4}R(\theta - \sin\theta)$. Let $d_0 = b_0^2 - 4a_0c_0$, then $d_0 = \frac{25}{16}\frac{1-\cos\theta}{\sin^2\theta}[8(1-\cos\theta)^2 + (1-\cos\theta)\sin^2\theta - 15(\theta - \sin\theta)\sin\theta]$. For $0 < \theta < \frac{\pi}{2}$, it can be verified that $c_0 > 0$ and $d_0 > 0$ by the Taylor expansion technique. Since $a_0 > 0$ and $b_0 < 0$, then there exist two positive roots to Eq. (16):

$$\lambda_1 = \frac{-b_0 + \sqrt{d_0}}{2a_0} \quad \text{and} \quad \lambda_2 = \frac{-b_0 - \sqrt{d_0}}{2a_0}.$$

In fact, these two roots can both serve for the construction of an interpolating curve, but the parameter speed for the curves derived from λ_1 and λ_2 are different.

With the quintic curve constructed above, the fitting error $e(t)$ along the z direction satisfies equations $e(t)=0$, $e'(t)=0$ and $e''(t)=0$ at $t=0$, $t=0.5$ and $t=1$. Any polynomials satisfying these equations and with degree less than 9 but not 9 all vanish. On the other hand, the range of the parameter t is proportional to the central angle 2θ but not the screw ratio of the helix, then the approximation order along the z direction is 9 with respect to the center angle of the circular arc. Because the curve $Q(t)$ interpolates the ends of the arc with G^2 contact and interpolates the midpoint of the arc with G^1 contact, from (Fang, 1998), the approximation order to the circular arc is 8. Even though the approximation order to the arc segment is less than the order to the height function, for most practical cases (such as $b \leqslant R$ in Section 5) the fitting error to the arc is always less than the error in the height direction or the both are with very high accuracy.

Though the curves derived from λ_1 and λ_2 are with the same approximation order, the experiments show that the parametrization of the curve derived from root λ_1 is more uniform than the curve derived from λ_2. Even more, the first curve is also with higher approximation accuracy to the helix than the second one, therefore $\lambda = \lambda_1$ is the suggested choice for the free parameter of the interpolating quintic Bézier curve.

4. Approximating a helix by rational Bézier curves

In the above section we have approximated the arc segment by a quintic polynomial and approximated the helix by a quintic Bézier curve, but the interpolating curve is not lying on the same cylinder surface with the helix except for a few points. In this section we represent the circular arc with rational Bézier curves and propose a new algorithm to approximate the helix on the same cylinder surface as the helix.

A circular arc with angle $0 < 2\theta < \pi$ in radian can be represented as a quadratic rational Bézier curve

$$R(t) = \frac{R_0 B_0^2(t) + R_1 w B_1^2(t) + R_2 B_2^2(t)}{B_0^2(t) + w B_1^2(t) + B_2^2(t)}, \tag{17}$$

where $w = \cos\theta$ is the weight and R_0, R_1 and R_2 are the control points. To approximate the helix with enough freedoms, it is necessary to raise the order of the curve $R(t)$. We adopt here the zigzag reparametrization technique proposed by Blanc and Schlick (1996) to increase the order of curve $R(t)$ to four first. The reparametrization function is

$$s(t) = \frac{pt + (1-p)t^2}{1 - 2(1-p)t + 2(1-p)t^2}, \tag{18}$$

where p is a positive free variable. The properties of this reparametrization can be found at (Blanc and Schlick, 1996); one advantage for circular arc representation by this method is that we have one freedom to adjust the parameter speed of the curve. We obtain a quintic rational Bézier representation of the arc segment by elevating the degree of the reparametrized curve to five with the traditional degree elevation method (Farin, 1990). The quintic rational Bézier curve can be denoted as

$$Q(t) = \frac{\sum_{i=0}^{5} Q_i w_i B_i^5(t)}{\sum_{i=0}^{5} w_i B_i^5(t)}, \tag{19}$$

where the control points Q_i $(i = 0, 1, \ldots, 5)$ and the weights w_i $(i = 0, 1, \ldots, 5)$ are as the follows:

$$
\begin{aligned}
Q_0 &= R_0,\\
Q_1 &= \frac{(1+2p)R_0 + 2wpR_1}{1+2p+2wp},\\
Q_2 &= \frac{(p+\frac{p^2}{2})R_0 + w(1+p+p^2)R_1 + \frac{p^2}{2}R_2}{(1+w)(p+p^2)+w},\\
Q_3 &= \frac{\frac{p^2}{2}R_0 + w(1+p+p^2)R_1 + (p+\frac{p^2}{2})R_2}{(1+w)(p+p^2)+w},\\
Q_4 &= \frac{2wpR_1 + (1+2p)R_2}{1+2p+2wp},\\
Q_5 &= R_2,\\
w_0 &= 1, \qquad w_1 = \frac{1+2(1+w)p}{5}, \qquad w_2 = \frac{(1+w)(p+p^2)+w}{5},\\
w_3 &= w_2, \qquad w_4 = w_1, \qquad w_5 = w_0.
\end{aligned}
$$

With the curve $Q(t)$ defined, we can now compute the end derivatives for the function $h(t)$. Let $l_0 = \|R_1 - R_0\|$, $l_1 = \|R_2 - R_1\|$ and $l_2 = \|R_2 - R_0\|$, then $l_0 = l_1 = R\tan\theta$ and $l_2 = 2R\sin\theta$. The first and the second order derivatives of $h(t)$ at $t = 0$ are $h_0' = 2wpl_0\frac{b}{R}$ and $h_0'' = [4l_0(w + wp - wp^2 - 2w^2p^2) + 2wp^2l_2]\frac{b}{R}$.

Based on the symmetry property of the arc segment and the rational Bézier curve $Q(t)$, it can be easily verified that $h'(t) = h'(1-t)$. Further more, we have $h_1' = h_0'$, $h_1'' = -h_0''$ and $h''(0.5) = 0$. To construct a space quintic rational Bézier curve approximating the helix, the function $z(t)$ should be as follows:

$$
z(t) = \frac{\sum_{i=0}^{5} z_i w_i B_i^5(t)}{\sum_{i=0}^{5} w_i B_i^5(t)}. \tag{20}
$$

It is clear that $z_0 = 0$, $z_5 = 2\theta b$, and the remaining coefficients can be determined by the first and the second derivatives at the ends, i.e., $z'(i) = h_i'$, $z''(i) = h_i''$ $(i = 0, 1)$, then we have

$$
\begin{aligned}
z_1 &= \frac{w_0}{5w_1}h_0' + z_0,\\
z_2 &= \frac{h_0''w_0^2 - (2w_0 - 10w_1)w_0h_0'}{20w_0w_2} + z_0,\\
z_3 &= \frac{h_1''w_5^2 + (2w_5 - 10w_4)w_5h_1'}{20w_3w_5} + z_5,\\
z_4 &= z_5 - \frac{w_5}{5w_4}h_1'.
\end{aligned}
$$

With the coefficients obtained above, and by substituting the formulae of h_0' and h_0'', we have

$$
z'\left(\frac{1}{2}\right) = \frac{6R\theta + (1+4p+p^2)[2R\theta + 2w(R\theta - l_0)] - wp^2l_2}{(1+w)(1+p)^2}\frac{b}{R}. \tag{21}
$$

On the other hand the length of the derivative at the midpoint of the curve $Q(t)$ can be computed as

$$
\|Q'(0.5)\| = \frac{4l_2}{(1+w)(1+p)}. \tag{22}
$$

In a similar way to the quintic polynomial curve, the function $z(t)$ satisfies equation $z'(t) = z'(1-t)$, and then we have $z(0.5) = (z_0 + z_5)/2$ and $z''(0.5) = 0$. The principal normal of the rational curve also matches that of the helix at the midpoint. To compute the values for the free variable p, we demand again $z'(0.5) = h'(0.5)$ so that the approximating curve matches the tangent direction at the midpoint of the helix segment while it interpolates the position of the point. This also means that the Frenet frame of the helix at the midpoint is interpolated. Then we have

$$\frac{z'(0.5)}{\|Q'(0.5)\|} = \frac{b}{R}. \tag{23}$$

Substituting Eqs. (21) and (22) into (23), we have

$$a_1 p^2 + b_1 p + c_1 = 0, \tag{24}$$

where $a_1 = 2R(\theta - \sin\theta)(1+\cos\theta)$, $b_1 = 8R[\theta(1+\cos\theta) - 2\sin\theta]$ and $c_1 = 2R[\theta(4+\cos\theta) - 5\sin\theta]$. Let $d_1 = b_1^2 - 4a_1c_1$, then $d_1 = 16R^2[3\theta(\theta\cos\theta - 2\sin\theta)(1+\cos\theta) + \sin^2\theta(11 - 5\cos\theta - \theta\sin\theta)]$. In a similar method as in Section 3, it can be proven that $a_1 > 0$, $b_1 < 0$, $c_1 > 0$ and $d_1 > 0$. Then there exist two positive roots to Eq. (24), both of which can be used to construct an interpolating curve. From the experiments we find that the curve derived by $p = \frac{-b_1 - \sqrt{d_1}}{2a_1}$ can be with even higher approximation accuracy. Even more, with the same reason as the approximation by quintic polynomial curves, the approximation order of the function $h(t)$ by the quintic rational function $z(t)$ is also 9.

Remarks. According to (Blanc and Schlick, 1996), Eq. (17) can be changed into a quartic rational Bézier curve $Q(t)$ by a reparametrization technique. Let the curve be

$$Q(t) = \frac{\sum_{i=0}^{4} Q_i w_i B_i^4(t)}{\sum_{i=0}^{4} w_i B_i^4(t)}, \tag{25}$$

then the control points Q_i $(i = 0, 1, \dots, 4)$ and the weights w_i $(i = 0, 1, \dots, 4)$ are as the follows:

$$\begin{aligned}
Q_0 &= R_0, \\
Q_1 &= \frac{R_0 + wR_1}{1+w}, \\
Q_2 &= \frac{p^2R_0 + 2w(1+p^2)R_1 + p^2R_2}{2(p^2 + p^2w + w)}, \\
Q_3 &= \frac{wR_1 + R_2}{1+w}, \\
Q_4 &= R_2,
\end{aligned}$$

$$w_0 = 1, \qquad w_1 = \frac{1+w}{2}p, \qquad w_2 = \frac{p^2 + p^2w + w}{3}, \qquad w_3 = w_1, \qquad w_4 = w_0.$$

We assume that the interpolating function $z(t)$ is

$$z(t) = \frac{\sum_{i=0}^{4} z_i w_i B_i^4(t)}{\sum_{i=0}^{4} w_i B_i^4(t)}. \tag{26}$$

Just like the quintic rational curve, we have $z_0 = 0$, $z_4 = 2\theta b$. The coefficients z_1 and z_3 can be computed based on the first derivatives of the function $h(t)$ at $t = 0$ and $t = 1$. We have

$$z_1 = \frac{w_0}{4w_1} h_0' + z_0,$$
$$z_3 = z_4 - \frac{w_4}{4w_3} h_1'.$$

Because z_2 is determined by the second derivatives of the function $z(t)$ at either ends, we can obtain two values for z_2 based on h_0'' or h_1'' as follows:

$$z_2^0 = \frac{h_0'' w_0 - (2w_0 - 8w_1) h_0'}{12w_2} + z_0,$$
$$z_2^1 = \frac{h_1'' w_4 - (2w_4 - 8w_3) h_1'}{12w_2} + z_4.$$

To be sure that the interpolating function $z(t)$ matches the second derivatives at both ends, we need $z_2^0 = z_2^1$. And because of $h_0' = h_1'$ and $h_0'' = -h_1''$, the equation $z_2^0 = z_2^1$ can be changed into

$$\frac{h_0'' - (2w_0 - 8w_1) h_0'}{6w_2} = 2\theta b. \tag{27}$$

By substituting h_0' and h_0'', Eq. (27) can be changed into

$$p^2 = \frac{\cos\theta(\theta - \tan\theta)}{(1 + \cos\theta)(\sin\theta - \theta)}. \tag{28}$$

The positive square root of Eq. (28) can be chosen as the parameter for the construction of an interpolating space curve. The error function $e(t)$ for the height approximation by quartic rational curves satisfies $e(0.5) = 0$ and $e(t) = 0$, $e'(t) = 0$, $e''(t) = 0$ at $t = 0$ and $t = 1$, then it can be obtained that the approximation order along the z direction is 7.

5. Examples and comparison

We have applied the new algorithm to the helix approximation with various center angles, the radii and the heights, and we just sample a few examples here to show the efficiency of the method. Within this paper, we compute the fitting error numerically at selected points of the curve. Given a point on the fitting curve, we compute the projection point on the xy-plane first and elevate the point on the helix segment along the z-axis.

In Fig. 2, a helix segment with $R = b = 100$ and $2\theta = 2\pi/3$ is approximated by a quintic polynomial curve. To illustrate the approximation more clearly, we plot the principal normal and binormal along the quintic curve of which the principal normal is with the length of the curvature radius and the binormal is with a constant value. We plot the fitting errors for the approximation to the helix by this quintic Bézier curve in Fig. 3, where the tangent angle between the Bézier curve and the helix, the normal angle between these two types of curves, the curvature difference, the torsion difference and the height difference are plotted along the curve. In addition to the fitting error for space curves, the fitting error for the corresponding arc approximation has also been plotted. The quintic Bézier curve does not lie on the same cylinder surface as the original helix, but the deviation from the surface is much lower than the height deviation.

Besides the approximation by a polynomial Bézier curve, we have also approximated the same helix segment in Fig. 2 by quintic rational Bézier curves based on our new algorithm and Seemann's algorithm.

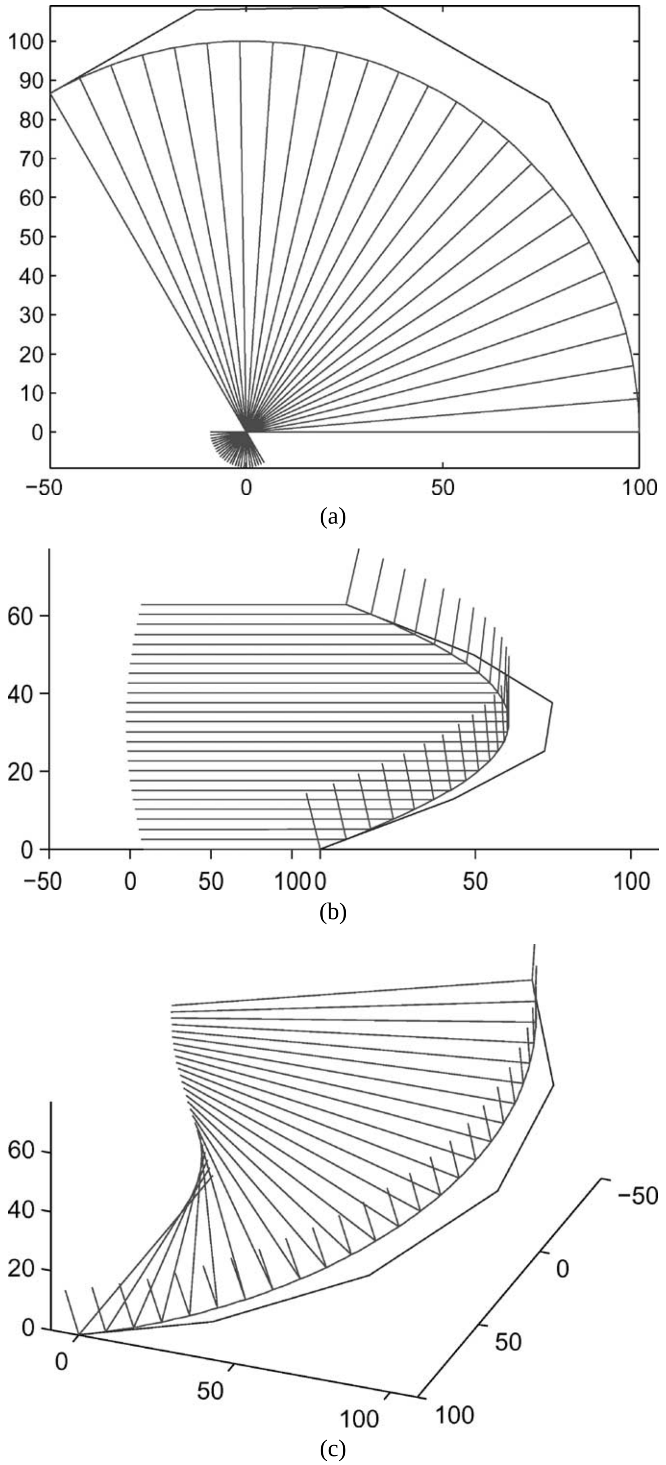

Fig. 2. A helix segment approximated by a quintic Bézier curve. (a) Top view; (b) side view; (c) perspective view.

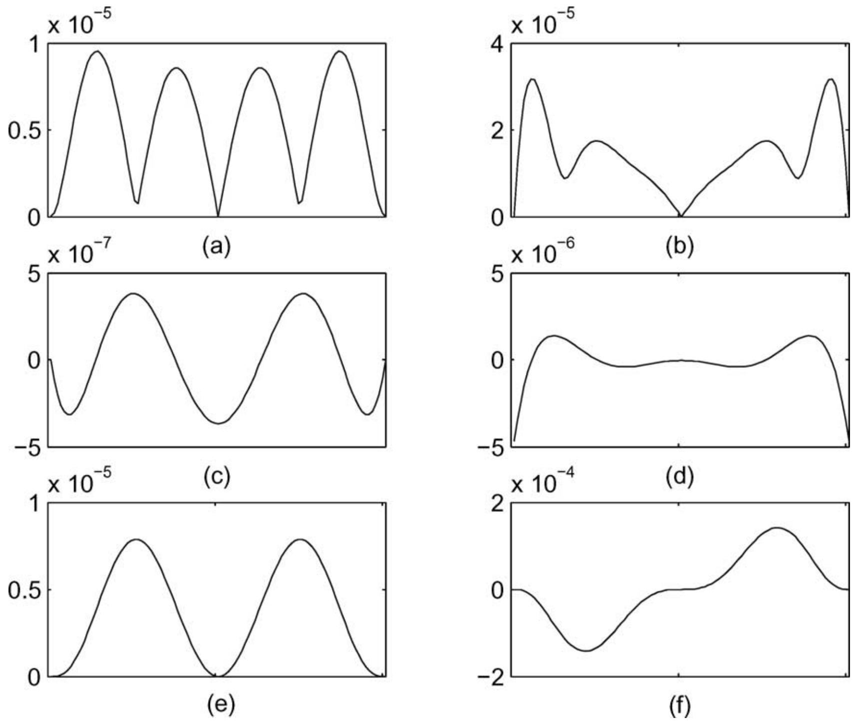

Fig. 3. Error plots for the approximation by a quintic Bézier curve. $R = b = 100$, $2\theta = 2\pi/3$. (a) The angle between the tangents of the Bézier curve and the helix; (b) the angle between the principal normal of the Bézier curve and the helix; (c) the curvature difference; (d) the torsion difference; (e) the fitting error for the arc; (f) the height difference along the z-axis.

Since the tangent directions are more accurate than the normal directions, and the rational Bézier curves are on the same cylinder surface as the original helix, we just plot the normal angle, the curvature difference, the torsion difference and the height difference for the approximation by rational curves (see Fig. 4). The fitting accuracy for quartic rational Bézier curves may be lower than for quintic rational Bézier curves, but when the circular angle is an acute angle, the approximation is still with a high accuracy. We have approximated a segment of helix with $R = b = 100$ and $2\theta = 2\pi/5$ by a quartic rational Bézier curve. For a comparison we have constructed another approximating curve based on Seemann's algorithm. The error plots for these two approximations are illustrated in Fig. 5.

From the error plots we can also see that the quintic polynomial curves, the quintic rational curves or even quartic rational curves can all be used to approximate the helices with high accuracy, as well as with G^2 interpolation at the ends. From the examples we can see that the approximation by quintic rational Bézier curves based on Seemann's method produces also G^2 contact at the ends, but not for quartic rational Bézier curves. To compare the fitting accuracy more efficiently, the maximum and the minimum fitting errors for the examples in this paper are listed in Table 1.

From Eqs. (16), (24) and (28), we notice that the solutions of λ and p are depending on the values of θ, but not on the parameter b. On the other hand, the coefficients of the interpolating function $z(t)$ at the three cases are all proportional to b, then the error function $e(t)$ is proportional to the variable b too. To show the convergence rate with respect to the central angle, we choose four helix segments with $R = b = 100$ and the central angles 2θ as $\frac{8}{9}\pi$, $\frac{4}{9}\pi$, $\frac{2}{9}\pi$ and $\frac{1}{9}\pi$, respectively. The maximum fitting error

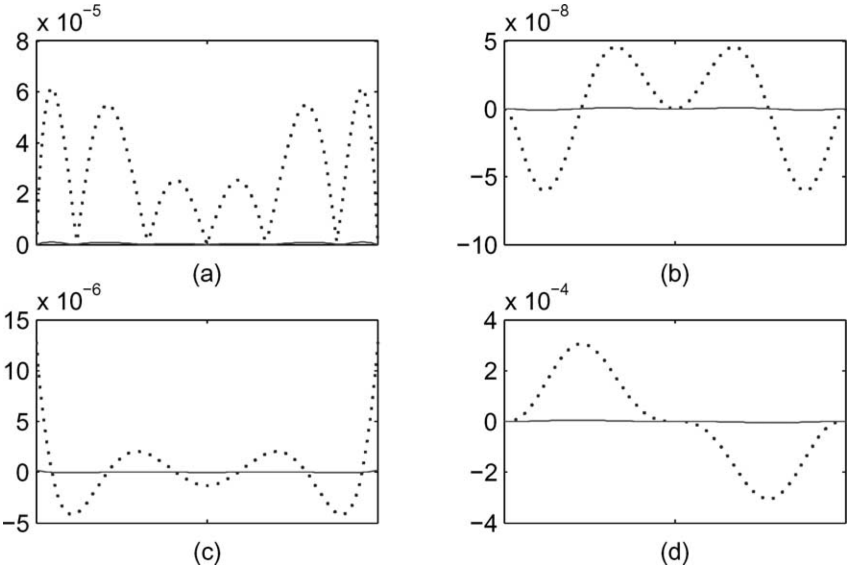

Fig. 4. Error plots for the approximation by quintic rational Bézier curves based on Seemann's method (dashed) and our new method (solid). $R = b = 100$, $2\theta = 2\pi/3$. (a) The normal angle; (b) the curvature difference; (c) the torsion difference; (d) the height difference.

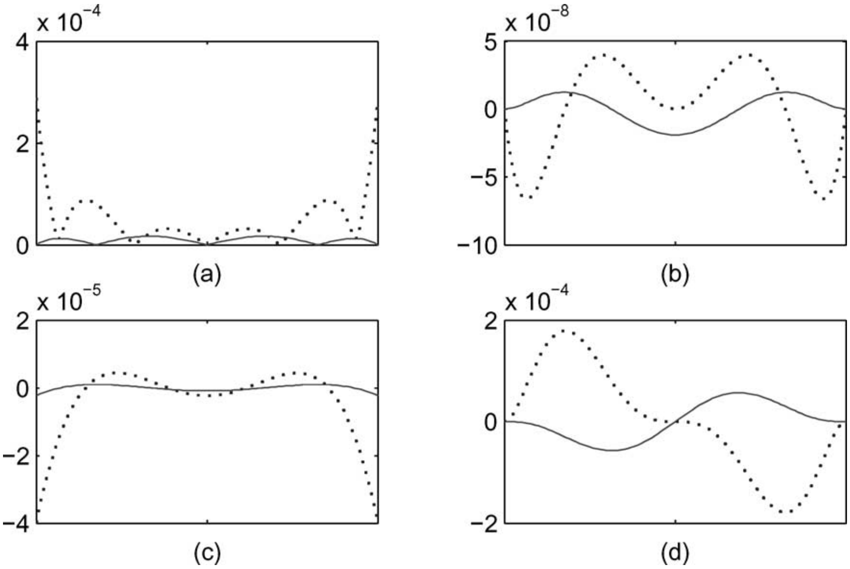

Fig. 5. Error plots for the approximation by quartic rational Bézier curves based on Seemann's method (dashed) and our new method (solid). $R = b = 100$, $2\theta = 2\pi/5$. (a) The normal angle; (b) the curvature difference; (c) the torsion difference; (d) the height difference.

along the z direction, the maximum fitting error for circular arc approximation by polynomials and the ratios of consecutive errors for the same type of approximating curves are listed in Table 2. The ratio $m(\phi)$ is computed as $m(\phi) = \log_2(e(2\phi)/e(\phi))$.

Table 1
The error range for the examples through Figs. 3–5

The fitting curves	Tangent angle	Normal angle	Curvature difference	Torsion difference	Height difference
YangBez5	0, 0.953×10^{-5}	0, 0.317×10^{-4}	-3.667×10^{-7}, 3.802×10^{-7}	-0.467×10^{-5}, 0.138×10^{-5}	-0.142×10^{-3}, 0.142×10^{-3}
YangRBez5	0, 0.988×10^{-7}	0, 0.963×10^{-6}	-0.979×10^{-9}, 0.771×10^{-9}	-0.655×10^{-7}, 1.901×10^{-7}	-0.503×10^{-5}, 0.503×10^{-5}
SmnRBez5	0, 0.603×10^{-5}	0, 0.613×10^{-4}	-0.602×10^{-7}, 0.455×10^{-7}	-0.418×10^{-5}, 1.278×10^{-5}	-0.305×10^{-3}, 0.305×10^{-3}
YangRBez4	0, 0.191×10^{-5}	0, 0.169×10^{-4}	-0.191×10^{-7}, 0.124×10^{-7}	-0.232×10^{-5}, 0.991×10^{-6}	-0.569×10^{-4}, 0.569×10^{-4}
SmnRBez4	0, 0.662×10^{-5}	0, 2.878×10^{-4}	-0.662×10^{-7}, 0.398×10^{-7}	-0.398×10^{-4}, 0.439×10^{-5}	-0.179×10^{-3}, 0.179×10^{-3}

Table 2
The maximum errors and the convergence ratios for the helix approximation

2θ	$\frac{8}{9}\pi$	$\frac{4}{9}\pi$	$\frac{2}{9}\pi$	$\frac{1}{9}\pi$
RBez5	7.135×10^{-5}	1.266×10^{-7} 9.138	2.414×10^{-10} 9.035	4.672×10^{-13} 9.013
RBez4	1.768×10^{-2}	1.195×10^{-4} 7.208	9.016×10^{-7} 7.051	6.982×10^{-9} 7.012
PBez5	1.862×10^{-3}	3.864×10^{-6} 8.982	7.224×10^{-9} 8.994	1.414×10^{-11} 8.996
Arc	7.718×10^{-5}	3.113×10^{-7} 7.953	1.226×10^{-9} 7.987	4.803×10^{-12} 7.997

6. Conclusion and discussion

In this paper we have presented an efficient new method for the approximation of helix segments by quintic polynomial Bézier curves, quintic and quartic rational Bézier curves. The Bézier curves not only interpolate the positions and curvatures at the ends of the helix, but also match the Frenet frames of the helix at the ends. By selecting a proper parametrization of the interpolating curves, the approximation order for the height functions along the helix axis is 9 for quintic curves and 7 for quartic rational curves. When the central angle for a helix segment is larger than π, it can be divided and approximated by two or more pieces of smooth connected Bézier curves.

There is one freedom λ or p for the error function $e(t)$ while approximating the helix segment by quintic polynomial curves or quintic rational curves. We have computed the values for these variables based on the assumption that $e'(0.5) = 0$. In fact we can reduce the fitting error further by minimizing the maximum of $|e(t)|$, but nonlinear equations will be solved and the computational cost will increase heavily. Another solution to set the freedoms for the quintic curves interpolation is based on the geometric Hermite interpolation method for general boundary data in (Yang, 2003) so that the fitting curves also interpolate the torsions at the ends of the helix. But the fitting accuracy for the height function and for the curvature function will be reduced if the end torsions are interpolated. In addition to interpolate the helices with high accuracies, the freedoms for the quintic rational curves can also be chosen with some other criteria such as that the fitting curve $Q(t)$ be with quasi-uniform parameter speed or C^1 continuity in homogeneous space (Blanc and Schlick, 1996; Fang, 2002). Then these properties can be well inherited by the interpolating space curves.

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