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Uniform hyperbolic polynomial B-spline curves

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Abstract

This paper presents a new kind of uniform splines, called *hyperbolic polynomial B-splines*, generated over the space $\Omega = \mathrm{span}\{\sinh t, \cosh t, t^{k-3}, t^{k-4}, \ldots, t, 1\}$ in which k is an arbitrary integer larger than or equal to 3. Hyperbolic polynomial B-splines share most of the properties as those of the B-splines in the polynomial space. We give the subdivision formulae for this new kind of curves and then prove that they have the variation dimishing properties and the control polygons of the subdivisions converge. Hyperbolic polynomial B-splines can take care of freeform curves as well as some remarkable curves such as the hyperbola and the catenary. The generation of tensor product surfaces by these new splines is straightforward. Examples of such tensor product surfaces: the saddle surface, the catenary cylinder, and a certain kind of ruled surface are given in this paper. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Uniform B-spline; C-B-splines; Exponential spline; Transcendental curves; Hyperbolic polynomial

1. Introduction

As a unified mathematic model with simple algorithms, NURBS (Non-Uniform Rational B-Splines) curves and surfaces have become the *de facto* standard in CAD/CAM (Hoschek and Lasser, 1993). However, as shown by Mainar et al. (2001), there still exist several limitations of the NURBS model, which keeps it from being applied conveniently and easily in some cases. For example, rational form may be unstable, and derivatives and integrals are hard to compute. Furthermore, it fails to represent some remarkable transcendental curves such as the helix and the catenary, which are often used in engineering. In recent years, several new spline curve and surface schemes have been proposed for geometric modeling in CAGD. For instance, Zhang (1996, 1997) introduced C-B-splines, which coincide with the helix splines proposed by Pottmann and Wagner (1994). C-curves admit exact representations

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for the ellipse (Zhang, 1996, 1999), the cycloid and the helix (Mainar et al., 2001), the sphere (Morin et al., 2001).

Pottmann (1993) noted that the space given by span$\{\sinh t, \cosh t, t, 1\}$ yields the exponential splines in tension, and Koch and Lyche (1989, 1991), Mazure (1999) obtained its normalized B-basis by applying the blossoming principle and a de Casteljau type algorithm in extended Chebyshev spaces (Pottmann, 1993). Indeed, C-B-splines and the exponential splines in tension are both symmetric Tchebycheffian B-splines (Wagner and Pottmann, 1994), which are natural generalizations of the polynomial B-splines (Schumaker, 1981).

By taking the hyperbolic functions into consideration, the exponential splines in tension admit an exact representations of some remarkable curves, such as the hyperbola and the catenary, without appealing to rational form. In addition, their derivatives and integrals are easy to compute. However, exponential splines in tension are not applicable to freeform polynomial curves of high orders, which severely restrict their applications in CAGD.

In this paper we begin by considering the uniform splines generated over the space span$\{\sinh t, \cosh t, t^{k-3}, t^{k-4}, \ldots, t, 1\}$ $(k \geqslant 3)$. We call such splines *hyperbolic polynomial B-splines of order* k. Such splines share most of the properties as those of the B-splines in polynomial space (Farin, 1997; Lane and Riesenfeld, 1980; Piegl and Tiller, 1997). We give explicit expressions for the hyperbolic polynomial B-spline basis function. In Section 3, we list the basic properties of the hyperbolic polynomial B-spline curves. The main part of this section is devoted to the subdivision formulae of this kind of curves. We then prove that the curves have the variation diminishing (V.D.) property and the control polygons of the subdivisions converge. As applications, we represent the hyperbola and the catenary with such curves in Section 4. In Section 5, we look into tensor product hyperbolic polynomial B-spline surfaces. Many algorithms of the curve scheme can be easily extended to the surface scheme. With this new surface model, we construct the saddle surface, the catenary cylinder, and a ruled surface with some prescribed boundary. A short conclusion is given in Section 6.

2. The basis of hyperbolic polynomial B-splines

2.1. An integral approach to the definition of the basis

Let $t_i = i\alpha$ $(i = 0, \pm 1, \pm 2, \ldots)$ (α is the interval length, $\alpha \geqslant 0$) be a set of knots which partition the parameter axis t uniformly. We denote the collection of piecewise hyperbolic polynomial splines of order k defined on $[t_i, t_{i+1}]$ $(i = 0, \pm 1, \pm 2, \ldots)$ by $\Omega_{k,\alpha}$, in which each function is $k-2$ times continuously differentiate at the knot t_i $(i = 0, \pm 1, \pm 2, \ldots)$ and on each interval $[t_i, t_{i+1}]$ $(i = 0, \pm 1, \pm 2, \ldots)$ it is a hyperbolic polynomial of order k. It can be easily checked that the usual operations of addition and scalar multiplication of functions in $\Omega_{k,\alpha}$ are closed, i.e., $\Omega_{k,\alpha}$ is a linear space. A basis of $\Omega_{k,\alpha}$ is called a *hyperbolic polynomial B-spline basis of order k* if the basis functions are nonnegative, form a partition of unity, and have minimal supports. In this section we shall construct a basis of $\Omega_{k,\alpha}$ for $k \geqslant 3$ and discuss the properties of the basis functions.

Theorem 1. *There exists no hyperbolic polynomial B-spline basis of the space* $\Omega_{2,\alpha}$.

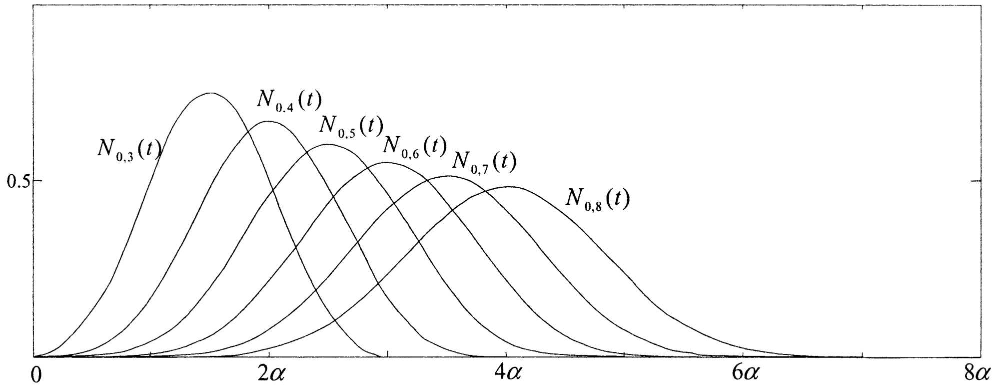

Fig. 1. Hyperbolic polynomial B-spline basis of order 3–8 ($\alpha = 1$).

Proof. Assume there exist a hyperbolic polynomial B-spline basis of $\Omega_{2,\alpha}$. From the property of partition of unity, all basis functions must sum to one. At the same time each basis function is a linear combination of $\{\sinh t, \cosh t\}$ on each interval, so 1 must be a linear combination of $\{\sinh t, \cosh t\}$, which contradicts the fact that $\{\sinh t, \cosh t, 1\}$ are linearly independent. This proves the proposition. □

To construct a basis of the space $\Omega_{k,\alpha}$ when $k \geqslant 3$, we first define a set of functions over $\Omega_{2,\alpha}$:

$$N_{0,2}(t) = \begin{cases} -\dfrac{\alpha}{2(\cosh\alpha - 1)} \sinh t, & 0 \leqslant t \leqslant \alpha, \\ -\dfrac{\alpha}{2(\cosh\alpha - 1)} \sinh(2\alpha - t), & \alpha \leqslant t \leqslant 2\alpha, \\ 0, & \text{elsewhere}, \end{cases} \tag{1}$$

and

$$N_{i,2}(t) = N_{0,2}(t - i\alpha) \quad (i = 0, \pm 1, \pm 2, \ldots). \tag{2}$$

For $k \geqslant 3$ let

$$N_{i,k}(t) = \frac{1}{\alpha} \int_{t-\alpha}^{t} N_{i,k-1}(x)\,\mathrm{d}x \quad (i = 0, \pm 1, \pm 2, \ldots). \tag{3}$$

It can be easily seen that $N_{i,k}(t)$ $(i = 0, \pm 1, \pm 2, \ldots)$ possesses the properties to be listed in Section 2.2, from which we conclude that $N_{i,k}(t)$ $(i = 0, \pm 1, \pm 2, \ldots)$ constitute a basis of $\Omega_{k,\alpha}$ for $k \geqslant 3$. Then, we call $N_{i,k}(t)$ $(i = 0, \pm 1, \pm 2, \ldots)$ the *hyperbolic polynomial B-spline basis of order k*. Hyperbolic polynomial B-spline basis of order 3–8 are illustrated in Fig. 1.

2.2. Properties of the basis

Some basic properties of the hyperbolic polynomial B-spline basis of order k given above are listed as follows. These properties can be easily derived from formulae (l)–(3).

(1) *Non-negativity*: $N_{i,k} \geqslant 0$, $t \in (-\infty, +\infty)$.

(2) *Local support*:

$$N_{i,k}\begin{cases} >0, & t\in(i\alpha,(i+k)\alpha),\\ =0, & \text{elsewhere.}\end{cases}$$

(3) *Partition of unity*: $\sum_i N_{i,k}(t)\equiv 1$.
(4) *Linear independence*: $N_{i,k}(t)$ $(i=0,\pm1,\pm2,\ldots)$ are linearly independent on $(-\infty,+\infty)$. In particular, $N_{i,k}(t), N_{i+1,k}(t),\ldots,N_{i+n,k}(t)$ $(n\geqslant k)$ are linearly independent on the interval $[(i+k-1)\alpha,(i+n+1)\alpha]$.
(5) *Derivative*: $N'_{i,k}(t)=\frac{1}{\alpha}(N_{i,k-1}(t)-N_{i+1,k-1}(t))$.

Proof.

$$\begin{aligned}N'_{i,k}(t) &= \left(\frac{1}{\alpha}\int_{t-\alpha}^{t} N_{i,k-1}(x)\,\mathrm{d}x\right)=\frac{1}{\alpha}\big(N_{i,k-1}(t)-N_{i,k-1}(t-\alpha)\big)\\ &= \frac{1}{\alpha}\big(N_{i,k-1}(t)-N_{i,k-1}(t-\alpha)\big).\end{aligned}$$

(6) *Symmetry*: $N_{i,k}(i\alpha+k\alpha-t)=N_{i,k}(i\alpha+t),\quad t\in[0,k\alpha]$.

2.3. Explicit expressions of the basis functions

From formulae (1)–(3) and by integration and induction, we can express the functions in the given basis explicitly.

Theorem 2. *Hyperbolic polynomial B-spline basis functions can be expressed as*:

$$N_{i,k}(t)=\begin{cases}\dfrac{\big(\sum_{j=i}^{i+k-1} d_k^j(t)B_{j,1}(\alpha,t)+\sum_{m=1}^{[(k-1)/2]}\sum_{j=0}^{2m} a_{j,2m}B_{j,k-2m}(\alpha,t)\big)}{2\alpha^{k-3}(\cosh\alpha-1)}, & k\text{ odd},\\[2ex] \dfrac{\big(\sum_{j=i}^{i+k-1} e_k^j(t)B_{j,1}(\alpha,t)+\sum_{m=1}^{[(k-1)/2]}\sum_{j=0}^{2m} a_{j,2m}B_{j,k-2m}(\alpha,t)\big)}{2\alpha^{k-3}(\cosh\alpha-1)}, & k\text{ even},\end{cases}\quad k\geqslant 3, \tag{4}$$

where $[x]$ is the greatest integer less than or equal to x, $B_{j,l}(\alpha,t)$ is the uniform B-spline basis function defined over $[j\alpha,(j+l)\alpha]$, and

$$d_k^j(t)=A_k^j\cosh(t-j\alpha)+C_k^j\cosh\big(t-(j+1)\alpha\big);$$
$$e_k^j(t)=A_k^j\sinh(t-j\alpha)+C_k^j\sinh\big(t-(j+1)\alpha\big);$$
$$a_{0,2}=1,\quad a_{1,2}=2\cosh\alpha,\quad a_{2,2}=-1,\quad \text{when } m=1; \tag{5}$$
$$a_{j,2m}=\big(A_{2m}^{j-1}-C_{2m}^{j}\big)\cosh\alpha-\big(A_{2m}^{j}-C_{2m}^{j-1}\big),\quad \text{when } m>1; \tag{6}$$

in which the coefficients A_k^j and C_k^j are given by

$$A_k^j=C_k^j=0,\quad \text{when } k\geqslant 3 \text{ and } j<i \text{ or } j>i+k-1; \tag{7}$$
$$A_3^i=C_3^{i+2}=1,\quad A_3^{i+1}=C_3^{i+1}=-1,\quad A_3^{i+2}=C_3^i=0,\quad \text{when } k=3; \tag{8}$$
$$A_k^j=A_{k-1}^j-A_{k-1}^{j-1},\quad C_k^j=C_{k-1}^j-C_{k-1}^{j-1},\quad \text{when } k>3 \text{ and } i\leqslant j\leqslant i+k-1. \tag{9}$$

Proof. By induction on k. When $k=3$, we have $m=1$. (4) follows easily from (5) and (8).

Assume (4) holds for $k=r-1$. When $k=r$, we only have to show that (4) holds for $l\alpha \leqslant t \leqslant (l+1)\alpha$, $\forall l \in [i, k-1]$. Without loss of generality, assume that r is odd. The proof is similar when r is even.

Simple algebraic manipulation yields

$$\begin{aligned}
N_{i,r}(t) &= \frac{1}{\alpha}\int_{t-\alpha}^{t} N_{i,r-1}(x)\,\mathrm{d}x \\
&= \frac{1}{2\alpha^{r-3}(\cosh\alpha - 1)}\Biggl(\int_{l\alpha}^{t}\bigl(A_{r-1}^{t}\sinh(x-l\alpha) + C_{r-1}^{l}\sinh(x-(l+1)\alpha)\bigr)\,\mathrm{d}x \\
&\quad + \int_{x-\alpha}^{l\alpha}\bigl(A_{r-1}^{l-1}\sinh(x-(l-1)\alpha) + C_{r-1}^{l-1}\sinh(x-l\alpha)\bigr)\,\mathrm{d}x \\
&\quad + \int_{t-\alpha}^{t}\sum_{m=1}^{[(r-2)/2]}\sum_{j=0}^{2m} a_{j,2m}B_{j,r-1-2m}(\alpha,x)\,\mathrm{d}x\Biggr) \\
&= \frac{1}{2\alpha^{r-3}(\cosh\alpha - 1)}\Biggl(\bigl(A_{r-1}^{l} - A_{r-1}^{l-1}\bigr)\cosh(t-l\alpha) + \bigl(C_{r-1}^{l} - C_{r-1}^{l-1}\bigr)\cosh\bigl(t-(l+1)\alpha\bigr) \\
&\quad + \bigl(A_{r-1}^{l-1} - C_{r-1}^{l}\bigr)\cosh\alpha - \bigl(A_{r-1}^{l} - C_{r-1}^{l-1}\bigr) + \sum_{m=1}^{[(r-2)/2]}\sum_{j=0}^{2m} a_{j,2m}B_{j,r-2m}(\alpha,t)\Biggr) \\
&= \frac{\bigl(\sum_{j=i}^{i+k-1} d_k^j(t)B_{j,1}(\alpha,t) + \sum_{m=1}^{[(k-2)/2]}\sum_{j=0}^{2m} a_{j,2m}B_{j,k-2m}(\alpha,t)\bigr)}{2\alpha^{k-3}(\cosh\alpha - 1)}.
\end{aligned}$$

This proves the theorem. □

3. Hyperbolic polynomial B-spline curves

The functions given in Section 2 serve as a basis for hyperbolic polynomial B-spline curves over the whole parameter space. However, for most geometric modeling, the span of the parameter t is usually restricted to a finite interval $[a, b]$ with $a < b$. Hence we shall confine our discussion to spline curve modeling on finite interval.

We denote the space of hyperbolic polynomial B-splines of order k defined over $[a, b]$ by $\Omega_{k,\alpha}[a, b]$. If $a = k\alpha$, $b = (n+1)\alpha$, then $N_{1,k}(t), N_{2,k}(t), \ldots, N_{n,k}(t)$ $(n \geqslant k)$ constitute a basis of the space $\Omega_{k,\alpha}[a, b]$. (See Fig. 2.) Therefore, a spline curve in $\Omega_{k,\alpha}[a, b]$ can be written as

$$p_k(t) = \sum_{i=1}^{n} P_i N_{i,k}(t), \quad k\alpha \leqslant t \leqslant (n+1)\alpha \ (n \geqslant k) \tag{10}$$

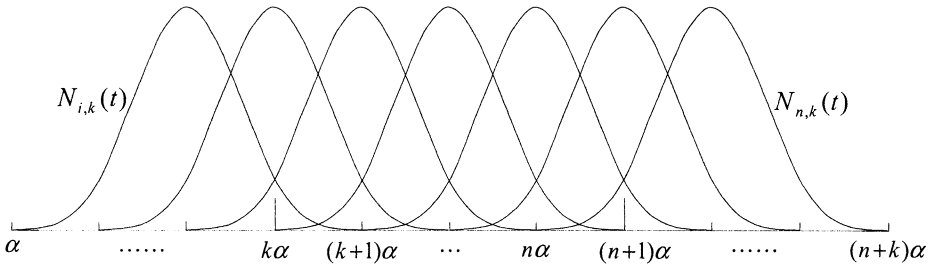

Fig. 2. $N_{1,k}(t), \dots, N_{n,k}(t)$ are basis of the space $\Omega_{k,\alpha}[k\alpha, (n+1)\alpha]$.

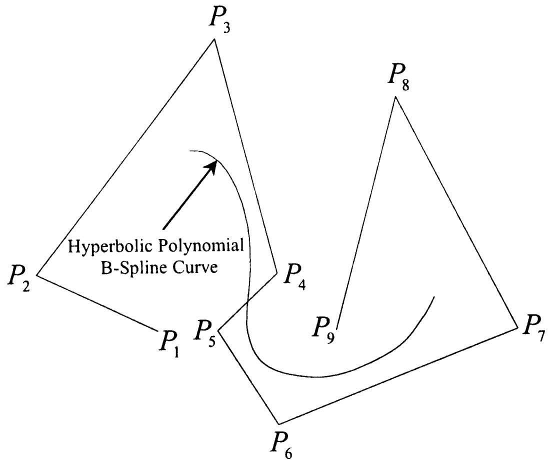

Fig. 3. Hyperbolic polynomial B-spline curve of order 6 ($\alpha = 1$).

where P_i $(i = 1, 2, \dots, n)$ are the control points, i.e., the control polygon is $P = [P_1, P_2, \dots, P_n]$. Fig. 3 illustrates a hyperbolic polynomial B-spline curve of order 6. From the figure, we see that the control polygon is approximated by the curve.

3.1. Properties of the curve

Similar to the B-spline curves, hyperbolic polynomial B-spline curves have the following properties.

Proposition 3.1 (Convex hull property). *A curve $p_k(t)$ $(i\alpha \leqslant t \leqslant (i+1)\alpha,\ i = k, \dots, n)$ defined on $[i\alpha, (i+1)\alpha]$ lies inside the convex hull H_i of the control points $P_{i-k+1}, \dots, P_i$, and the entire curve $p_k(t)$ given in* (10) *lies inside $H = \bigcup_{i=k}^{n} H_i$, which is union of H_i. This follows, since the hyperbolic polynomial B-spline basis functions are nonnegative and they sum to one.*

Proposition 3.2 (Geometric invariance). *Because $p_k(t)$ is an affine combination of the control points P_i $(i = 1, \dots, n)$, its shape is independent of the choice of coordinate system.*

Proposition 3.3 (Local control property). *Change of one control point will alter at most k segments of the original hyperbolic polynomial B-spline curve of order k. Hence local adjustment can be made without disturbing the rest of the curve.*

Proposition 3.4 (Symmetry). *As illustrated by the curve in Fig.* 3, *it is clear that the control points can be labeled $P_1, P_2, \ldots, P_n$ or $P_n, P_{n-1}, \ldots, P_1$ without changing the shape of the curve. They differ only in the direction in which they are traversed. If we do not consider the direction of a curve, we have*:

$$\sum_{i=1}^{n} P_i N_{i,k}(i\alpha + t) = \sum_{i=1}^{n} P_{n-i} N_{i,k}(i\alpha + k\alpha - t), \quad t \in [0, k\alpha].$$

Proposition 3.5. *The derivative of a hyperbolic polynomial B-spline curve*:

$$\frac{\mathrm{d}}{\mathrm{d}t} p_k(t) = \frac{1}{\alpha} \sum_{i=2}^{n} N_{i,k-1}(t) \Delta P_i \quad (k\alpha \leqslant t \leqslant (n+1)\alpha)$$

where $\Delta P_i = P_i - P_{i-1}$.

3.2. Subdivision formulae for hyperbolic polynomial B-splines

In this section, we shall discuss the subdivision formulae for hyperbolic polynomial B-spline curves. We then show that the curves have the V.D. property.

Let $N_{i,k}(\alpha, t)$ $(i = 0, \pm 1, \pm 2, \ldots)$ denote a basis function of order k with uniform partition interval α on the parameter axis as defined in (l)–(3). If we partition the parameter axis t with unit interval length $\alpha/2$, that is, we take $t_i' = i\alpha/2$ $(i = 0, \pm 1, \pm 2, \ldots)$ as knots, we have a new set of bases with this new set of knots. We denote these new bases of order k by $N_{i,k}(\alpha/2, t)$ $(i = 0, \pm 1, \pm 2, \ldots)$.

By definition, $\Omega_{k,\alpha}[k\alpha, (n+1)\alpha]$ is a space consisting of hyperbolic polynomial B-spline curves of order k which are $(k-2)$ times continuously differentiable at the knots $t_i = i\alpha$ $(i = 0, \pm 1, \pm 2, \ldots)$. Similarly, $\Omega_{k,\alpha/2}[k\alpha, (n+1)\alpha]$ is a space consisting of hyperbolic polynomial B-spline of order k that are $(k-2)$ times continuously differentiable at the knots $t_i' = i\alpha/2$ on the interval $[k\alpha, (n+1)\alpha]$. It is clear that $\Omega_{k,\alpha}[k\alpha, (n+1)\alpha]$ is a subspace of $\Omega_{k,\alpha/2}[k\alpha, (n+1)\alpha]$, and that curves in $\Omega_{k,\alpha}[k\alpha, (n+1)\alpha]$ can be expressed by $N_{i,k}(\alpha/2, t)$. When a curve in $\Omega_{k,\alpha}[k\alpha, (n+1)\alpha]$ is expressed by $N_{i,k}(\alpha, t)$ and then by $N_{i,k}(\alpha/2, t)$, the relationship between their control points is elucidated in the following theorem. (See Figs. 4 and 5.)

Theorem 3 (Subdivision). *Let $p_k(t)$ be a hyperbolic polynomial B-spline curve of order $k \geqslant 3$ with interval size α and control polygon $P = [P_1, \ldots, P_n]$, $n \geqslant k$, that is,*

$$p_k(t) = \sum_{i=1}^{n} P_i N_{i,k}(\alpha, t)$$

with $N_{i,k}(\alpha, t)$ given by (1)–(3) *and $t \in [k\alpha, (n+1)\alpha]$. Then*

$$p_k(t) = \sum_{i=1}^{2n-k+1} P_i^k N_{i+k,k}(\alpha/2, t)$$

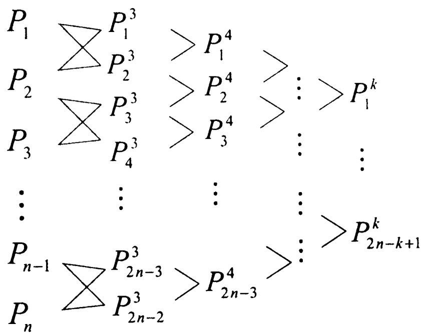

Fig. 4. The recursion of the control points on subdivision.

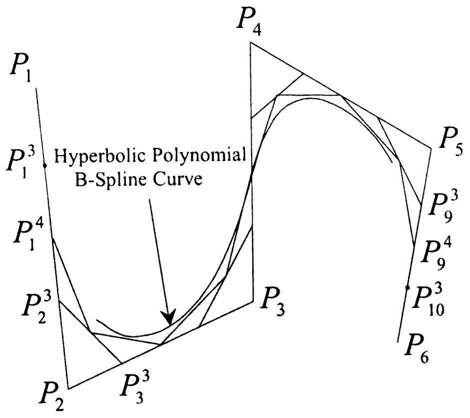

Fig. 5. Subdivision of hyperbolic polynomial B-spline curve ($\alpha = 1$).

where P_i^k is defined recursively by

$$P_i^k = \left(P_i^{k-1} + P_{i+1}^{k-1}\right)/2, \quad i = 1, 2, \dots, 2n-k+1,\ k > 3, \tag{11}$$

and for $i = 0, 1, 2, \dots, 2n-2$,

$$P_i^3 = \begin{cases} ((1+2\cosh(\alpha/2))P_{(i+1)/2} + P_{(i+3)/2})/(2(1+\cosh(\alpha/2))), & i \text{ odd},\\ (P_{i/2} + (1+2\cosh(\alpha/2))P_{i/2+1})/(2(1+\cosh(\alpha/2))), & i \text{ even}. \end{cases} \tag{12}$$

Proof. By induction on k. When $k = 3$,

$$\begin{aligned}
\sum_{i=1}^{n} P_i N_{i,3}(\alpha, t) &= \sum_{i=1}^{n} P_i \cdot \left(\frac{1}{\alpha}\int_{t-\alpha}^{t} N_{i,2}(\alpha, x)\,\mathrm{d}x\right)\\
&= \sum_{i=1}^{n} P_i\left(\frac{1}{\alpha}\int_{t-\alpha}^{t}\left(\frac{1}{1+\cosh(\alpha/2)}N_{2i,2}(\alpha/2, x) + \frac{2\cosh(\alpha/2)}{1+\cosh(\alpha/2)}N_{2i+1,2}(\alpha/2, x)\right.\right.\\
&\qquad\left.\left. + \frac{1}{1+\cosh(\alpha/2)}N_{2i+2,2}(\alpha/2, x)\right)\mathrm{d}x\right)\\
&= \sum_{i=1}^{n} P_i\left(\frac{1}{2(1+\cosh(\alpha/2))}N_{2i,3}(\alpha/2, t) + \frac{1+2\cosh(\alpha/2)}{2(1+\cosh(\alpha/2))}N_{2i+1,3}(\alpha/2, t)\right.\\
&\qquad\left. + \frac{1+2\cosh(\alpha/2)}{2(1+\cosh(\alpha/2))}N_{2i+2,3}(\alpha/2, t) + \frac{1}{2(1+\cosh(\alpha/2))}N_{2i+3,3}(\alpha/2, t)\right).
\end{aligned}$$

Since

$$N_{2,3}(\alpha/2, t) = N_{3,3}(\alpha/2, t) = N_{2n+2,3}(\alpha/2, t) = N_{2n+3,3}(\alpha/2, t) = 0$$

for $t \in [3\alpha, (n+1)\alpha]$, we have

$$\begin{aligned}\sum_{i=1}^{n} P_i N_{i,3}(\alpha,t) &= \sum_{i=1}^{n-1}\biggl(\frac{(1+2\cosh(\alpha/2))P_i+P_{i+1}}{2(1+\cosh(\alpha/2))}N_{2i+2,3}(\alpha/2,t)\\ &\quad+\frac{P_i+(1+2\cosh(\alpha/2))P_{i+1}}{2(1+\cosh(\alpha/2))}N_{2i+3,3}(\alpha/2,t)\biggr)\\ &= \sum_{i=1}^{2n-2} P_i^3 N_{i+3,3}(\alpha/2,t).\end{aligned}$$

Thus (12) holds. Now assume (11) holds for all l with $3 \leqslant l < k$. For $l = k$, we have

$$\sum_{i=1}^{n} P_i N_{i,k}(\alpha,t) = \sum_{i=1}^{n} P_i\left(\frac{1}{\alpha}\int_{t-\alpha}^{t} N_{i,k-1}(\alpha,x)\,\mathrm{d}x\right) = \frac{1}{\alpha}\int_{t-\alpha}^{t}\left(\sum_{i=1}^{n} P_i N_{i,k-1}(\alpha,x)\,\mathrm{d}x\right) \tag{13}$$

for $t \in [k\alpha, (n+1)\alpha]$. Now by our inductive hypothesis, (13) reduces to

$$\begin{aligned}\sum_{i=1}^{n} P_i N_{i,k}(\alpha,t) &= \frac{1}{\alpha}\int_{t-\alpha}^{t}\left(\sum_{i=1}^{2n-k+2} P_i^{k-1} N_{i+k-1,k-1}(\alpha/2,x)\right)\mathrm{d}x\\ &= \frac{1}{2}\left(\frac{1}{\alpha/2}\left(\int_{t-\alpha/2}^{t}+\int_{t-\alpha}^{t-\alpha/2}\right)\sum_{i=1}^{2n-k+2} P_i^{k-1} N_{i+k-1,k-1}(\alpha/2,x)\,\mathrm{d}x\right)\\ &= \frac{1}{2}\left(\sum_{i=1}^{2n-k+2} P_i^{k-1} N_{i+k-1,k}\left(\frac{\alpha}{2},t\right)+\sum_{i=1}^{2n-k+2} P_i^{k-1} N_{i+k-1,k}\left(\frac{\alpha}{2},t-\frac{\alpha}{2}\right)\right).\end{aligned} \tag{14}$$

Since

$$N_{k,k}(\alpha/2,t) = N_{2n+1,k}(\alpha/2,t-\alpha/2) = 0$$

for $t \in [k\alpha, (n+1)\alpha]$, after dropping terms involving these basis functions and rearranging the remaining terms in (14), we have

$$\begin{aligned}\sum_{i=1}^{n} P_i N_{i,k}(\alpha,t) &= \frac{1}{2}\left(\sum_{i=2}^{2n-k+2} P_i^{k-1} N_{i+k-1,k}(\alpha/2,t)+\sum_{i=1}^{2n-k+1} P_i^{k-1} N_{i+k-1,k}(\alpha/2,t-\alpha/2)\right)\\ &= \sum_{i=1}^{2n-k+1}\left(P_i^{k-1}+P_{i+1}^{k-1}\right)/2\cdot N_{i+k,k}(\alpha/2,t)\\ &= \sum_{i=1}^{2n-k+1} P_i^{k}\cdot N_{i+k,k}(\alpha/2,t).\end{aligned}$$

This proves the theorem. □

This theorem shows how the new control polygon can be obtained from the old control polygon after one round of subdivision. By iteration of the subdivision process, we generate a sequence of control polygons.

To simplify notation, we use $B_k(P;1,n)(t)$ to denote $p_k(t)$ with

$$P=\Phi^0[P]=[P_1,P_2,\ldots,P_n]$$

as control polygon. Let

$$\Phi^1[P]=\left[P_1^{k,1},P_2^{k,1},\ldots,P_n^{k,1}\right]$$

be the control polygon after one round of subdivision, where $P_i^{k,1}=P_i^k$ is given by (11) and (12). Inductively, let

$$\Phi^l[P]=\Phi\left[\Phi^{l-1}P\right]=\left[P_1^{k,l},P_2^{k,l},\ldots,P_{r(l,k)}^{k,l}\right]$$

for $r(l,k)=2^{l+1}(n-k+1)+k-1$, $l\geqslant 1$, denote the control polygon after l rounds of subdivision. We next show that when the number of subdivisions increases, the sequence of control polygons converges to the spline curve.

Theorem 4. *Let $B_k(P;1,n)(t)$ and $\Phi^l[P]$ be defined as above. Then*

$$\lim_{l\to\infty}\Phi^l[P]=B_k(P;1,n)(t).$$

Proof. From Theorem 3 and by simple induction on l we have

$$B_k\left(\Phi^l[P];1,n\right)(t)=B_k(P;1,n)(t).$$

Now let $M=\max_i|P_{i+1}-P_i|$. It is easy to get

$$\left|P_{i+1}^k-P_i^k\right|\leqslant\frac{1}{1+\cosh(\alpha/2)}M.$$

Therefore

$$\left|P_{i+1}^{k,l}-P_i^{k,l}\right|\leqslant\frac{1}{(1+\cosh(\alpha/2))\cdots(1+\cosh(\alpha/2^l))}M.$$

Since $0<\alpha<\pi$, $0<\alpha/2<\pi/2$, we have

$$\left|P_{i+1}^{k,l}-P_i^{k,l}\right|\leqslant\frac{1}{(1+\cosh(\alpha/2))^l}M,$$

that is,

$$\lim_{l\to\infty}\left|P_{i+1}^{k,l}-P_i^{k,l}\right|=0.$$

Hence

$$\lim_{l\to\infty}\left|P_{i+j}^{k,l}-P_i^{k,l}\right|=0 \tag{15}$$

for any $i\in\{1,\ldots,r(l,k)\}$, $j=1,\ldots,k$, $i+j\leqslant r(l,k)$.

From the convex hull property, for any $t_0\in[k\alpha,(n+1)\alpha]$, we know that $B_k(P;1,n)(t_0)$ lies within the convex hull of $P_i^{k,l},P_{i+1}^{k,l},\ldots,P_{i+k}^{k,l}$ for some i. Together with (15), we conclude that

$$\lim_{l\to\infty}\Phi^l[P]=B_k(P;1,n)(t).\qquad\square$$

Theorem 4 ensures that recursive subdivision of control polygon leads to its corresponding hyperbolic polynomial B-spline curve. The following theorem shows that such curve possesses the variation diminishing (V.D.) property, which is crucial for work in CAD by preventing the curve from wiggling too much.

Theorem 5 (V.D. property). *No plane intersects a hyperbolic polynomial B-spline curve more often than it intersects the corresponding control polygon.*

Proof. For any arbitrary selected plane P, the points of intersection between the plane P and the control polygons will not increase after subdivision. Because the sequence of control polygons converge to the hyperbolic polynomial B-spline after repeated subdivisions, so the V.D. property holds. □

Theorem 6 (Convexity preserving). *If the control polygon is convex, then the corresponding hyperbolic polynomial B-spline curve is also convex.*

Proof. The convex control polygon preserves convexity after each round of subdivision, so from the convergence property, the corresponding hyperbolic polynomial B-spline curve is also convex. □

4. Representing the hyperbola and the catenary

In this section we shall show how to represent the hyperbola and the catenary in hyperbolic polynomial B-spline forms. Here we use hyperbolic polynomial B-splines of order 4, that is, splines over the space spanned by $\{\sinh t, \cosh t, t, 1\}$. From (1) and (2), the hyperbolic polynomial B-splines $N_{i,4}(t)$ of order 4 are:

$$N_{i,4}(t) = \frac{1}{2\alpha(\cosh\alpha - 1)} \times \begin{cases} \sinh(t - i\alpha) - (t - i\alpha), & i\alpha \leqslant t \leqslant (i+1)\alpha, \\ -2\sinh(t-(i+1)\alpha) - \sinh(t-(i+2)\alpha) \\ \quad + (2\cosh\alpha + 1)(t-(i+1)\alpha) - \alpha, & (i+1)\alpha \leqslant t \leqslant (i+2)\alpha, \\ \sinh(t-(i+2)\alpha) + 2\sinh(t-(i+3)\alpha) \\ \quad - (2\cosh\alpha + 1)(t-(i+2)\alpha) + 2\alpha\cosh\alpha, & (i+2)\alpha \leqslant t \leqslant (i+3)\alpha, \\ -\sinh(t-(i+4)\alpha) + t - (i+4)\alpha, & (i+3)\alpha \leqslant t \leqslant (i+4)\alpha. \end{cases} \tag{16}$$

A segment of a hyperbolic polynomial B-spline curve of order 4 can be expressed as:

$$p_4(t) = \sum_{i=1}^{4} N_{i,4}(t) P_i, \quad 4\alpha \leqslant t \leqslant 5\alpha. \tag{17}$$

4.1. Representing the hyperbola

By appropriate choice of coordinates, we can express one branch of the hyperbola in the parametric form

$$x(t) = a\cosh t, \quad y(t) = b\sinh t, \qquad -\infty < t < \infty,$$

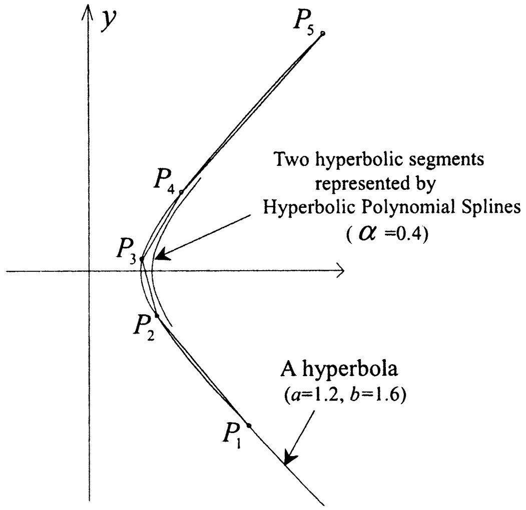

Fig. 6. Representation of two hyperbolic segments by hyperbolic polynomial B-spline.

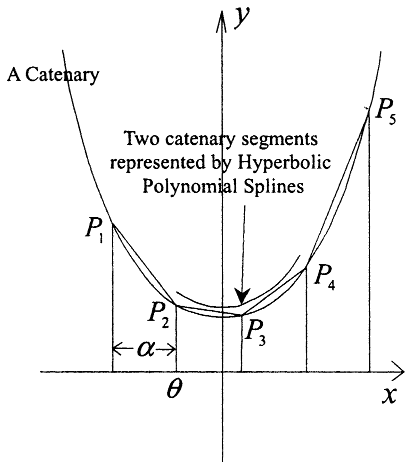

Fig. 7. Represented of two catenary segments by hyperbolic polynomial B-splines.

where a and b are respectively the real semi-axis and the imaginary semi-axis of the hyperbola. Let P_1, P_2, P_3 and P_4 with respective coordinates (see Fig. 6)

$$\begin{aligned} x_1 &= a\cosh(-\alpha+\theta), & y_1 &= b\sinh(-\alpha+\theta),\\ x_2 &= a\cosh(\theta), & y_2 &= b\sinh(\theta),\\ x_3 &= a\cosh(\alpha+\theta), & y_3 &= b\sinh(\alpha+\theta),\\ x_4 &= a\cosh(2\alpha+\theta), & y_4 &= b\sinh(2\alpha+\theta), \end{aligned} \tag{18}$$

be chosen as control points on this hyperbola with $\alpha > 0$ and θ an arbitrary real number.

Theorem 7. *The corresponding hyperbolic polynomial B-spline with these control points represents a branch of the hyperbola of the form*

$$x(t) = \frac{a\sinh\alpha}{\alpha}\cosh(t+\theta), \qquad y(t) = \frac{b\sinh\alpha}{\alpha}\sinh(t+\theta), \tag{19}$$

where $4\alpha \leqslant t \leqslant 5\alpha$, $\alpha > 0$ *and* θ *is an arbitrary real number.*

Proof. By substituting (16) and (18) into (17), we get (19). This is a hyperbola in parametric form. □

In Fig. 6, two segments of hyperbolic polynomial B-splines with five control points are used to represent a branch of the hyperbola.

4.2. Representing the catenary

The catenary is a transcendental curve of the form $y = \alpha \cosh(x/b)$. Similar to the representation of the hyperbola, we can choose four control points on the catenary and the corresponding hyperbolic polynomial B-spline is also a catenary. For instance, in Fig. 7, we choose the four points

$$\begin{array}{ll} x_1 = b(-\alpha + \theta), & y_1 = a\alpha \cosh(-\alpha + \theta)/\sinh\alpha, \\ x_2 = b\theta, & y_2 = a\alpha \cosh(\theta)/\sinh\alpha, \\ x_3 = b(\alpha + \theta), & y_3 = a\alpha \cosh(\alpha + \theta)/\sinh\alpha, \\ x_4 = b(2\alpha + \theta), & y_3 = a\alpha \cosh(2\alpha + \theta)/\sinh\alpha, \end{array} \tag{20}$$

as control points. Obviously, these four points are on the catenary $y = \frac{a\alpha}{\sinh\alpha}\cosh(x/b)$. Substituting (16) and (20) into (17), we get another catenary

$$x(t) = b(t + \theta), \qquad y(t) = a\cosh(t + \theta)$$

in hyperbolic polynomial B-spline form in which $4\alpha \leqslant t \leqslant 5\alpha$.

5. Hyperbolic polynomial B-surfaces

In the previous sections, we have discussed the curve scheme and used it to represent the hyperbola and the catenary. In this section, we extend this scheme to tensor product surfaces. We then give several examples for illustration.

Exactly as in the construction of B-spline tensor product surfaces from B-spline curves, we can construct hyperbolic polynomial B-spline surfaces from hyperbolic polynomial B-spline curve by

$$s(u, v) = \sum_{i=1}^{n}\sum_{j=1}^{m} N_{i,k}(u)N_{j,k}(v)P_{i,j}, \quad u \in [k\alpha, (n+1)\alpha],\ v \in [k\alpha, (m+1)\alpha],$$

where $P = [P_{i,j}]$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m)$ are the control meshes. Many properties of the curves can be extended to the surfaces. For example, the convex hull properties and the convexity preserving properties also hold for the surface scheme; the subdivision formulae can be used in both directions, etc.

In Fig. 8(a), we construct a ruled surface interpolating two prescribed boundaries defined by a parabola and by a hyperbola. Suppose that P_{2j}, P_{3j} $(j = 1, 2, \ldots, 5)$ are the control points of these two curves. Let $[P_{ij}]$ $(i = 1, \ldots, 4;\ j = 1, \ldots, 5)$ be the control mesh, in which $P_{1j} = 2P_{2j} - P_{3j},\ P_{4j} = 2P_{3j} - P_{2j}$ $(j = 1, 2, \ldots, 5)$, the resulting surface

$$\sum_{i=1}^{4}\sum_{j=1}^{5} N_{i,k}(u)N_{j,k}(v)P_{i,j} \quad (u \in [4\alpha, 5\alpha],\ v \in [5\alpha, 6\alpha])$$

is a ruled surface interpolating the given curves. Using the same method as in Fig. 8(a), the hyperbolic cylinder and the catenary cylinder can be similarly generated. Furthermore, we can easily construct blending surface of two given surfaces. Fig. 8(b) is a blending surface with patches of hyperbolic cylinder on one side and patches of catenary cylinder on the other side. In Fig. 8(c), there is a slideway generated by taking patches of saddle surface and catenary cylinder respectively on each side. Fig. 8(d) demonstrates the model of a mouse constructed by the hyperbolic polynomial B-spline surface scheme.

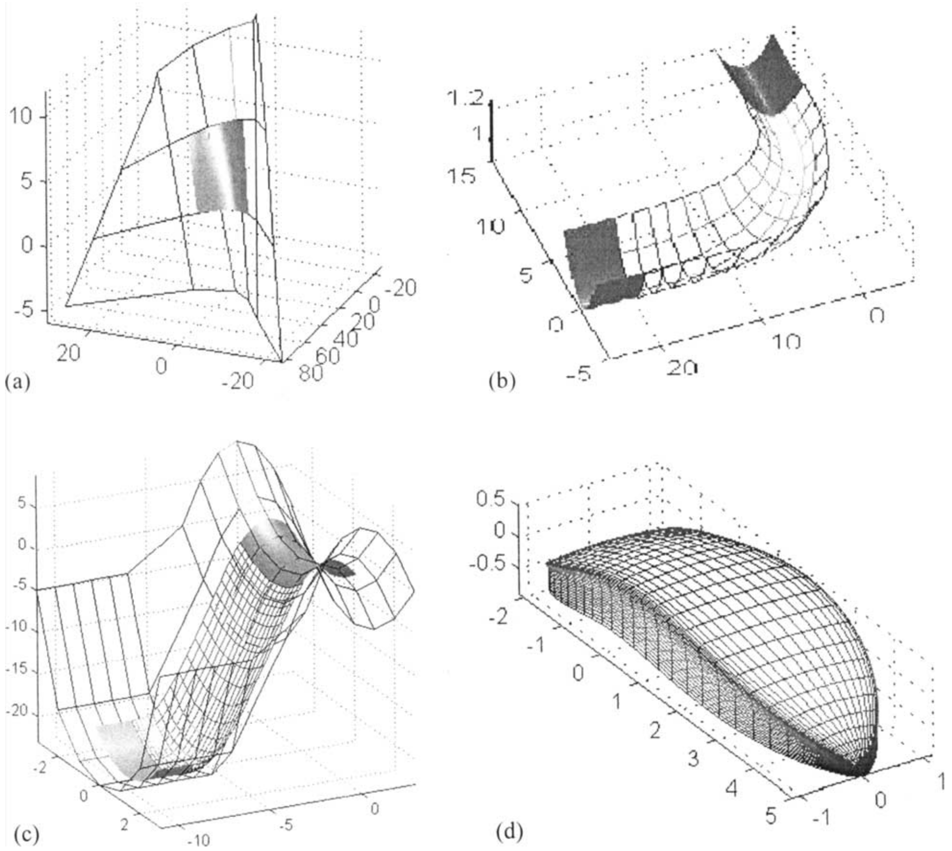

Fig. 8. Using tensor product patches for constructing some special surfaces and geometric models. (a) Ruled surface with parabola and hyperbola as its generatrices. ((b), (c)) Blending surfaces of two special patches. (d) Constructing a 'mouse'.

6. Conclusion

In this paper we have obtained general piecewise hyperbolic polynomial B-spline basis of order $k \geqslant 3$ over the space $\Omega = \mathrm{span}\{\sinh t, \cosh t, t^{k-3}, t^{k-4}, \ldots, t, 1\}$, from which we easily obtain hyperbolic polynomial B-spline curves model. Exponential splines in tension developed by Pottmann (1993) are special kind of hyperbolic polynomial B-splines of order 4. Hyperbolic polynomial B-spline curves not only inherit the advantage of polynomial curves, but also take on the characteristics of hyperbolic functions. Hyperbolic polynomial B-spline curves share nearly all the properties that uniform B-splines have. In addition to the polynomial curves, they also provide exact representations of some remarkable transcendental curves such as the catenary and hyperbola. The subdivision formulae of this kind of curves are given. The convergence property makes it possible for us to get the curve by recursive subdivisions. Furthermore, we can use tensor product to construct hyperbolic polynomial B-spline surfaces. We expect that hyperbolic polynomial B-spline model can be employed as a new powerful tool for constructing freeform curves and surfaces in CAGD.

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References

Farin, G., 1997. Curves and Surfaces for Computer Aided Geometric Design, 4th edn. Academic Press, San Diego, CA.

Hoschek, J., Lasser, D., 1993. Fundamentals of Computer Aided Geometric Design. AK Peters, Wellesley, MA.

Koch, P.E., Lyche, T., 1989. Exponential B-splines in tension, in: Chui, C.K., Schumaker, L.L., Ward, J.D. (Eds.), Approximation Theory VI. Academic Press, New York, pp. 361–364.

Koch, P.E., Lyche, T., 1991. Construction of exponential tension B-splines of arbitrary order, in: Laurent, P.J., Le Méhauté, A., Schumaker, L.L., Ward, J.D. (Eds.), Curves and Surfaces. Academic Press, Boston, MA, pp. 225–258.

Lane, J.M., Riesenfeld, R.F., 1980. A theoretical development for the computer generation and display of piecewise polynomial surfaces. IEEE Trans. Pattern Anal. Machine Intell. PAMI-2 (1), 35–46.

Mainar, E., Peña, J.M., 1999. Corner cutting algorithms associated with optimal shape preserving representations. Computer Aided Geometric Design 16, 883–906.

Mainar, E., Peña, J.M., Sânchez-Reyes, J., 2001. Shape preserving alternatives to the rational Bézier model. Computer Aided Geometric Design 18, 37–60.

Mazure, M.L., 1999. Chebyshev–Bernstein bases. Computer Aided Geometric Design 16, 649–669.

Morin, G., Warren, J., Weimer, H., 2001. A subdivision scheme for surface of revolution. Computer Aided Geometric Design 18, 483–502.

Piegl, L., Tiller, W., 1997. The NURBS Book, 2nd edn. Springer, Berlin.

Pottmann, H., 1993. The geometry of Tchebycheffian spines. Computer Aided Geometric Design 10, 181–210.

Pottmann, H., Wagner, M.G., 1994. Helix splines as example of affine Tchebycheffian splines. Adv. Comput. Math. 2, 123–142.

Schumaker, L.L., 1981. Spline Functions: Basic Theory. Wiley, New York, pp. 363–499.

Wagner, M.G., Pottmann, H., 1994. Symmetric Tchebycheffian B-spline schemes, in: Laurent, P.J., Le Méhauté, A., Schumaker, L.L. (Eds.), Curves and Surfaces in Geometric Design. AK Peters, pp. 225–258.

Zhang, J.W., 1996. C-curves: an extension of cubic curves. Computer Aided Geometric Design 13, 199–217.

Zhang, J.W., 1997. Two different forms of C-B-splines. Computer Aided Geometric Design 14, 31–41.

Zhang, J.W., 1999. C-Bézier curves and surfaces. Graphical Models and Image Processing 61, 2–15.