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Surface interpolation of meshes by geometric subdivision

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Abstract

Subdivision surfaces are generated by repeated approximation or interpolation from initial control meshes. In this paper, two new non-linear subdivision schemes, face based subdivision scheme and normal based subdivision scheme, are introduced for surface interpolation of triangular meshes. With a given coarse mesh more and more details will be added to the surface when the triangles have been split and refined. Because every intermediate mesh is a piecewise linear approximation to the final surface, the first type of subdivision scheme computes each new vertex as the solution to a least square fitting problem of selected old vertices and their neighboring triangles. Consequently, sharp features as well as smooth regions are generated automatically. For the second type of subdivision, the displacement for every new vertex is computed as a combination of normals at old vertices. By computing the vertex normals adaptively, the limit surface is G^1 smooth. The fairness of the interpolating surface can be improved further by using the neighboring faces. Because the new vertices by either of these two schemes depend on the local geometry, but not the vertex valences, the interpolating surface inherits the shape of the initial control mesh more fairly and naturally. Several examples are also presented to show the efficiency of the new algorithms.
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1. Introduction

Subdivision surface is an efficient geometric modeling tool that is generated by repeated approximation or interpolation from an initial control mesh [1–6,21]. For each time of subdivision, the old mesh will be split and the vertices of the mesh will be refined for a new mesh. With ease to implement and flexibility to represent a variety of complex geometric shapes, subdivision surfaces have been used widely in the fields of computer aided geometric design and computer graphics. Besides surface modeling, subdivision surfaces also find applications in digital surface processing [19].

In the past few decades, there is an abundance of literature dealing with the problem of mesh subdivision and smooth surface generation. According to whether the original meshes will be interpolated or not, subdivision surfaces can be classified into two categories, interpolatory subdivision surfaces and approximate subdivision surfaces. The first type of subdivision surfaces compute and add new vertices to the old set of vertices for each step of subdivision [5,12,13,16]. The second type of subdivision surfaces compute all vertices for a new mesh and replace the old mesh by a new mesh for each subdivision [2,4,14,15]. On the other hand, subdivision surfaces can also be grouped into stationary subdivision surfaces and non-stationary subdivision surfaces just according to the criterion that the subdivision rules will be the same or will be changed during the subdivision processes.

Many of current subdivision schemes are spline based and new vertices are computed as linear combinations of old vertices. The combination coefficients are often derived by generalizing some blending functions or interpolation functions in discrete form to vertices with general valences. Because the blending functions and interpolation functions are generally with limited supports, new vertices

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for a subdivision surface can then be computed locally. These schemes will be referred as linear subdivision schemes. Though linear subdivision schemes can be used to generate smooth surfaces, when some features are desired on the final subdivision surfaces, special rules or parameters should often be applied [11,18].

In this paper, we investigate a geometric based method for surface interpolation of meshes by subdivision. Instead of a simple linear combination of old vertices, every new interpolated vertex will be computed according to the local geometry of the mesh. For each step of subdivision we compute and add one new vertex corresponding to every edge of the mesh and replace each old triangle by four new sub-triangles. Depending on the formulae for new vertices, two new subdivision schemes, face based subdivision scheme and normal based subdivision scheme will be introduced.

As for the face based subdivision scheme, we observe that each intermediate mesh is a piecewise linear approximation to the final surface, and then we can compute new vertices based on the local geometry of the mesh. To compute a new vertex corresponding to an edge, the edge itself and the planes determined by the neighboring triangles of the edge end points will be used to predict the new vertex. Then, we compute the new vertex as the solution to the least square fitting problem of old vertices and old planes. By choosing the fitting coefficients properly, smooth regions as well as sharp features implied by the original control mesh can be generated automatically.

To obtain a global smooth surface, we propose another simple yet efficient new subdivision scheme, the normal based subdivision scheme, for surface interpolation. The displacement vector from the midpoint of an edge to the new vertex is given as a combination of normal vectors at the edge end points. It can be shown that when the normal vectors are refined adaptively for each time of subdivision the limit surface will be smooth. Moreover, the shape of the interpolating surface can be improved further by taking into account the influence of the neighboring triangles.

The geometric based subdivision method has several distinguished properties which make it attractive for surface modeling and geometry processing.

- The subdivision rules depend on the local geometry of the mesh. Then the interpolating surface inherits the initial shape of the control mesh fairly and naturally.
- For the face based subdivision method, smooth regions as well as sharp features can be generated automatically during subdivision.
- For the normal based subdivision scheme, not only the limiting surface is smooth but also the surface normal can be easily controlled.

The organization of the paper is as follows. In Section 2, we will introduce some related work. We will propose face based subdivision scheme and normal based subdivision scheme in Sections 3 and 4, respectively. The examples and comparisons with some existing methods are presented in Section 5. In Section 6, we will conclude the paper.

2. Related work

In addition to polynomial surfaces (see Ref. [8] and references therein), subdivision surfaces are another kind of powerful tools for surface interpolation [5,16,20]. A well known subdivision algorithm for surface interpolation of triangle meshes is the butterfly subdivision scheme proposed by Dyn et al. [5]. By this scheme, the limit surface will be smooth provided that the initial control mesh is regular, i.e. the valences of all interior vertices are six. For those extraordinary vertices with valences other than six, the limit surface will not be smooth at the vertices. To obtain an overall smooth interpolating surface, Zorin et al. [23] have presented a modified butterfly subdivision algorithm which can deal with irregular vertices well. Based on the analysis of the convergence of the subdivision surfaces [24], new special rules are defined for extraordinary vertices.

From another point of view, the butterfly subdivision scheme is parameterization dependent. If a bicubic parametric surface has been used to interpolate a local regular mesh, all with uniform parameterization of the vertices, one can then derive the subdivision scheme by picking a proper point from the bicubic surface. Because a butterfly subdivision surface is a discrete analogue to bicubic surface interpolation with uniform parameterization, then the final surface may have undesirable undulations when the triangle sizes of the original mesh are not regular or the mesh has ugly changing local shapes. For the same reason, modified butterfly subdivision surfaces also suffer from some unnecessary undulations.

Recently, Labsik and Greiner [16] proposed a $\sqrt{3}$-subdivision scheme for surface interpolation of triangular meshes. Every triangle will be split into nine sub-triangles with two times of subdivision. Similar to the butterfly subdivision scheme, the interpolatory $\sqrt{3}$-subdivision scheme has also been derived from bicubic surface fitting. An subdivision scheme was first designed to interpolate meshes with regular vertices and then special rules were presented to deal with irregular vertices for the generation of overall smooth surfaces. Same as butterfly subdivision scheme, this scheme also suffers from the problem of surface undulation caused by uniform parameterization.

Besides the stationary subdivision schemes, smooth and fair interpolatory surfaces can also be computed implicitly or by non-stationary subdivision schemes. Halstead et al. [9] proposed an algorithm for Catmull–Clark surface interpolation by solving a large linear system. Kobbelt and Schröder [15] have presented a variational subdivision scheme for surface interpolation under the objective of the fairness of final surfaces. Surfaces by these methods are often fairer than those by explicit subdivision schemes. However, the computation for these algorithms are costly.

In addition to smoothness and fairness, sharp feature generation and normal control at some points or some edges are also important topics for surface modelling. Hoppe et al. [11] have proposed an algorithm for piecewise smooth surface reconstruction based on the design of special subdivision rules along feature edges and feature corners. Sederberg et al. [18] have generalized the knot insertion technique of NURBS surfaces to meshes with arbitrary topology, and features can be generated along smooth surface generation with special selection of some parameters. As to the problem of normal control along surface edges, the subdivision rules should often be repaired locally to meet this special requirement [1]. Though these methods can be used to generate surfaces with sharp features or prescribed normals, the interpolation of a given mesh with automatic preservation of features is still an open problem.

Different from spline based subdivision schemes, the new subdivision schemes in this paper are geometric and non-linear. The key ingredient of the new schemes is that new vertices are computed relating to the local geometry of the mesh. For face based subdivision scheme, new vertices are computed as the solutions to weighted least square fitting of old vertices and neighboring planes. For normal based subdivision scheme, normal vectors for every intermediate mesh are computed adaptively and displacements for new vertices are computed as normal combinations. A related work in the literature, where a point is fitted to a set of planes, is the full range approximation and quadric error metric algorithm for mesh simplification [7,10,17]. Our method differs from that for mesh simplification greatly. We aim to construct fine meshes from coarse meshes and every new vertex is the solution to the least square fitting problem of planes as well as vertices.

3. Face based subdivision of meshes

In this section, we will discuss how to construct interpolating subdivision surfaces depending on the triangles and vertices of the meshes but not the vertex valences. The subdivision scheme derived from local geometry and the shape analysis of the interpolating surfaces will be presented in Sections 3.1 and 3.2, respectively.

3.1. Face based subdivision scheme

There are two basic steps for surface interpolation of meshes by subdivision, topology split and new vertex computation. Within this paper we present subdivision schemes with regular topology split, i.e. split every edge into two edges and split every triangle into four sub-triangles (see Fig. 1). In this section, we discuss how to compute new vertices using face based subdivision scheme.

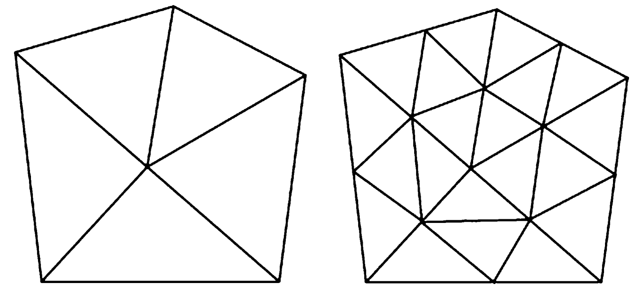

Fig. 1. Topology split of a triangular mesh.

Without loss of generality, we can assume that $p_1^k p_2^k$ is an edge on a mesh, π_0 and π_1 are two triangles sharing the edge. Before presenting the subdivision scheme, we first show a few special cases how to compute a new point corresponding to the edge. (a) If every neighboring triangle of vertex p_1^k or of vertex p_2^k is either coplanar with π_0 or coplanar with π_1, then the edge is a feature edge or a flat edge (with zero Gauss curvature). At this time, the interpolated vertex can be simply chosen as the midpoint of the edge. (b) If the edge and all its neighboring triangles form a local convex shape, i.e. there exists a plane passing through the edge and all neighboring triangles lie just on one side of the plane, then the new vertex should be picked on the other side of the plane and the mesh will be smoothed when the new vertex has been added.

From a geometric point of view, when the angles or the maximum angle between neighboring triangles have been reduced, the refined mesh is smoothed. However, if we compute new edge vertices just according to the goal that the angles between neighboring triangles are made smaller, this will arouse the problem of solving heavy non-linear equations. With the observation that all neighboring triangles of an edge approximate the local surface well, an alternative solution to choose the interpolated point is to compute a weighted least square fitting problem of the triangles and vertices.

For the convenience of the following description, we assume that the vertices adjacent to p_1^k or p_2^k are p_j^k ($j=3,4,\ldots,l$). The planes passing through the triangles π_0 or π_1 are still denoted as π_0 and π_1, the planes determined by the other neighboring triangles are denoted as π_j ($j=2, 3,\ldots, l-1$) (see Fig. 2). It is clear that the intersection of planes π_0 and π_1 is just the line passing through the edge $p_1^k p_2^k$, any point with minimum total distance to these two planes lies on this line. On the other hand, the interpolated edge point should be close to the midpoint of the edge too. If we find a point with minimum total distance to two planes π_0, π_1 as well as to two end vertices p_1^k, p_2^k, we will obtain the midpoint of the edge.

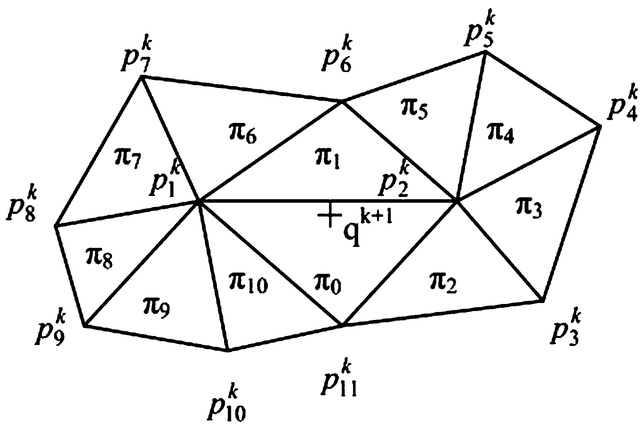

Fig. 2. Compute a new vertex for interpolation.

Let $p_{i_1}^k$, $p_{i_2}^k$ and $p_{i_3}^k$ be the three vertices of a triangle, the unit normal vector to the triangle can be computed as

$$n_i = \frac{(p_{i_2}^k - p_{i_1}^k) \times (p_{i_3}^k - p_{i_1}^k)}{||(p_{i_2}^k - p_{i_1}^k) \times (p_{i_3}^k - p_{i_1}^k)||}. \tag{1}$$

Now the plane passing these three points can be denoted as $A_i p = 0$, where $A_i = (a_i\ b_i\ c_i\ d_i)$ stands for the coefficient and $p = (x_p\ y_p\ z_p\ 1)^{\mathrm{T}}$ is any point on the plane. The coefficients a_i, b_i and c_i are just the three components of the unit normal n_i and $d_i = -n_i p_{i_1}$. Moreover, if p is an arbitrary point in space, the distance from the point to the plane can be obtained as $|A_i p|$.

Assume that the equations for planes π_i $(i=0, 1,\ldots, l-1)$ are $A_i p = 0$ $(i=0, 1,\ldots, l-1)$, respectively, and assume that $q^{k+1} = (x\,y\,z\,1)^{\mathrm{T}}$, then the absolute distance from q^{k+1} to each plane π_i is $|A_i q^{k+1}|$. Similarly, the points p_1^k and p_2^k can also be represented in homogenous space as $p_1^k = (x_1\ y_1\ z_1\ 1)^{\mathrm{T}}$ and $p_2^k = (x_2\ y_2\ z_2\ 1)^{\mathrm{T}}$. The distance from q^{k+1} to p_1^k and the distance from q^{k+1} to p_2^k are $||q^{k+1} - p_1^k||$ and $||q^{k+1} - p_2^k||$, respectively. Now, we can compute the point q^{k+1} according to the criterion that the sum of the weighted distances to all neighboring planes and to two end vertices reaches a minimum. For ease of computation we use the distance square instead of the absolute distance itself. Then we have

$$F(q^{k+1}) = \sum_{i=0}^{l-1} \alpha_i (A_i q^{k+1})^2 + \sum_{j=1}^{2} \beta_j (q^{k+1} - p_j^k)^2, \tag{2}$$

where α_i $(i=0, 1,\ldots, l-1)$, β_1 and β_2 are the weights.

Eq. (2) is a quadric equation with respect to the unknown q^{k+1}, and this equation can be rewritten as

$$F(q^{k+1}) = (q^{k+1})^{\mathrm{T}} \left(\sum_{i=0}^{l-1} \alpha_i A_i^{\mathrm{T}} A_i \right) q^{k+1} + (q^{k+1})^{\mathrm{T}} \times (\beta_1 Q_1 + \beta_2 Q_2) q^{k+1 = (q^{k+1})^{\mathrm{T}} Q q^{k+1}} \tag{3}$$

where

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ -x_1 & -y_1 & -z_1 & x_1^2 + y_1^2 + z_1^2 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 1 & 0 & 0 & -x_2 \\ 0 & 1 & 0 & -y_2 \\ 0 & 0 & 1 & -z_2 \\ -x_2 & -y_2 & -z_2 & x_2^2 + y_2^2 + z_2^2 \end{pmatrix}$$

and $Q = \sum_{i=0}^{l-1} \alpha_i A_i^{\mathrm{T}} A_i + \beta_1 Q_1 + \beta_2 Q_2$. Because the matrixes Q_1, Q_2 and $A_i^{\mathrm{T}} A_i$ are all symmetric matrixes, so is the matrix Q. Let the matrix Q be represented with all its elements, we have

$$Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{12} & q_{22} & q_{23} & q_{24} \\ q_{13} & q_{23} & q_{33} & q_{34} \\ q_{14} & q_{24} & q_{34} & q_{44} \end{pmatrix}.$$

To solve the equation $(q^{k+1})^{\mathrm{T}} Q q^{k+1} = \min$ is equivalent to solving the system of equations $\partial F(q^{k+1})/\partial x = 0$, $\partial F(q^{k+1})/\partial y = 0$ and $\partial F(q^{k+1})/\partial z = 0$. This will lead to the following system of linear equations

$$\begin{pmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{12} & q_{22} & q_{23} & q_{24} \\ q_{13} & q_{23} & q_{33} & q_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4}$$

From Eq. (4) we have the solution as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{pmatrix}^{-1} \begin{pmatrix} -q_{14} \\ -q_{24} \\ -q_{34} \end{pmatrix}. \tag{5}$$

No doubt, different choices of the weights in Eq. (2) will lead to different results. A large weight means much more influence of the corresponding triangle or vertex to the new point. We present here a simple and practical choice of the coefficients. With efficient description of local shapes of a mesh, we choose coefficients based on areas of triangles and angles between pairs of neighboring triangles. The bigger a triangle, the more influence it will have on the local shape of the interpolating surface. As to the angle, an ideal choice is the angle between a neighboring triangle and the tangent plane at the new vertex. A triangle with smaller angle can always be used to predict the new vertex more accurately. For practical computation, we estimate this angle from dihedral angles between neighboring triangles.

Suppose that the unit normal corresponding to triangle π_i is n_i $(i=0, 1,\ldots, l-1)$, then we can first compute the angles θ_{i_0}, θ_{i_1} between π_i with π_0 or π_1, i.e. $\theta_{i_0} = \cos^{-1}(n_i n_0)$

and $\theta_{i_1} = \cos^{-1}(n_i n_1)$. For $i=0$, 1, we compute the angles as $\theta_i = 0.25\theta_{i_0} + 0.25\theta_{i_1} + 0.1$. For $1<i<l$, the angle is picked as $\theta_i = 0.75\min(\theta_{i_0}, \theta_{i_1}) + 0.25\max(\theta_{i_0}, \theta_{i_1}) + 0.1$. The item 0.1 is used to keep the angle from being zero. If θ_i is small, it implies that the corresponding triangle can predict the new vertex better than other triangles with larger angles.

Let a_i be the area of the triangle π_i, a simple rule for the choice of the coefficient α_i is that it should be proportional to a_i and anti-proportional to θ_i. Thus we have

$$\alpha_i = \frac{a_i/\theta}{\sum_{j=0}^{l-1} a_j/\theta_j}$$

($i=0$, 1,..., $l-1$). As for the choice of β_1 and β_2, they can always be set an equal number so that the solution to Eq. (2) is just the midpoint of the edge $p_1^k p_2^k$ when all the neighboring triangles are coplanar. One method to choose β_1 or β_2 is based on an initial estimation of the solution and recompute β from this estimation. We will explain this method in detail in Section 4. Another method is to use a fixed number for each time of subdivision and a default choice for these two weights is 0.2.

3.2. Property analysis

In this section, we give a brief property analysis to face based subdivision surfaces. Sufficient conditions for feature preservation will be given and the condition for mesh smoothing will also be discussed.

Let $f_i(X) = X^T A_i X + 2B_i X + c_i (i=0,1,\dots,m)$ be a set of quadric functions of X, where the variable $X=(x, y, z)^T$, and $A_i = A_i^T$, B_i are 3×3 and 1×3 coefficient matrixes, respectively. Let $f(X) = \sum_{i=0}^m s_i f_i(X)$, we have the following theorem.

Theorem 1. *Let X_i be the solution to equation $f_i(X)=\min$ ($i=0$, 1,..., m), $\bar{X}$ be the solution to $f(X)=\min$, then $\bar{X}$ is an affine combination of all X_is.*

Proof. It can be derived that the solution to equation $f_i(X)=\min$ is $X_i = -A_i^{-1}B_i$, the solution of $f(X)=\min$ is $\bar{X} = -A^{-1}(\sum_{i=0}^m s_i B_i)$, where $A = \sum_{i=0}^m s_i A_i$. Thus, we have $\bar{X} = \sum_{i=0}^m s_i(A^{-1}A_i)X_i$. This proves the theorem. □

From Theorem 1 we can see the solution $\bar{X}$ will approach X_i when the coefficient s_i has been increased. As an application, Eq. (3) can be rewritten as follows

$$F(q^{k+1}) = f_F(q^{k+1}) + f_E(q^{k+1}),$$

where $f_F(q^{k+1}) = \sum_{i=0}^{l-1} \alpha_i(A_i q^{k+1})^2$ and $f_E(q^{k+1}) = \sum_{j=1}^2 \beta_j (q^{k+1} - p_j^k)^2$. From a geometric point of view, the solution to $f_F(q^{k+1})=\min$ is the generalized intersection of all neighboring planes and the solution to $f_E(q^{k+1})=\min$ is the midpoint of the edge $p_1^k p_2^k$, then the final solution to $F(q^{k+1})=\min$ is their affine combination.

Theorem 2. *Let $A_i p=0$ ($i=0$, 1,...,m) be a set of planes, and α_i ($i=0$, 1,...,m), β be a set of positive numbers, if each of these planes passes through points p_1^k and p_2^k, then the solution to $\sum_{i=0}^m \alpha_i(A_i p)^2 + \beta[(p-p_1^k)^2 + (p-p_2^k)^2] = \min$ is $\frac{1}{2}(p_1^k + p_2^k)$.*

Proof. When p_1^k and p_2^k are both lying on a plane $A_i p=0$, we have $A_i p_1^k = 0$ and $A_i p_2^k = 0$. Moreover, the line $p_1^k p_2^k$ lies on the plane too. It is clear that the solution to $(p-p_1^k)^2 + (p-p_2^k)^2 = \min$ is $p_m = \frac{1}{2}(p_1^k + p_2^k)$, then $A_i p_m = 0$ ($i=0$, 1,..., m). Substituting the point p_m, we have

$$\sum_{i=0}^m \alpha_i(A_i p_m)^2 + \beta[(p_m - p_1^k)^2 + (p_m - p_2^k)^2]$$
$$= \beta[(p_m - p_1^k)^2 + (p_m - p_2^k)^2].$$

This means that both objective functions on either side of the equality have reached the same minimum at point p_m. □

From Theorems 1 and 2, we can see that flat regions as well as sharp features can be preserved during the surface interpolation processes. Firstly, when an edge $p_1^k p_2^k$ lies on a local flat region, the interpolated point is just the midpoint of the edge (see Fig. 3(a)). In this way, flat regions cannot only be preserved, but also the subdivision is uniform. Secondly, if the edge $p_1^k p_2^k$ is a sharp edge, then all neighboring triangles can be grouped into two planes and these two planes intersect along the edge $p_1^k p_2^k$ (see Fig. 3(b)). From Theorem 2, the interpolated vertex is the midpoint of the edge too. So the sharp edges can be preserved well during surface subdivision.

Besides sharp edges, sharp corners are also important surface features. Without loss of generality, we can assume that p_1^k is a feature corner that is the intersection of three

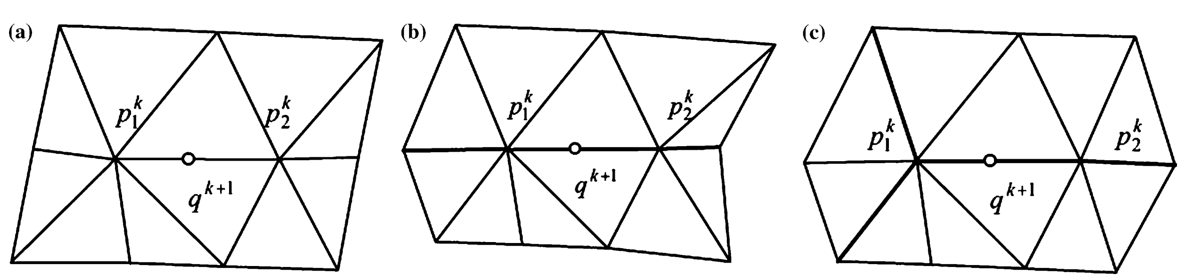

Fig. 3. Feature preservation during surface interpolation.

planes. In this case, all neighboring triangles passing the point p_1^k or p_2^k lie on these three planes where two planes intersect along the edge $p_1^k p_2^k$ (see Fig. 3(c)). As discussed at the end of Section 3.1, the coefficients for the triangles on the planes passing the edge $p_1^k p_2^k$ are always larger than the coefficients for other triangles passing the corner. Then the solution to equation $F(q^{k+1}) = \min$ is very close to the edge $p_1^k p_2^k$ and the sharp corner p_1^k will be preserved well during surface interpolation.

Before discussing how a mesh can be smoothed during subdivision, we present first the definition of the angle between a triangle and an edge and the definition of ridge edges. In the following text we denote the plane passing through a triangle π_i as π_i too.

Definition 1. Assume that a triangle π_i and an edge $p_1^k p_2^k$ share a common vertex such as p_1^k, the acute angle between the plane π_i and the line $p_1^k p_2^k$ is ϕ (see Fig. 4a). Let l_c be a line on the plane π_i and perpendicular to $p_1^k p_2^k$, then the angle between the edge $p_1^k p_2^k$ and the triangle π_i can be defined as ϕ if the triangle and the projection of p_2^k on the plane lie on different sides of l_c; otherwise, the angle will be defined as $\pi - \phi$. If $p_1^k p_2^k$ is an edge of the triangle π_i, the angle between the edge and the triangle is defined zero.

Definition 2. Assume that $p_1^k p_2^k$ be an arbitrary edge on a local oriented mesh, triangles π_i ($i=0, 1, \ldots, l-1$) are neighboring to vertex p_1^k or p_2^k, we define the edge $p_1^k p_2^k$ a ridge edge if there exists an oriented plane π_e passing through the edge so that all the triangles lie on one side of the plane and the angles between pairs of oriented planes π_i and π_e ($i=0, 1, \ldots, l-1$) are all acute angles (see Fig. 4b).

From Definition 2, if $p_1^k p_2^k$ is a ridge edge and all the neighboring triangles lie below an oriented plane π_e as in the definition, we can see that every closed volume bounded by at least four of the neighboring planes lies above the plane π_e. The closed volume bounded by all the planes π_i ($i=0, 1, \ldots, l-1$) is the maximum volume and we refer this volume as the bounding set formed by the neighboring planes. Moreover, the bounding set is a convex set above the plane π_e and the edge $p_1^k p_2^k$ is an edge of the bounding set. As the location of the new vertex corresponding to a ridge edge, we have the following theorem.

Theorem 3. *If $p_1^k p_2^k$ is a ridge edge, q^{k+1} be the solution to the quadric quadric equation (2), then there exists a plane passing through the edge so that all the neighboring triangles of $p_1^k p_2^k$ lie on one side of the plane and the point q^{k+1} lies on the other side of the plane.*

Proof. As in Definition 2, if $p_1^k p_2^k$ is a ridge edge, there exists a plane π_e passing through the edge. All the neighboring triangles π_i ($i=0, 1, \ldots, l-1$) lie at one side (below) the plane π_e and the bounding set lies above the plane. We will show that q^{k+1} lies at the same side of the plane π_e with the bounding set. We prove this assertion by reduction to absurdity.

Besides the bounding set, every three planes partition the space into a set of trihedrons. Let q be an arbitrary point below the plane π_e, we can then project q onto the planes π_i ($i=0, 1, \ldots, l-1$), with q_i ($i=0, 1, \ldots, l-1$) as the projections. By connecting q to q_i, we obtain a set of line segments as qq_i ($i=0, 1, \ldots, l-1$). We can show that all these line segments lie on one side of a plane π_q through the point q. To choose such a plane π_q, we choose first a trihedron with minimum trihedral angle among all the trihedrons in which the point q lies. Then, the three connection lines from q to its projection on three planes of the chosen trihedron form another trihedron. This new trihedron is a dual trihedron of the original. Because the original chosen trihedron has minimum trihedral angle, the connection line from q to its projection on any other plane lies within the dual trihedron. With the dual trihedron, we can find a plane π_q through the point q and all the line segments qq_is lie at one side of π_q.

When we want to find a plane π_q through the point q so that all line segments qq_is as well as qp_1^k and qp_2^k lie at one side of π_q, we can just add two more planes through p_1^k or p_2^k with qp_1^k and qp_2^k as the normals, respectively. In the same principle as above we will find a plane π_q through the point q, and qq_i ($i=0, 1, \ldots, l-1$), qp_1^k and qp_2^k lie at one side of π_q. If we move the point q a little along the normal of π_q toward the edge $p_1^k p_2^k$, all these line segments will be shortened. This means that q is not the solution to the quadric quadric equation (2). The theorem is proven. □

From Theorems 1 and 3, we can see that q^{k+1} will approach the midpoint of the edge $p_1^k p_2^k$ by increasing the coefficient β ($\beta_1 = \beta_2 = \beta$). Then, with proper choice of the coefficient, the maximum angle from the edge $p_1^k q^{k+1}$ to the triangles neighboring to p_1^k will be less than the maximum angle from the edge $p_1^k p_2^k$ to the same set of triangles. Similarly, the maximum angle from the edge $p_2^k p_1^k$ to the triangles neighboring to p_2^k will be reduced when

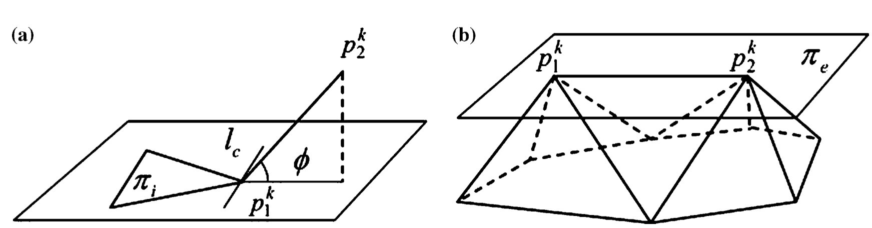

Fig. 4. (a) The angle between a triangle and an edge. (b) A ridge edge and its neighboring triangles.

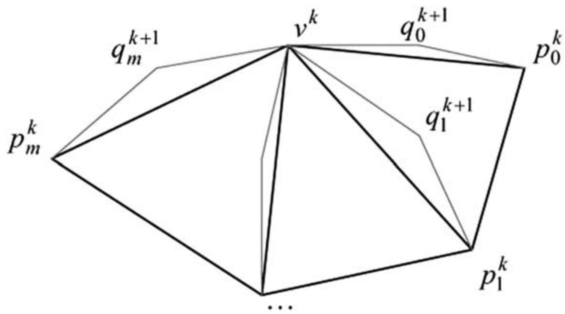

Fig. 5. Vertex smoothing during subdivision.

the edge $p_2^k p_1^k$ has been replaced by $p_2^k q^{k+1}$. Though a concrete choice of the parameters for mesh smoothing is beyond the scope of this paper, we present a theorem for local mesh smoothing under proper choices of the combination parameters.

Theorem 4. *If every edge abutting a mesh vertex is a ridge edge, then the mesh will be smoothed at the vertex by face based subdivision.*

Proof. As shown in Fig. 5, v^k is a fixed vertex on a mesh and $v^k p_i^k$ ($i=0, 1,\ldots, m$) are the set of edges connecting the vertex v^k and its neighboring vertices. For a ridge edge $v^k p_i^k$, there exists a plane along the edge and the interpolated vertex q_i^{k+1} lies at the other side of the plane with respect to the local mesh. The maximum angle between the new edge $v^k q_i^{k+1}$ and the neighboring triangles of v^k will be reduced with proper choice of coefficients for the quadric equation. When the old triangles have been replaced by the new sub-triangles, the maximum angle from the new edge $v^k q_i^{k+1}$ to the new set of neighboring triangles of v^k is less than the maximum angle from the old edge to the original neighboring triangles. This just means that the local mesh at vertex v^k has been smoothed after subdivision. □

For other types of edges within a mesh, some of them may become ridge edges after a few steps of subdivision and the mesh will be smoothed during subdivision. For general types of edges, because we compute the interpolated vertex in a least square fitting sense of local triangles and vertices, then new added vertices are always close to the old edges and the shape of the original control mesh can be kept well.

4. Surface interpolation under normal constraint

In this section, we will present a new algorithm for smooth surface interpolation under normal or tangent plane constraint. At first we introduce a normal based subdivision scheme for smooth surface interpolation. Then an improved geometric subdivision algorithm is presented with the combination of normal based subdivision and shape optimization.

4.1. Normal based subdivision scheme

The basic idea for normal based subdivision is that the displacement vector for every new vertex is computed as a combination of normal vectors at old vertices, and the vertex normals are computed adaptively for each time of subdivision. We will show that the limit surface is G^1 smooth with proper choice of a free parameter.

It is clear that the computation of the normal at a vertex is equivalent to the determination of the tangent plane at the vertex. Moreover, the chord tangent angles between edges and tangent planes at the edge end points can be used to measure the smoothness of meshes. Since there are many different ways for normal selection of meshes, the rule for normal based subdivision is that all chord tangent angles around a fixed vertex are close to each other. As discussed in the following text, this rule can help to achieve the smoothness of the limit surfaces.

In this paper, we compute normal vector to every vertex as a weighted average of normals of its neighboring triangles. Though the chord tangent angles are not exactly equal, but the computation is easy and the results are satisfying. Suppose that π_j ($j=0, 1,\ldots, m_i-1$) are the triangles sharing the vertex p_i^k, for each triangle π_j the angle at the vertex p_i^k is ϕ_j and the normal of the triangle is n_j, then the normal at the vertex p_i^k can be estimated as

$$n_i^k = \frac{\sum_{j=0}^{m_i-1} \phi_j n_j}{\| \sum_{j=0}^{m_i-1} \phi_j n_j \|}. \tag{6}$$

With normal vector at p_i^k defined, the face tangent angle between the triangle π_j and the tangent plane at p_i^k can be computed as the angle between the normals n_i^k and n_j. Moreover, all edges on a mesh can then be classified into two categories according to the local shapes defined by the normals and the edges.

Definition 3. Let $p_i^k p_j^k$ be an arbitrary edge on a mesh, the normal vectors at vertices p_i^k and p_j^k are n_i^k and n_j^k, respectively. Let $d_i^k = n_i^k(p_i^k - p_j^k)/2$ and $d_j^k = n_j^k(p_j^k - p_i^k)/2$, the edge $p_i^k p_j^k$ is defined as a convex edge when $d_i^k d_j^k > 0$ or defined as an inflection edge otherwise.

Let v^k be a vertex on the mesh after kth subdivision, p_i^k ($i=0, 1,\ldots, l-1$) are its first round neighboring vertices, let the unit normals at v^k and p_i^k ($i=0, 1,\ldots, l-1$) are n_v^k and n_i^k ($i=0, 1,\ldots, l-1$), respectively, then the new point q_i^{k+1} corresponding to edge $v^k p_i^k$ is computed as

$$q_i^{k+1} = \frac{1}{2}(v^k + p_i^k) + w(d_v^k n_v^k + d_i^k n_i^k), \tag{7}$$

where $d_v^k = \frac{1}{2}(v^k - p_i^k)n_v^k$ and $d_i^k = \frac{1}{2}(p_i^k - v^k)n_i^k$. The parameter w here is a positive free parameter which will be used to control the convergence and smoothness of the subdivision surface.

Let φ_i^k be the unsigned angle between the chord $v^k p_i^k$ and the tangent plane at v^k and γ_i^k be the unsigned angle between

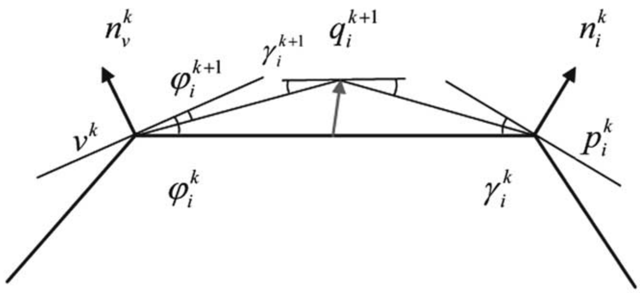

Fig. 6. Compute the interpolated vertex by adding a displacement vector.

the chord $p_i^k v^k$ and the tangent plane at p_i^k (see Fig. 6), we have $\varphi_i^k < \pi/2$ and $\gamma_i^k < \pi/2$. Because the vectors n_v^k, n_i^k and $v^k p_i^k$ are approximately coplanar with enough times of subdivision, we can always treat convex or inflection edges as on planes. If the edge $v^k p_i^k$ is a convex edge, the displacement vector $d_v^k n_v^k + d_i^k n_i^k$ points toward either of the tangent planes at the edge end points. If the edge $v^k p_i^k$ is an inflection edge and $\varphi_i^k + \gamma_i^k < \pi/2$, the vector $d_v^k n_v^k + d_i^k n_i^k$ will point toward the tangent plane at v^k or the tangent plane at p_i^k with larger chord tangent angle. For other types of inflection edges, the displacement vector points toward the tangent plane with smaller chord tangent angle. This is similar to normal based subdivision for curve design [22]. If $\varphi_i^k + \gamma_i^k > \pi/2$ the displacement vector lies on the side with smaller chord tangent angle, the interpolating surface may be local self-intersected near some sharp corners. This can be remedied just by inversing the displacement vector or choosing a fixed point on the inflection edge.

Without taking account of the sign, the absolute value of d_v^k or d_i^k can be reformulated as

$$|d_v^k| = \left|\frac{1}{2}(v^k - p_i^k)\right| \sin \varphi_i^k, \tag{8}$$

$$|d_i^k| = \left|\frac{1}{2}(p_i^k - v^k)\right| \sin \gamma_i^k. \tag{9}$$

From Eqs. (8) and (9) we have

$$\varphi_i^k \approx \sin \varphi_i^k = \frac{|d_v^k|}{\left|\frac{1}{2}(v^k - p_i^k)\right|},$$

$$\gamma_i^k \approx \sin \gamma_i^k = \frac{|d_i^k|}{\left|\frac{1}{2}(p_i^k - v^k)\right|}.$$

To illustrate the convergence and smoothness of the limit surface, we should analyze the smoothness at every initial and intermediate vertex. For this analysis, we have the following theorem.

Theorem 5. *Let v^k be a vertex on the mesh after kth refinement, p_i^k $(i=0, 1,\ldots,l-1)$ are its first round neighboring vertices. If all face tangent angles around v^k are less than $\pi/2$ and at least one abutting edge is a convex edge, then there exists a proper choice of w and the limit surface is convergent and smooth at the vertex v^k.*

Proof. To prove the convergence and smoothness of the limit surface at the vertex v^k, we should only prove that the maximum chord tangent angle $\max_{0\le i\le l-1}\{\varphi_i^k\}$ will approach zero when k goes to infinity.

At first, we estimate the new chord tangent angle at q_i^{k+1}. Because we compute the normal vector or the tangent plane at q_i^{k+1} with all the chord tangent angles at the vertex as close to each other, we can then assume that the tangent plane at q_i^{k+1} is approximately paralleling to the edge $v^k q_i^{k+1}$ (see Fig. 6). Let γ_i^{k+1} be the angle between the chord $v^k q_i^{k+1}$ and the tangent plane at q_i^{k+1}, then we have

$$\gamma_i^{k+1} \approx \frac{\left|q_i^{k+1} - \frac{1}{2}(v^k + p_i^k)\right|}{\left|\frac{1}{2}(p_i^k - v^k)\right|} \le \frac{w|d_v^k|}{\left|\frac{1}{2}(p_i^k - v^k)\right|} + \frac{w|d_i^k|}{\left|\frac{1}{2}(p_i^k - v^k)\right|}$$
$$= w \sin \varphi_i^k + w \sin \gamma_i^k \le 2w \max\{\varphi_i^k, \gamma_i^k\}.$$

To estimate the angle φ_i^{k+1}, we just compute the angle according to the following two cases.

(i) $(d_v^k n_v^k, d_i^k n_i^k) > 0$

In this case, the angle between $d_v^k n_v^k$ and $d_i^k n_i^k$ is an acute angle. From the principle of triangle, we have

$$\frac{\left|q_i^{k+1} - \frac{1}{2}(v^k + p_i^k)\right|}{\left|\frac{1}{2}(p_i^k - v^k)\right|} = \frac{w|d_v^k n_v^k + d_i^k n_i^k|}{\left|\frac{1}{2}(p_i^k - v^k)\right|}$$
$$\ge \max\left\{\frac{w|d_v^k|}{\left|\frac{1}{2}(p_i^k - v^k)\right|}, \frac{w|d_i^k|}{\left|\frac{1}{2}(p_i^k - v^k)\right|}\right\} \approx w \max\{\varphi_i^k, \gamma_i^k\}.$$

So, we have

$$\varphi_i^{k+1} \le \varphi_i^k - w \max\{\varphi_i^k, \gamma_i^k\} \le (1-w)\max\{\varphi_i^k, \gamma_i^k\}.$$

(ii) $(d_v^k n_v^k, d_i^k n_i^k) < 0$

In this case, the angle between $d_v^k n_v^k$ and $d_i^k n_i^k$ is an obtuse angle. From the principle of triangle again, we have

$$\frac{\left|q_i^{k+1} - \frac{1}{2}(v^k + p_i^k)\right|}{\left|\frac{1}{2}(p_i^k - v^k)\right|}$$
$$= \frac{w|d_v^k n_v^k + d_i^k n_i^k|}{\left|\frac{1}{2}(p_i^k - v^k)\right|} \ge \frac{w||d_v^k| - |d_i^k||}{\left|\frac{1}{2}(p_i^k - v^k)\right|} \approx w|\varphi_i^k - \gamma_i^k|.$$

From the definition of displacement vector for inflection edge, the vector lies on the same side with larger chord tangent angle. Then, we have $\varphi_i^{k+1} \le \varphi_i^k - w|\varphi_i^k - \gamma_i^k|$ when $\varphi_i^k > \gamma_i^k$, and we have $\varphi_i^{k+1} \le \varphi_i^k + w|\varphi_i^k - \gamma_i^k|$ otherwise. From these two inequalities, we have

$$\varphi_i^{k+1} \le (1-w)\varphi_i^k + w\gamma_i^k.$$

By summarizing case (i) and case (ii), we have

$$\varphi_i^{k+1} \le (1-w)\max\{\varphi_i^k, \gamma_i^k\} + w s_i^k \gamma_i^k,$$

where $s_i^k = 0$ when $(d_v^k n_v^k, d_i^k n_i^k) > 0$ and $s_i^k = 1$ when $(d_v^k n_v^k, d_i^k n_i^k) < 0$. Let $r = \sum_{i=0}^{l-1} s_i^k$, and by summing all

φ_i^{k+1}, we have

$$\sum_{i=0}^{l-1} \varphi_i^{k+1} \le (1-w)\sum_{i=0}^{l-1} \max\{\varphi_i^k, \gamma_i^k\} + w\sum_{i=0}^{l-1} s_i^k \gamma_i^k \le [l(1-w)+rw] \max_{0\le i\le l-1}\{\varphi_i^k, \gamma_i^k\}.$$

Dividing by l on either side of the inequality, we have

$$\frac{1}{l}\sum_{i=0}^{l-1} \varphi_i^{k+1} \le \left(1-w+\frac{r}{l}w\right) \max_{0\le i\le l-1}\{\varphi_i^k, \gamma_i^k\}.$$

The abutting edges to a fixed vertex are refined after each subdivision, and some inflection edges will be fixed after a few times of subdivision, then we can assume that $0\le r<l$. If $w>0$, we have $1-w+(r/l)w<1$. On the other hand, when we choose $w<0.5$, we have $\gamma_i^{k+1} < 2w \max\{\varphi_i^k, \gamma_i^k\} < \max\{\varphi_i^k, \gamma_i^k\}$. When the normal vector at vertex v^k has been refined for new subdivision, all the chord tangent angles at the vertex will be close to their average value. Denoting the new chord tangent angles at v^k still as φ_i^{k+1} ($i=0, 1,\dots, l-1$), then we have

$$\max_{0\le i\le l-1}\{\varphi_i^{k+1}, \gamma_i^{k+1}\} \le c \max_{0\le i\le l-1}\{\varphi_i^k, \gamma_i^k\},$$

where $c=\max\{2w, 1-w+(r/l)w\}$. For $0<w<0.5$, we have $c<1$ and $\lim_{k\to\infty} \max_i\{\varphi_i^k\}=0$. This just implies that the limit position of tangent planes at the vertex v^k exists and all abutting chords converge to the limit plane. The theorem is proven. □

From the proof of the theorem we can see that if the normals at some vertices of an initial control mesh are fixed during subdivision then the subdivision surface converges and interpolates the fixed normals in the end. For practical construction of subdivision surfaces, we can always choose w between 0.2 and 0.4 for fast convergence and smoothness purposes. Moreover, we have the following corollary.

Corollary. *If there exists a tangent plane to every vertex of an initial mesh and all face tangent angles are less than π/2, the interpolating surface by normal based subdivision is G^1 smooth.*

4.2. Surface interpolation under tangent plane constraint

The normal based subdivision scheme presented in Section 4.1 can be used to construct smooth interpolating surfaces. However, since any surface depends on the local normal vectors, then the surface fairness may be sensitive to normal noises. To improve the fairness of an interpolating surface further, we compute every edge vertex under the constraint of tangent planes at the edge end points as well as the neighboring triangles. The edge vertex computed by normal based subdivision is used as an initial estimation of the new interpolated vertex. With this estimation, the parameter of the objective function for the new vertex will be computed explicitly.

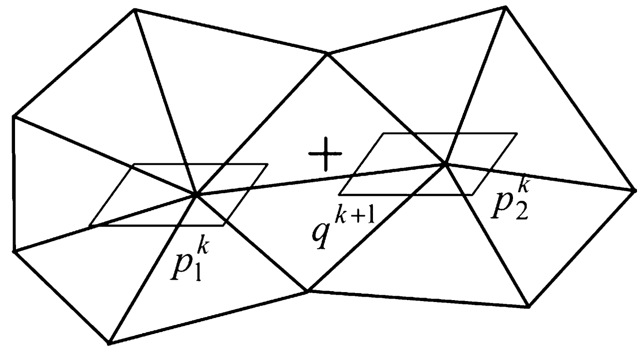

Fig. 7. Interpolation under normal constraint.

As shown in Fig. 7, when we want to compute an edge vertex along the edge $p_1^k p_2^k$, the tangent planes at vertices p_1^k and p_2^k will be used to predict the point position together with the planes passing through the neighboring triangles. In a similar way as face based subdivision, we compute the new edge vertex by minimizing an objective function. To keep the interpolated point from deviating the midpoint of the edge too much, another constraint is that the distance from the new vertex to either end of the edge should be as small as possible. Suppose that the neighboring planes passing the points p_1^k or p_2^k are $A_i p=0$ ($i=0, 1,\dots, l-1$), the two tangent planes are $A_i p=0$ ($i=l, 1,\dots, l+1$), then the object function which defines the edge vertex can be formulated as

$$F(q^{k+1}) = \sum_{i=0}^{l+1} \alpha_i (A_i q^{k+1})^2 + \beta \sum_{i=1}^{2} (q^{k+1} - p_i^k)^2, \tag{10}$$

where α_is and β are positive coefficients.

In a similar way to Section 3.1, we pick values for α_is based on the angles between pairs of planes and areas of neighboring triangles. Let θ_{i_0} and θ_{i_1} be the angles between the triangle π_i with the tangent planes at vertex p_1^k and p_2^k, respectively, then the deviation angle of the triangle π_i can be chosen as $\theta_i = 0.85\theta_{i_0} + 0.15\theta_{i_1} + 0.1$ or $\theta_i = 0.15\theta_{i_0} + 0.85\theta_{i_1} + 0.1$ depending on whether π_i is neighboring to p_1^k or neighboring to p_2^k. If π_i passes through the edge $p_1^k p_2^k$, the deviation angle is chosen as $\theta_i = 0.5\theta_{i_0} + 0.5\theta_{i_1} + 0.1$. The deviation angles for tangent planes at p_1^k and p_2^k are both chosen a constant such as 0.05. Let $a_l = a_{l+1}$ be the average area of all neighboring triangles, the coefficient α_i can be chosen proportional to a_i/θ_i and under the constraint $\sum_{i=0}^{l+1} \alpha_i = 1$.

To determine the parameter β, we choose an initial interpolating point by normal based subdivision scheme

$$\bar{q}^{k+1} = \frac{1}{2}(p_1^k + p_2^k) + wd_1 n_1^k + wd_2 n_2^k,$$

where $d_1 = \frac{1}{2}(p_1^k - p_2^k)n_1^k$ and $d_2 = \frac{1}{2}(p_2^k - p_1^k)n_2^k$. We choose here $w=0.25$ for initial vertex computation. Let $\bar{q}^{k+1} = (\bar{x}^{k+1}, \bar{y}^{k+1}, \bar{z}^{k+1})$, substituting the coordinates of

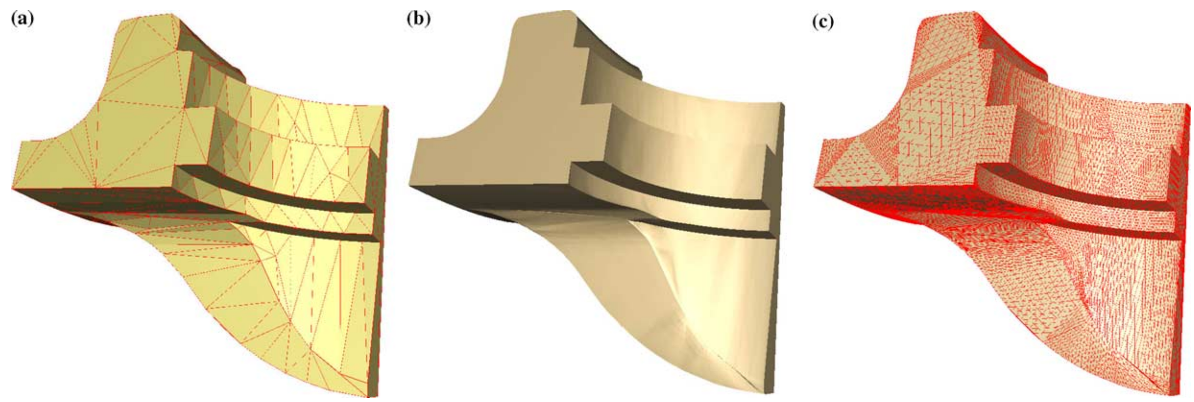

Fig. 8. (a) Initial control mesh; (b) face based interpolating surface; and (c) the mesh of the interpolating surface.

$\bar{q}^{k+1}$ into Eq. (4), we have

$$\begin{pmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{12} & q_{22} & q_{23} & q_{24} \\ q_{13} & q_{23} & q_{33} & q_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{x}^{k+1} \\ \bar{y}^{k+1} \\ \bar{z}^{k+1} \\ 1 \end{pmatrix} = \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ 1 \end{pmatrix}. \quad (11)$$

By expressing all elements q_{ij} $(1 \le i \le 3, i \le j \le 4)$ in terms of the original planes and vertices, we have

$$\varepsilon_x = \sum_{i=0}^{l+1} \alpha_i a_i (A_i \bar{q}^{k+1}) - \beta(x_1^k + x_2^k - 2\bar{x}^{k+1}),$$

$$\varepsilon_y = \sum_{i=0}^{l+1} \alpha_i b_i (A_i \bar{q}^{k+1}) - \beta(y_1^k + y_2^k - 2\bar{y}^{k+1}),$$

$$\varepsilon_z = \sum_{i=0}^{l+1} \alpha_i c_i (A_i \bar{q}^{k+1}) - \beta(z_1^k + z_2^k - 2\bar{z}^{k+1}).$$

Then we can choose a value for the parameter β according to the following equation

$$\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2 = \min. \quad (12)$$

With simple calculation, the solution to Eq. (12) is $\beta = AB/B^2$, where $A = \sum_{i=0}^{l+1} \alpha_i \times A_i \bar{q}^{k+1} \times n_i$ and $B = p_1^k + p_2^k - 2\bar{q}^{k+1}$. It can be easily verified that if A is parallel to B the solution to Eq. (10) will just be $\bar{q}^{k+1}$. To avoid being a negative number or to keep the interpolating surface as natural as possible, we choose $\beta = 0.4$ when $AB/B^2 < 0.4$ or choose $\beta = 2.0$ when $AB/B^2 > 2.0$.

When the parameter β has been selected, Eq. (10) is just a quadric equation with respect to the unknown q^{k+1}. By using the same method as in Section 3.1, the equation $F(q^{k+1}) = \min$ can be solved explicitly.

5. Examples and comparison

We have tested the new algorithms with a lot of different types of geometric models and we sample a few here to show the results. The comparisons with butterfly subdivision scheme and modified butterfly method are also presented.

In the first example the original control mesh is a simplified fandisk model (see Fig. 8(a)). There are some salient features on the mesh, moreover, the triangle sizes differ from each other greatly. To interpolate the mesh by

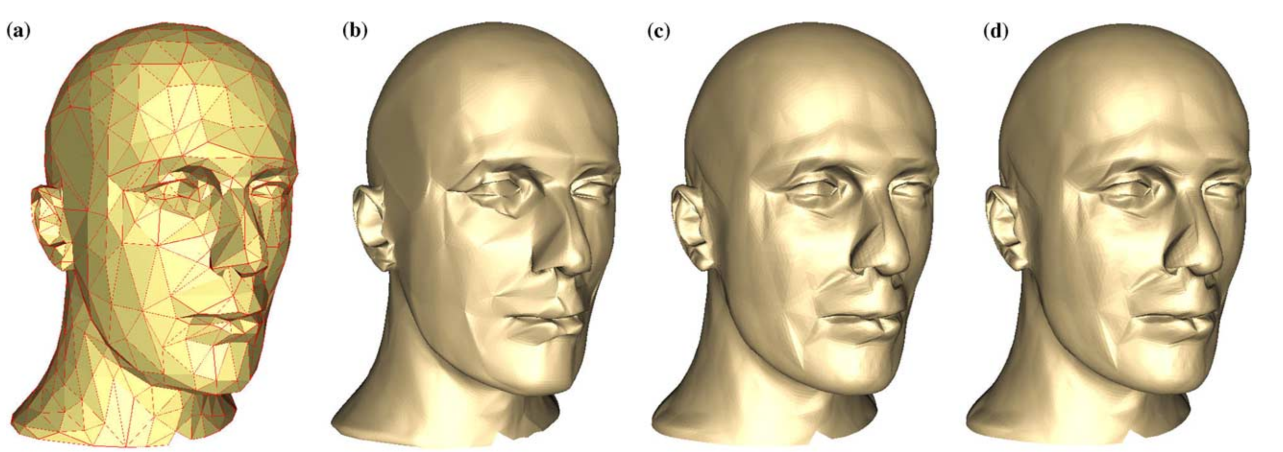

Fig. 9. (a) Initial control mesh; (b) butterfly subdivision surface; (c) normal based subdivision; and (d) surface interpolation under tangent plane constraint.

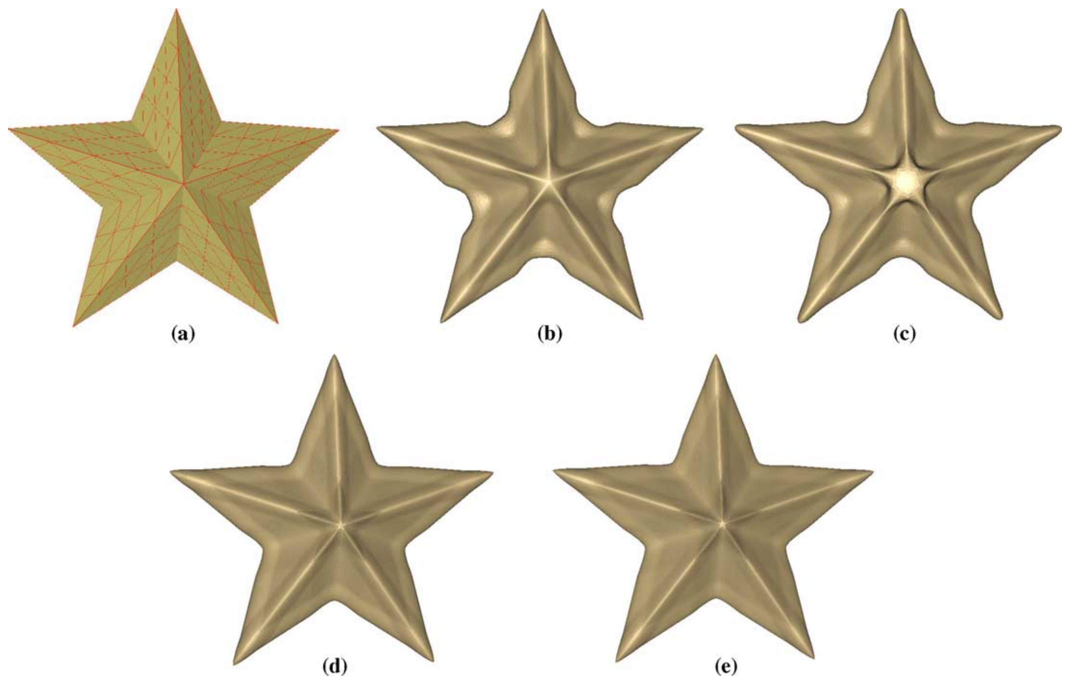

Fig. 10. (a) Initial control mesh; (b) interpolation using butterfly lgorithm; (c) interpolation using modified butterfly algorithm; (d) normal based subdivision; and (e) interpolation under tangent plane constraint.

a piecewise smooth surface with sharp edges and sharp corners preservation, we use the face based subdivision scheme. Without special care of the features, we obtain an interpolating surface after three times of subdivision in Fig. 8(b). All surfaces within this paper are flat shaded to show the smoothness or features on the surface. From the figures we can see that the curved regions become smoother while flat regions and sharp features are preserved well. To view the regularity of the interpolating surface, the subdivided mesh is illustrated together with the shaded surface (see Fig. 8(c)). Although the triangle sizes of the original mesh are ugly irregular, the final surface still looks well.

In the second example, we interpolate a head model using butterfly subdivision scheme, normal based subdivision scheme and subdivision under tangent plane constraint, respectively. The butterfly subdivision surface after three times of subdivision is shown in Fig. 9(b). It can be noticed that there appear several singular points in the surface. In Fig. 9(c) and (d) are two interpolating surfaces using normal based subdivision method and subdivision under tangent plane constraint. There is hardly any unnecessary undulation and distortion in the interpolating surfaces. Because we compute subdivision parameter for surface in Fig. 9(d) based on initial estimation of new vertices using normal based subdivision, these two surfaces are very similar.

In the last two examples we compare the new subdivision method with butterfly subdivision method. A solid star model with uniform triangulation is presented in Fig. 10(a). In Fig. 10(b) and (c) are interpolating surfaces by butterfly subdivision scheme and modified butterfly subdivision scheme after three times of subdivision. In Fig. 10(d) and (e) are interpolating surfaces using normal based subdivision scheme and subdivision under tangent plane constraint. These last two figures also show that the fairness of the interpolating surface has been improved under the tangent plane constraint compared to using the normal based

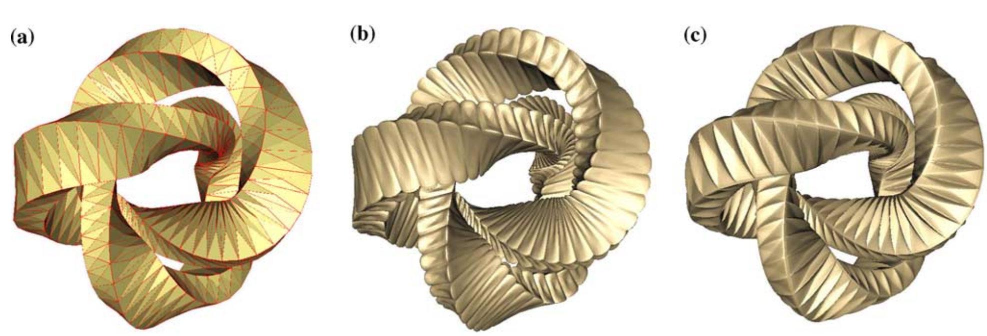

Fig. 11. (a) Initial control mesh; (b) interpolation using butterfly algorithm; and (c) interpolation using normal based subdivision scheme.

subdivision algorithm directly. Comparing the surfaces by new method with surfaces by butterfly subdivision algorithm, the deformation and undulation near sharp edges and concave corners have been removed efficiently. In Fig. 11, we present another interesting example for surface interpolation using butterfly subdivision scheme (Fig. 11(b)) and normal based subdivision scheme (Fig. 11(c)). Both examples in Figs. 10 and 11 show that the surfaces obtained by the new method look more fairly and naturally than some previous methods.

6. Conclusions and discussions

We have proposed two new subdivision schemes, face based subdivision scheme and normal based subdivision scheme, for surface interpolation of triangle meshes. Face based subdivision scheme computes new vertices by solving a least square fitting problem. Several interesting properties of this scheme have been investigated. As a result, flat regions are always smoothed while features are preserved well during subdivision.

Normal based subdivision scheme is another simple scheme for surface interpolation. The limit surfaces by this scheme are G^1 smooth. Moreover, the surface shapes can be improved further under the constraints of tangent planes and neighboring triangles together. The examples we have tested also show that the shapes of the interpolating surfaces using this new method are more fair and natural than using some traditional subdivision methods.

There are several interesting topics that deserve further study in the future. For face based subdivision, how to choose the fitting coefficients for optimal piecewise smooth surface construction is an interesting topic. For normal based subdivision, the new problem is how to design a convexity preserving subdivision scheme for surface interpolation. We have computed the new vertex corresponding to every edge with regular topology split, but we believe that this method also works for new vertex computation corresponding to some other types of topology split.

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