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Efficient circular arc interpolation based on active tolerance control

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Abstract

In this paper, we present an efficient sub-optimal algorithm for fitting smooth planar parametric curves by G^1 arc splines. To fit a parametric curve by an arc spline within a prescribed tolerance, we first sample a set of points and tangents on the curve adaptively as well as with enough density, so that an interpolation biarc spline curve can be with any desired high accuracy. Then, we construct new biarc curves interpolating local triarc spirals explicitly based on the control of permitted tolerances. To reduce the segment number of fitting arc spline as much as possible, we replace the corresponding parts of the spline by the new biarc curves and compute active tolerances for new interpolation steps. By applying the local biarc curve interpolation procedure recursively and sequentially, the result circular arcs with no radius extreme are minimax-like approximation to the original curve while the arcs with radius extreme approximate the curve parts with curvature extreme well too, and we obtain a near optimal fitting arc spline in the end. Even more, the fitting arc spline has the same end points and end tangents with the original curve, and the arcs will be jointed smoothly if the original curve is composed of several smooth connected pieces. The algorithm is easy to be implemented and generally applicable to circular arc interpolation problem of all kinds of smooth parametric curves. The method can be used in wide fields such as geometric modeling, tool path generation for NC machining and robot path planning, etc. Several numerical examples are given to show the effectiveness and efficiency of the method.

Keywords: Circular arc interpolation; Arc spline; Active tolerance; Data reduction; CAD/CAM

1. Introduction

Arc spline is a kind of geometric curve made of circular arcs and straight line segments. The offset of an arc spline is another arc spline, and it is often used as the description of tool path of CNC machinery as well as an efficient modeling tool [1–4]. Because of its simplicity, the arc spline method is easy to use and computationally efficient in shape modeling and other applications. In programming the tool path of CNC machinery, fewer arc segments can help to improve the production efficiency by reducing the number of instructions and tool motions. On the other hand, geometric shapes are often represented and modeled by parametric curves and surfaces in the fields of computer aided design and computer graphics. Then, computing smooth circular arc splines interpolating parametric curves with as few as possible segments has great significance for wide applications such as geometric modeling, CAD/CAM and robot path planning.

The crux for the approximation of a parametric curve by an arc spline lies at the fact that it is difficult to estimate the distance between these two types of curves analytically [5,6]. To find the maximum distance, one has to find the zeros of the derivative of the square of the Euclidean distance between a parametric curve and the corresponding arc segment. For example, we have to solve an equation with order $2n-1$ or $4n-1$ while approximating a parametric polynomial curve or a rational curve with order n by an arc spline. Solving high order equations will make the algorithm not robust and inefficient for most applications.

In this paper we present a novel numerical algorithm for computing the interpolation arc segments. Though it is difficult to compute the maximum deviation of an arc segment from the parametric curve analytically, the error bound can be estimated for an interpolation biarc curve when the sampled step is chosen small enough [7]. Then we can construct an interpolation biarc spline curve with high accuracy by sampling points and tangents on the curve with enough density. To obtain an interpolation arc spline with as few as possible segments within a prescribed tolerance, we compute the optimal interpolation biarc curve within the prescribed tolerance in Tchebycheff norm. Then we replace the corresponding part of the arc spline by the optimal biarc curve. Though the biarc number can be lowest by interpolating various parts of the curve with

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different optimal biarc curves, the purpose of this paper is to interpolate an original parametric curve with low number of circular arcs. In fact, when we interpolate a part of the curve with an optimal biarc curve, the second segment of the biarc associated with some successive arcs can always be approximated by another new biarc curve within the tolerance. Then, we just choose the first segment of the optimal fitting biarc curve as the interpolation circular arc to the original parametric curve, except at the end of the curve where we choose both of the two segments.

To compute the optimal interpolation biarc curve of original curve, we compute tight tolerances or active tolerances for triarc spirals and replace the spirals by the interpolation biarc curves within the active tolerances recursively. An interpolation biarc curve becomes the optimal interpolation biarc curve, until the triarc span that is consisting of this biarc curve with a successive arc segment is not a spiral or a spiral with no interpolation biarc curve within the permitted tolerance.

The algorithm has the following features:

- *Generality*: The algorithm constructs the interpolation circular arcs by lowering the segment number of a high accuracy fitting biarc spline. Then the algorithm can be used to approximate any type of smooth parametric curves.
- *Efficiency*: We obtain the interpolation circular arcs along with the data sampling and biarc curve fitting process, and compute all the parameters for the fitting arcs explicitly. The time needed for the whole procedure is just proportional to the sampling density.
- *Near optimality*: We lower the number of segments of an arc spline by interpolating local triarc spans with biarcs recursively and orderly. The result fitting arcs are minimax-like approximation to the original curve, and the number of segments of final interpolation arc spline always reaches or almost reaches the lowest.

All these benefits of the algorithm make it an efficient and practical method for geometric modeling and processing.

The organization of the paper is as follows. In Section 2 we will give a brief review of some related work and Section 3 will be devoted to the discussion of high accuracy biarc curve interpolation problem. We will derive the explicit formulae and give the algorithm for circular arc interpolation in Section 4. Some examples will be illustrated in Section 5 and we will conclude the algorithm in the last section.

2. Related work

Research on the topic of smooth circular arc interpolation for parametric curves has been active in recent years [5–13]. According to the ways how the algorithms solve the problem, the existing algorithms can be classified roughly into three categories: The first kind of algorithms construct interpolating arc splines by solving optimization problem; the algorithms that can be classified into the second category try to find analytic solutions for the construction of fitting arcs, but they are designed just for a few special kinds of parametric curves such as quadratic Bézier curves; the third kind of algorithms construct arc spline with relatively lower number of segments by fitting a set of sampled points or based on multiresolution analysis of an arc spline curve.

Marciniak and Putz [2] have proved that minimax approximation generates arc spline with lowest number of arc segments within given accuracy for a spiral. However, the algorithm is not efficient and the problem of how to deal with end points constraint has not been mentioned. Qiu et al. [5] have improved this algorithm by solving numerical equations with end constraint, but dividing a general smooth curve into spirals is computationally expensive. Though the interpolation for each spiral is optimal, but it may not mean the fitting arc spline is optimal for a curve consisting of several spirals.

Ong et al. [8] have employed an optimization procedure to find biarc spline fitting a B-spline curve by minimizing the area bounded by the B-spline and the arc spline curve. Meek and Walton [9] have studied the problem of fitting quadratic NURBS curve by arc spline. Ahn et al. [6] and Yong et al. [10] have given several algorithms for approximating quadratic Bézier curves by G^1 arc splines. The main drawback of these algorithms is that they cannot be extended to approximate general types of parametric curves.

Meek and Walton [11] have fitted a biarc spline curve to a set of discrete point by computing biarcs adaptively. Yeung and Walton [3] have fitted an arc spline to a set of sampled points from the curve instead of piecewise linear curve for NC machining. Though the arc spline can improve the machining quality than piecewise linear curve, there still remain big room for improving the fitting quality and machining efficiency with efficient algorithms. Wallner [12] has proposed a method for lowering the number of segments of an arc spline based on multiresolution analysis, but this method cannot control the approximating error explicitly.

In a recent paper [13], we have presented an active tolerance algorithm for approximating NURBS curves by arc splines. In this paper we generalize the algorithm for approximating arbitrary types of smooth parametric curves by arc splines and improve the former algorithm in several aspects. Firstly, a precise formula for circular arc interpolation is presented. On the second hand, we derive exact formulae for robust and tight active tolerance computation. Thirdly, a unified algorithm is presented for circular arc interpolation along with the whole curve, irrespective of the profile with increasing or with decreasing monotone curvature. All tests we have experimented show that the new algorithm can generate arc spline with lowest or almost lowest number of segments.

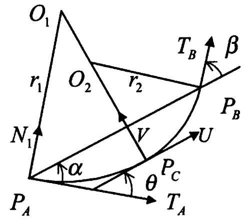

Fig. 1. Biarc curve construction.

3. High accuracy biarc curve fitting

In this section, we will first briefly review the definition of biarc curve and the estimation of error bound of biarc curve fitting, then we will discuss how to sample points and construct interpolation biarc curves within prescribed accuracy based on error bound formula.

3.1. Biarc curve

A biarc curve consists of two smoothly connected arc segments that interpolate two end points and the end tangents [1,14]. Assume that P_A and P_B be two distinct points associated with two unit tangent vectors T_A and T_B at the points. Let φ be the angle from the positive direction of x-axis to vector T_A, then the vector T_A can be represented as $T_A = (\cos\varphi, \sin\varphi)$. In this paper, we define an angle positive if it is counter-clockwise and negative otherwise. We can assume that O_1 and O_2 be the centers of two arc segments with radii r_1 and r_2, respectively. Let P_C be the joint point of the two arcs and U is the unit tangent vector at the joint point. If we denote the angle between vector T_A and vector U as θ, then we have $U = (\cos(\theta + \varphi), \sin(\theta + \varphi))$. In addition, we can represent the unit normal vector at point P_A as $N_1 = (-\sin\varphi, \cos\varphi)$ and the normal vector at point P_C as $V = (-\sin(\theta + \varphi), \cos(\theta + \varphi))$ (see Fig. 1). Let α and β be the angle from T_A to P_AP_B and the angle from P_AP_B to T_B, respectively, and let $l = \|P_B - P_A\|$, then the radii, the joint point and the centers for the two arcs can be computed as follows [14]:

$$r_1 = \frac{l}{2\sin((\alpha + \beta)/2)} * \frac{\sin((\beta - \alpha + \theta)/2)}{\sin(\theta/2)}$$

$$r_2 = \frac{l}{2\sin((\alpha + \beta)/2)} * \frac{\sin((2\alpha - \theta)/2)}{\sin((\alpha + \beta - \theta)/2)}$$

$$P_C = P_A + r_1(N_1 - V)$$

$$O_1 = P_A + r_1N_1$$

$$O_2 = P_C + r_2V \tag{1}$$

In above equations, the angle θ is a free variable, and various choices of the angle will generate different biarc curves. If slope angles α and β have the same sign, we can then construct a C_shaped biarc curve; otherwise construct an S_shaped biarc curve. For variable θ, we choose $\theta = \alpha$ for C_shaped biarc curve and $\theta = (3\alpha - \beta)/2$ for S_shaped biarc curve. This choice can help to keep the shape of the arc spline compatible with the original curve well [14,15].

3.2. Estimate the fitting error

When we construct a biarc curve interpolating a pair of sampled points and tangents on a parametric curve, we can estimate the fitting error as follows. For two end points P_A and P_B of a spiral associated with two unit tangent vectors T_A and T_B at the points, two circular arcs C_A and C_B that join points P_A and P_B and matches vector T_A or T_B, respectively, are two bounding circular arcs [7,9]. In Ref. [7], Meek and Wolton have proven that a spiral segment be enclosed by the two bounding circular arcs derived from the curve if the length of the curve approaches zero. Even more, the distance from the spiral to the interpolating biarc curve approaches 1/13.5 of the distance between the two bounding circular arcs.

For most applications, a smooth curve is always consisting of a finite set of spirals of which the curvatures are monotone. Then we can believe that most small segments are spirals if we divide a parametric curve by sampling points with enough density. Let α and β be the angles defined as above, then the maximum distance between the two bounding circular arcs is

$$d_B = \frac{\|P_B - P_A\|}{2}\left|\tan\frac{\alpha}{2} - \tan\frac{\beta}{2}\right|,$$

and the maximum distance d_M from the spiral to the interpolation biarc curve is approximately $d_B/13.5$. Since detecting whether a curve segment has been enclosed by the corresponding bounding circular arcs or not is not a trivial work, we choose to compute the distances from a few sampled points on the curve to the arc segment and choose the maximum distance d_N as the deviation. Because we have divided the parametric curve into many small intervals, we can sample three to five points in each interval to estimate the deviation. So, the distance between the parametric curve and the interpolation biarc curve can be estimated as $d = \max(d_M, d_N)$.

3.3. Sampling points

If we sample points on a smooth curve with too large step, the fitting biarc curve may not be with the desired accuracy. On the other hand, too much detailed sampling may be computationally expensive even though it can increase the fitting accuracy. To control the fitting error compactly, we sample points on the smooth curve adaptively. For each step, we first sample a new point with a fixed step and check the distance from the interpolation biarc curve to the parametric curve. If the maximum distance is within the prescribed accuracy, we accept the interpolation biarc curve; otherwise, we sample another point on the curve with a smaller parameter step.

Without loss of generality, we can assume that the parameter curve is defined on the interval [0, 1], and we choose the fixed step as $\Delta t = 0.02$. In fact, various choices of this fixed step do not influence the fitting results much because of high accuracy requirement. Let P_{l-1} be one point on the parametric curve $P(t)$ with parameter t_{l-1}, to choose next point P_l we can first set parameter $t_l = t_{l-1} + \Delta t$. If the deviation is within the prescribed accuracy, we set $P_l = P(t_l)$; otherwise, we will check another sampled point by choosing the parameter with a shorter step, i.e. $t_l = t_{l-1} + 0.65\Delta t$. This procedure continues until an accepted point is reached. If the permitted error bound for circular arc interpolation is τ, a positive real number less than $0.01 * \tau$ can be set as a well working accuracy for high accuracy biarc curve interpolation. In all the examples cited in this paper we set the fitting accuracy as $0.002 * \tau$. Because the initial fitting biarc spline curve is with so high accuracy to the original parametric curve, we will not distinguish these two types of curves and we mean the high accuracy interpolation arc spline curve when we refer to the original curve without particular declaration in the following sections. An advantage of the high accuracy biarc spline representation is that we can compute the distance from an arc segment or a point to the original curve more easily.

4. Smooth circular arc interpolation

Though the arc spline constructed above is with high accuracy to the original parametric curve, but the arc segments are always too many to be used efficiently and should be reduced as much as possible for most applications. To obtain interpolation circular arcs within a given tolerance, we replace local spiral spans by the corresponding interpolating biarc curves within active tolerances. We will give the circular arc interpolation formula for spirals with decreasing curvature in the following subsections, and the formula for spirals with increasing curvature can be obtained similarly.

4.1. The strategy

For an arc spiral consisting of three segments or a triarc spiral, there are a family of biarc curves interpolating the same end points and end tangents of the spiral [14]. To find a new biarc curve within a prescribed deviation, we construct the two segments of the biarc curve by expanding the arc segment with smallest radius and shrinking the arc segment with biggest radius of the triarc spiral, with two end points and end tangents of the spiral fixed. The magnitude of the expansion or shrinking of one new arc segment is decided by its deviation from the spiral and the another arc segment can be decided based on continuity constraint. Then the entire new biarc curve will be within the permitted tolerance of original curve if the deviation of second arc segment is no larger than τ, either.

For an arc span consisting of more than three segments of which the radii are monotone increasing, we can construct optimal fitting biarc curves by checking triarc spirals recursively and orderly. We begin the procedure by choosing the first three arcs and construct an interpolation biarc curve. When we replace the first three arcs by the interpolation biarc curve, we will obtain a new arc span with one segment fewer. The first three segments of this new arc span will again form a new triarc spiral, and we can fit this new spiral by another new biarc curve. To guarantee that the deviation from this new biarc curve to the original curve is not larger than τ, we compute a new tighter tolerance for the construction of this new biarc curve. This procedure continues until the active tolerance vanishes and the maximum deviation from the latest biarc curve to the original curve reaches τ. At the time, the interpolation biarc curve is the optimal approximation to the original curve under the mean of minimax approximation. From another point of view, we obtain the optimal biarc curve by expanding the first segment and shrinking the last segment of the corresponding arc span within the given tolerance. Since the second segment of the optimal biarc can be expanded with another end fixed in following steps, we leave the first segment of the biarc as an interpolation circular arc to the parametric curve and check a new triarc spiral that is consisted of the second segment of the biarc curve associated with another two successive arcs from the original fitting arc spline.

If there exist inflection points in the triarc span or the triarc span is not a spiral, we leave the first segment as one interpolation arc, and check another triarc span with a new arc segment added. One advantage of the method is that it can keep the shape of the final interpolation arc spline compatible with the original curve well. On the other hand, the algorithm can be implemented more easily and conveniently.

4.2. Fitting triarc spiral by biarc curve

Without loss of generality, we can assume that O_0, O_1 and O_2 be three smooth connected arc segments of a triarc spiral with increasing radii r_0, r_1 and r_2, respectively. The spiral starts at point P_0 and ends at P_3 with two joint points P_1 and P_2. To determine a new biarc curve interpolating points P_0, P_3 and the tangents at these two points within controllable deviation, we construct these two new arcs by expanding r_0 and shrinking r_2 with end points P_0 and P_3 and the tangents at these two points fixed (see Fig. 2). Let

$$V_0 = \frac{O_0 - P_0}{\|O_0 - P_0\|} \quad \text{and} \quad V_2 = \frac{O_2 - P_3}{\|O_2 - P_3\|},$$

then the centers for these two new arc segments are $O_a = O_0 + tV_0$ and $O_b = O_2 - uV_2$, respectively, where t and u are the expansion or shrink magnitudes. We obtain the radii

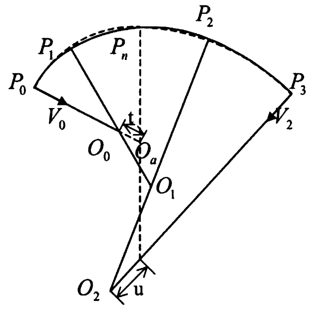

Fig. 2. Approximate a triarc spiral by a biarc curve.

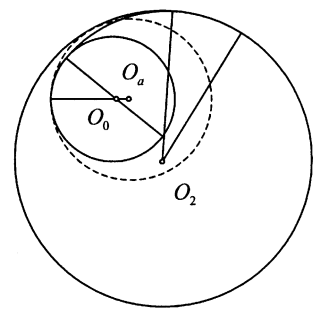

Fig. 4. Expand arc O_0 within circle O_2.

for these two new arcs as $r_a = r_0 + t$ and $r_b = r_2 - u$, respectively.

To guarantee that arc segments O_a and O_b are G^1 continuous at the joint point, the radii for the two arcs should satisfy the following equation:

$$r_a + \|O_a - O_b\| = r_b \tag{2}$$

By expressing O_a, O_b, r_a and r_b as the functions of t and u, we have:

$$\|O_2 - O_0 - uV_2 - tV_0\| = r_2 - r_0 - u - t \tag{3}$$

Let $D = O_2 - O_0$ and $t_0 = r_2 - r_0$, then Eq. (3) can be transformed into

$$u = \frac{X_0 + X_1 t}{Y_0 + Y_1 t} \tag{4}$$

where $X_0 = t_0^2 - D^2$, $X_1 = 2(DV_0 - t_0)$, $Y_0 = 2(t_0 - DV_2)$ and $Y_1 = 2(V_0 V_2 - 1)$. Once parameter t or u has been decided, the two new arcs can then be obtained. Let

$$V_n = \frac{O_a - O_b}{\|O_a - O_b\|},$$

then the point jointing arc O_a and O_b is $P_n = O_a + r_a V_n$.

Since we construct arc O_a by expanding arc O_0, the maximum deviation of arc O_a from the triarc spiral is the maximum distance between arc O_a and arc O_1, i.e. $d_m = r_a + \|O_1 - O_a\| - r_1$ (see Fig. 3). To make sure that arc O_a is within the permitted tolerance of the original curve, we assume that d_m is no larger than the active tolerance ϵ_i at

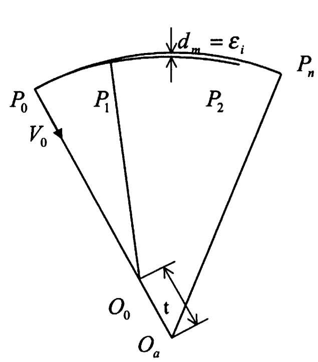

Fig. 3. Tolerance constraint for arc expansion.

point P_1, and then we have

$$d_m = r_a + \|O_1 - O_a\| - r_1 \le \epsilon_i \tag{5}$$

By substituting the equations $r_a = r_0 + t$ and $O_a = O_0 + tV_0$ into Eq. (5), we have

$$t \le t_b = \frac{(r_1 - r_0 + \epsilon_i)^2 - (O_1 - O_0)^2}{2[r_1 - r_0 + \epsilon_i + (O_1 - O_0)V_0]} \tag{6}$$

To be sure that the joint point of the biarc curve lies in the sector region of the triarc spiral, we expand arc O_0 inside circle O_2 (see Fig. 4), then we have

$$\|O_a - O_2\| + r_a \le r_2 \tag{7}$$

From Eq. (7) we have

$$t \le t_c = \frac{t_0^2 - D^2}{2(t_0 - DV_0)} \tag{8}$$

The maximum value for t can now be chosen as $t = \min(t_b, t_c)$. It can be easily verified that both t_b and t_c are larger than or equal to zero, then we conclude that t is non-negative. It is clear that $r_a \le r_b$ based on Eq. (2), and $r_b \le r_2$ since arc O_b is inside circle O_2. Then the new arc spline is still an arc spiral when a triarc span of the original spiral has been replaced by the corresponding biarc curve.

The distance from arc O_a to the original curve is within the permitted bound by the control of active tolerances, then we should compute the distance from arc O_b to the original curve to confirm whether the biarc curve can be accepted or not. Because the fitting error is very small compared with arc radii and the original high accuracy fitting arc spline is dense with arc segments, we choose to compute the distances from those joint points that lie in the sector region of arc O_b to the new biarc curve for the estimation of the distance between the original curve and arc O_b. If the maximum deviation d_b is less than τ, we will replace the triarc spiral by the biarc curve; otherwise, set arc O_0 as an interpolation segment and choose arcs O_1, O_2 and another new arc segment for further approximation.

4.3. Computing the active tolerance

The tolerance along the original curve is the prescribed constant τ, but tolerance will vary along the new arc spline

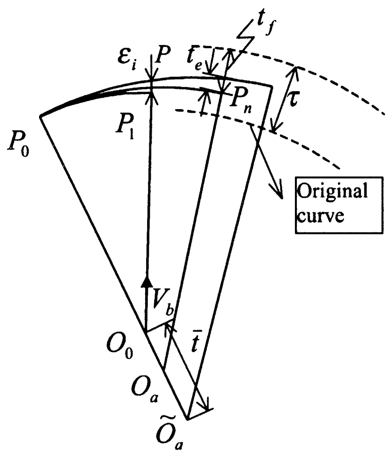

Fig. 5. Compute the active tolerance.

when we replace some arc spans with interpolating biarc curves. To control the deviation of first arc segment of new interpolation biarc curve, we use the active tolerance at the first joint point P_1 of the triarc span for this purpose. When we compute arc O_a by expanding the radius of arc O_0, a conservative estimation is that the maximum deviation between arc O_a and the triarc span is no more than the active tolerance at P_1. When the arc interpolation procedure starts, the first arc can be expanded or shrunk with maximum deviation τ, and we set the active tolerance $\epsilon_0 = \tau$; otherwise, the active tolerances should be recomputed according to the geometry of the fitting biarc curve.

If we enlarge the radius of arc O_0 with magnitude $\bar{t}$ along V_0 with point P_0 fixed so that point P_1 has been moved ϵ_i away from its current position, then the radius and center for this expanded arc are $\tilde{r}_a = r_0 + \bar{t}$ and $\tilde{O}_a = O_0 + \bar{t}V_0$, respectively. Let

$$V_b = \frac{P_1 - O_0}{\|P_1 - O_0\|},$$

then the point on arc $\tilde{O}_a$ that is ϵ_i distant away from point P_1 can be represented as $P = O_0 + (r_0 + \epsilon_i)V_b$ (see Fig. 5). Thus the vector from $\tilde{O}_a$ to point P can be represented as $P - \tilde{O}_a = -\bar{t}V_0 + (r_0 + \epsilon_i)V_b$. Since the length of vector $P - \tilde{O}_a$ is $r_0 + \bar{t}$, then we have

$$(r_0 + \bar{t})^2 = [-\bar{t}V_0 + (r_0 + \epsilon_i)V_b]^2 \tag{9}$$

From Eq. (9), we have

$$\bar{t} = \frac{2r_0\epsilon_i + \epsilon_i^2}{2[r_0 + (r_0 + \epsilon_i)V_0V_b]} \tag{10}$$

If r_0 has been enlarged by the magnitude $\bar{t}$, then arc O_0 will be expanded to its maximum permitted extent at the position of point P_1. Accordingly, the expansion magnitude of arc O_a at point P_n should be no more than $t_e = r_0 + \bar{t} - \|P_n - (O_0 + \bar{t}V_0)\|$ during successive approximation steps.

On the other hand, the expansion of arc O_a should satisfy the condition that the distance from new position of point P_n to the original curve should be less than or equal to τ. For an arc span without inflection, an outer offset of the curve is another arc spline of which the radius of each arc segment has been increased by a constant value. If a point lies on one outer offset curve we mean that the point lies at the convex side of the original curve; similarly, we can define inner offset curve and concave side for an arc span if the curve or the point lies on another side of the original curve. We denote the signed distance from point P_n to the original curve as t_m, t_m is positive when the point P_n lies at the convex side of the curve and t_m is negative otherwise. Then the distance from point P_n to the outer offset of original curve is $t_f = \tau - t_m$ (see Fig. 5). To be sure that arc O_a can be further expanded within the permitted tolerance of original curve, we choose $t_f = \tau$ when P_n lies at the concave side of the original curve. We will explain this assertion in next paragraph. Then the permitted tolerance at point P_n is $\epsilon_{i+1} = \min(t_e, t_f, \tau)$. If some ϵ_{i+1} becomes negative, we set $\epsilon_{i+1} = 0$ to keep the first arc from expanding any more, and construct the fitting biarc curve by shrinking the radius of last arc only.

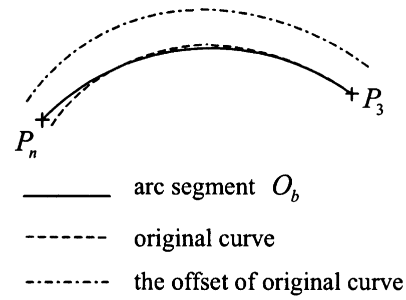

Fig. 6. Active tolerance for arc O_b.

The active tolerance ϵ_{i+1} at point P_n satisfies the inequality $0 \le \epsilon_{i+1} \le \tau$. If point P_n lies at the convex side of arc O_b, then we have $\epsilon_{i+1} < \tau$. Any point that lies at the convex side of arc O_b and is within ϵ_{i+1} distance to the arc is also within the permitted tolerance to the original curve (see Fig. 6). If point P_n lies at the concave side of original curve, then the whole segment of arc O_b lies at the concave side of original curve. At this time, we have $\epsilon_{i+1} = \tau$, an outer offset arc segment which is τ distance away to arc O_b is also within the permitted tolerance region of original curve. For these reasons the expansion magnitude can be well controlled by the active tolerances during the successive approximation steps.

For a curve with monotone increasing curvature radius, we construct interpolating biarc curves for selected triarc spirals recursively. The radius of an original arc O_0 will be expanded until the active tolerance vanishes, and the procedure continues until the maximum deviation of arc O_b from the original curve has reached its maximum value within tolerance τ. Then we obtain an optimal fitting biarc curve. When we set arc O_a as an interpolation circular arc to the original curve, we approximate arc O_b associated with successive arcs by expanding the radius of arc O_b.

Because arc O_b is first obtained by shrinking the radius of an original arc segment with final end fixed, then, if we expand this shrunk arc with another end unchanged, the permitted expansion tolerance at the movable end point is τ (such as point P_3 in Fig. 6). If that is the case, we set the new active tolerance as τ. If an interpolation circular arc is obtained by shrinking the radius with one end fixed and expanding the radius with another end fixed, it deviates the original curve at both sides. Though the former maximum deviation on one side may decrease a little when the arc is expanded toward the maximum deviation on the other side, the interpolation circular arc is still minimax-like approximation to the original curve and it is a sub-optimal approximation to the curve.

If the radius of an arc segment is local minimal in a spline, the arc segment will first be expanded with final end fixed in a triarc spiral with increasing curvature and then will be expanded further with another end fixed in a triarc spiral with decreasing curvature. Since the second expansion may increase the initial deviation, in a similar way as Eq. (10), we compute the maximum bound that can keep the increased deviation within permitted tolerance. Then we set the initial active tolerance for second expansion as the minor one between the obtained bound and τ. Similar result holds for arcs with local maximum curvature radius. If there exist inflection points on a triarc span, the first segment will be used as the interpolation circular arc, and we set τ as the active tolerance for next triarc span. This algorithm will make the circular arc interpolation procedure more natural without the need to divide the parametric curve into spirals and lower the number of final fitting arcs much more than approximating each spiral separately.

4.4. The algorithm

Given a smooth parametric curve, we construct high accuracy interpolation biarc curves by sampling points and tangents on the curve adaptively. Along with this process we lower the number of arcs by checking every three consecutive arc segments on the spline orderly within active tolerances.

If some triarc span forms a spiral with increasing or decreasing curvature, we will construct an interpolation biarc curve. We delete one segment and replace the other two segments of the span by the biarc curve when the biarc curve is within the permitted tolerance, and then we continue the procedure with next approximation step by adding a consecutive arc segment. If there is no biarc curve that can replace the current triarc span or the span is not a spiral, the first segment of the span will be kept as an interpolation arc segment and we continue the procedure by checking a new triarc span with a new arc segment added.

We stop the procedure until there is no sampled point on the parametric curve and the last segment of fitting arcs becomes the interpolation arc. Then, the final arc spline is the fitting arc spline to the original parametric curve. The sketch of the algorithm for smooth circular arc interpolation is given as follows:

Algorithm: Smooth Circular Arc Interpolation
input: a smooth parametric curve $P(t)$ and tolerance τ
output: a smooth circular arc spline within τ
construct a biarc curve interpolating sampled points $P(t_0)$, $P(t_1)$ and tangents $T(t_0)$, $T(t_1)$;
choose first two arcs from the spline as arc1 and arc2;
set active tolerance $\epsilon_0 = \tau$;
while (($t < 1$) or (next arc! = NULL))
{
  if (next arc == NULL)
    {select $t = t_{l+1}$ and construct biarc interpolating $P(t_l)$ and $P(t_{l+1})$ }
  arc0 = arc1;
  arc1 = arc2;
  arc2 = next arc;
  if (the three arcs form a triarc spiral)
  {
    Compute arcs arc_a and arc_b based on ϵ_i;
    Compute the deviation d_b of arc_b to the original curve;
    if ($d_b < \tau$)
    { delete arc0;
      arc1 = arc_a;
      arc2 = arc_b;
      compute the active tolerance ϵ_{i+1};
    } else $\epsilon_{i+1} = \tau$;
  } else compute and reset ϵ_{i+1};
}
output the final arc spline.

5. Examples

The algorithm is implemented on a SGI octane workstation with MIPS R10000 and 128 MB memory. We have applied our algorithm on a number of parametric curves and obtain satisfying approximating results. For all the examples we have experimented, we obtain interpolation arc splines in real time. We present a few examples here to show the efficiency of the new method.

The parametric curve in example 1 is a quadratic Bézier curve with control points (1.0, 1.0), (2.0, 1.0), (4.5, 2.75). This is a segment of spiral and we obtain an interpolating arc spline with 19 circular arcs within tolerance 10^{-5} by sub-optimal interpolation algorithm (see Fig. 7(a)). We have also obtained an arc spline with 26 arcs within the same tolerance by interpolating the curve with a set of smooth connected optimal biarc curves (see Fig. 7(b)). For this and the following examples, the thick curves indicate the fitting arc splines while the thin curves stand for the error plots which have been enlarged along the normal of the parametric curve. In the second example, we approximate

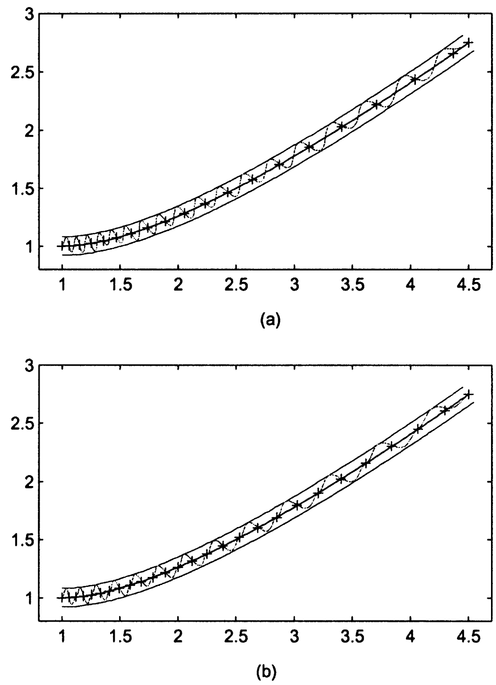

Fig. 7. Approximating a quadratic Bézier curve by arc splines $\tau = 10^{-5}$ (a) Sub-optimal circular arc interpolation; (b) Optimal biarc curve interpolation.

a segment of sine curve defined on interval [0, 2π] by arc splines within different tolerances (see Fig. 8). Notice that there exist inflection point and curvature extremes on the curve. However, we have computed the interpolation circular arcs without dividing the curve into spirals. We have designed a cam profile by a rational Bézier curve of degree six (see Fig. 9). The control points and the weights for this Bézier curve are (1, 1), (1, 2.3), (3.4, 1.9), (4, 1), (3.4, 0.1),

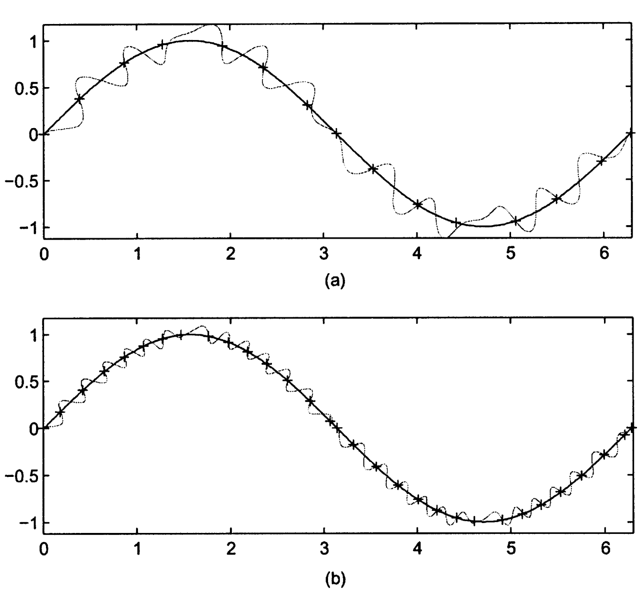

Fig. 8. Approximating sine curve segment by arc splines. (a) $\tau = 10^{-3}$; (b) $\tau = 10^{-4}$.

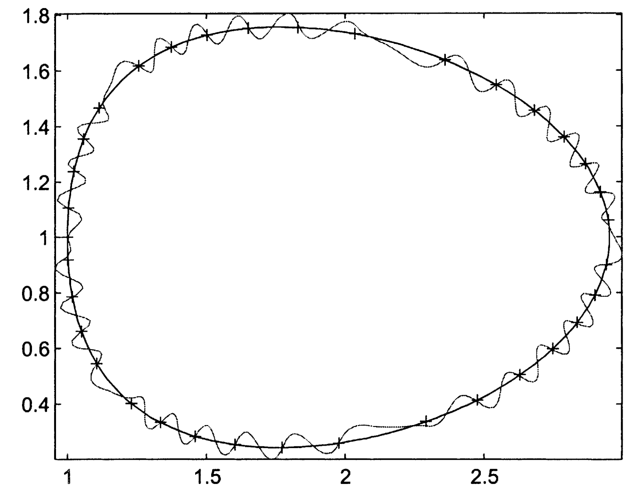

Fig. 9. Smooth circular arc interpolation for a cam profile. $\tau = 0.5 \times 10^{-4}$.

(1, −0.3), (1, 1) and 1, 1.2, 0.8, 1, 0.8, 1.2, 1 respectively. Based on the control of active tolerances, the deviation of the interpolation arcs is well distributed.

In the last two examples, we have approximated two cubic B-spline curves by arc splines. The points in Tables 1 and 2 are the control points for these two B-spline curves. The knots for each curve are uniform except that the end knots are with multiple 4 so that the B-spline curve can interpolate the ends of the control polygon. In Fig. 10, we have designed the B-spline curve as a face profile. Though the curve is with many inflections and curvature extremes, we obtain the fitting circular arcs in real time. In the last example, we try to approximate the knot shape symbol for Olympic game 2008 by a cubic B-spline curve (see Fig. 11). When we interpolate the curve by circular arcs, we compute two offsets of the arc spline with distance as the magnified tolerance. This example also shows that the deviation of final fitting arcs can be well controlled. The tolerances, the error magnification factors, the segments numbers and the computation time for every example are illustrated in Table 3.

6. Conclusions

In this paper we have presented a simple general algorithm for approximating smooth parametric curves by G^1 arc splines. While most traditional methods have formulated the problem as solving nonlinear optimization

Table 1
Control points of the uniform cubic B-spline curve for face profile

No.	Control points	No.	Control points	No.	Control points
0	1.10, 5.42	4	2.53, 2.30	8	3.42, 0.92
1	2.86, 4.98	5	3.37, 1.20	9	2.77, 0.34
2	1.34, 3.85	6	2.88, 1.33	10	1.23, 1.05
3	5.30, 2.00	7	2.56, 1.20		

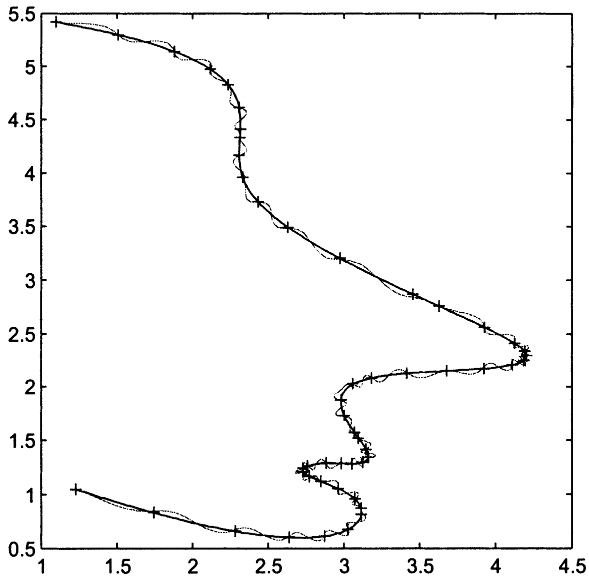

Fig. 10. Approximating a B-spline curve by arc spline. $\tau = 0.5 \times 10^{-3}$.

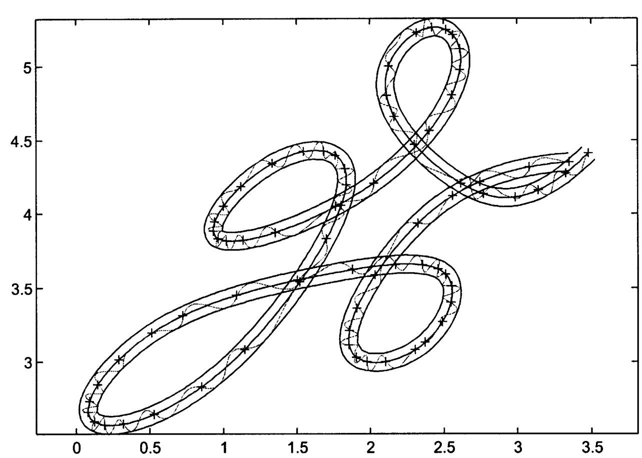

Fig. 11. Approximating a B-spline curve by arc spline and computing two offsets of the arc spline with the magnified tolerance. $\tau = 0.3 \times 10^{-3}$.

Table 2
Control points of the uniform cubic B-spline curve for the knot shape

No.	Control points	No.	Control points	No.	Control points
0	3.48, 4.41	7	0.59, 3.60	14	2.73, 3.75
1	3.07, 4.00	8	1.34, 4.52	15	2.42, 2.99
2	2.55, 4.15	9	2.17, 4.43	16	1.64, 2.93
3	1.91, 4.88	10	0.97, 2.68	17	2.17, 3.90
4	2.45, 5.42	11	−0.20, 2.42	18	2.78, 4.31
5	2.77, 5.01	12	0.43, 3.31	19	3.35, 4.35
6	2.08, 4.08	13	1.69, 3.61		

equations, in this paper we have computed the fitting circular arcs explicitly and efficiently. Based on the control of active tolerances, we obtain sub-optimal fitting circular arcs by expanding or shrinking some former interpolation arcs. Not only can we compute circular arcs interpolating spirals in the form of minimax-like approximation, but also we can fit arcs to curves with curvature extremes in a natural and efficient way. Then, the method is practical and useful for many applications such as geometric modeling, CNC machining, etc.

Besides the approximation of a smooth parametric curve by arc spline, the method presented in this paper can also be used to lower the segment number of arc splines obtained by some other methods [16,17]. The algorithm in this paper has only been designed for the problem of planar circular arc interpolation now, it will be a meaningful thing to extend the active tolerance method for spatial circular arc interpolation and some other piecewise linear or circular approximation problems which are traditionally solved by optimization method in the future.

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Table 3
The tolerance and the arc numbers for all the examples

Curves	Tolerances (τ)	Magnification factors	Arc number	Time (s)
quadratic Bézier curve	0.1×10^{-4}	5000	19	0.066
	0.1×10^{-4}	5000	26	0.065
Sine curve	0.1×10^{-2}	200	14	0.029
	0.1×10^{-3}	1000	30	0.068
Cam profile	0.5×10^{-4}	1000	35	0.24
Face profile	0.5×10^{-3}	100	50	0.19
Knot shape	0.3×10^{-3}	200	72	0.36

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