Fitting and fairing Hermite-type data by matrix weighted NURBS curves

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ABSTRACT

This paper proposes techniques to fit and fair sequences of points together with normals or tangents at the points by matrix weighted NURBS curves. Given a set of Hermite-type data, a matrix weighted NURBS curve is constructed by choosing the input points as control points and computing the weight matrices using the normals or tangents. Unlike traditional B-spline or NURBS curves that have only linear precision, matrix weighted NURBS curves with point-normal or point-tangent control pairs have almost circular or helical precision. Matrix weighted NURBS curves constructed from Hermite-type data can be fair and fit the input points closely when the original data were regularly sampled from curves with smoothly varying tangents and curvatures. If the original data are non-uniformly spaced or noisy, fair fitting curves can still be obtained by repeatedly sampling points from previously constructed curves and constructing new matrix weighted NURBS curves using the resampled data.

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1. Introduction

Fitting discrete points by smooth parametric curves have many applications in geometric modeling [1,2], CNC machining [3], data compression [4,5] and pattern recognition [6], etc. In this paper we focus on fitting and fairing ordered points in 2D or 3D space together with given or estimated derivatives at the points by matrix weighted rational B-spline curves. Matrix weighted NURBS (non-uniform rational B-spline) curves are natural extensions of traditional NURBS curves but additional degrees of freedom within the matrix weights permit novel ways to control the shapes of the curves [7]. By proper definition of the matrix weights the extended NURBS curves can be used to fit and fair Hermite-type data efficiently.

A B-spline curve that fits a point set can usually be obtained by solving a linear system with fixed parametrization of the points [2] or by solving a nonlinear system when the parametrization is variable [8]. If the initial points are noisy and need not to be interpolated exactly, a fair fitting B-spline curve can be obtained by minimizing an energy functional that is defined by integrals of curve derivatives or curvature derivatives [9]. Even though functionals defined by curvature derivatives can give high quality fitting curves, it usually suffers high computational costs. Besides unified approaches for fitting and fairing, B-spline curves can also be faired independently after fitting or construction. Typical algorithms for B-spline curve fairing include knot removal [10,11], local or global energy minimization [12,13] and multi-scale filtering [14,15], etc.

Compared with B-spline curves, NURBS curves are more flexible in shape representation [1,2,16]. It is promising to reconstruct NURBS curves and surfaces from measured data for reverse engineering [17]. If all control points, weights and knots are free variables during the fitting process, one has to employ numerical optimization techniques to find the solutions due to the nonlinearity of the fitting functional with respect to the free variables [18,19]. It may be more complex to optimize fairness functionals defined by derivatives or curvatures of NURBS curves of general degrees along with data fitting. To our knowledge, there is few report on fairing of NURBS curves and surfaces of arbitrary degrees.

Low order NURBS curves and intrinsically defined curves have easily computed or explicit curvature profiles. These kinds of curves have frequently been used for data fitting or fair shape design. A curvature continuous conic spline can be constructed from a convex G^1 continuous conic spline by adjusting the tangents at the joint points and the curve can be faired further by repositioning the joint points [20]. Recently, a G^2 quadratic B-spline curve has been used to fit scanned data with proper choices of knots [21]. A planar cubic B-spline curve can also be faired by using target curvature [22]. Intrinsically defined curves such as clothoid or Euler spirals are fair and can be employed for data fitting or interpolation [23,24]. Low order NURBS curves have limited continuity orders and can only be used for fitting planar points. Though intrinsically defined curves are fair themselves, how to fair given points by this kind of curves is not clear.

Hermite-type data such as point-normal pairs or point-tangent pairs can help to model functional or fair shapes [25]. Interpolating a sequence of points together with prescribed derivatives

or curvatures at the points by a B-spline curve has been studied in [26,27]. Differently from B-spline curve interpolation which needs iterations to solve large systems, matrix weighted rational B-spline curves can be constructed from the given data directly. Matrix weighted NURBS curves with point-normal control pairs have almost circular precision and the curves can be used to fit uniformly sampled points and normals in a plane very well [7]. But for noisy planar points or spatial data the constructed matrix weighted NURBS curves with point-normal control pairs may not be fair or deviate from the control points much. In this paper we first present a robust method for computing the weight matrices from the given points and normals such that the obtained matrix weighted NURBS curves have the same precision but are less sensitive to data noise. To better fit points in 3D space we propose to construct matrix weighted NURBS curves with point-tangent control pairs. When only discrete points are available, estimating unit tangents at the points is much easier and more robust than estimating normal vectors in 3D space. Matrix weighted NURBS curves with point-tangent control pairs have almost helical precision and can be used to fit spatial data more accurately than curves with point-normal control pairs.

Besides direct data fitting, an algorithm for fairing Hermite-type data by matrix weighted NURBS curves will be given. Motivated by the facts that matrix weighted NURBS curves with point-normal control pairs or point-tangent control pairs have almost circular or helical precision and smooth planar or spatial curves can be approximated by smoothly connected circular arcs or helical segments, we propose to fair ordered points in 2D or 3D space by repeatedly constructing new matrix weighted NURBS curves using points sampled from previously constructed curves. The normals or tangents at the sampled points can be evaluated from the curves or estimated adaptively from the sampled points. This algorithm is simple to implement and can give high quality fitting curves in the end. Note that B-spline or NURBS curves have only linear precision and repeatedly constructed B-spline or NURBS curves with control points sampled from previously obtained curves will shrink to a point or tend to be a straight line. B-spline or NURBS curves have to be faired using more complex algorithms.

The remaining of the paper is structured as follows. In Section 2 we review the definition of matrix weighted NURBS curves with point-normal control pairs. A robust scheme for computing the matrix weights against data noise will be given. Section 3 is devoted to the construction of matrix weighted NURBS curves with point-tangent control pairs. Nice properties such as almost helical precision will be discussed. Practical algorithms for fitting and fairing discrete data by matrix weighted NURBS curves in 2D or 3D space will be given in Section 4. Section 5 presents several interesting examples for fitting or fairing by the proposed techniques and the paper is concluded with a brief summary in Section 6.

2. Matrix weighted NURBS curves with point-normal control pairs

This section recalls the definition of matrix weighted NURBS curves with point-normal control pairs. A robust method for computing the weight matrices for data fitting by matrix weighted NURBS curves will be given.

Suppose that $P_0, P_1, \ldots, P_n$ are a sequence of points lying in $\mathbb{R}^d$ and $\mathbf{n}_i \in \mathbb{R}^d$, $i = 0, 1, \ldots, n$, are unit normal vectors specified at the points. Assume that $\mathbf{t} = \{t_0, t_1, \ldots, t_{n+k}\}$ is a non-decreasing sequence and no k numbers are the same except for at two ends, and $N_{i,k}(t)$, $i = 0, 1, \ldots, n$, are the B-splines of order k (degree $k - 1$) defined on the knot vector $\mathbf{t}$. Assume $\omega_i > 0$, $\mu_i > -1$, $i = 0, 1, \ldots, n$, are a set of predefined parameters. Let I be the

identity matrix of order d. A matrix weighted NURBS curve with point-normal control pairs is given by [7]

$$Q(t) = \left[\sum_{i=0}^{n} M_i N_{i,k}(t) \right]^{-1} \sum_{i=0}^{n} M_i P_i N_{i,k}(t), \ t \in [t_{k-1}, t_{n+1}], \quad (1)$$

where

$$M_i = \omega_i(I + \mu_i \mathbf{n}_i \mathbf{n}_i^T), \quad i = 0, 1, \ldots, n, \quad (2)$$

are the weight matrices and the capital T means the transpose of a column vector.

Because the matrices given by Eq. (2) and their convex combinations are nonsingular, the matrix function $M(t) = \sum_{i=0}^{n} M_i N_{i,k}(t)$ is reversible and the curve $Q(t)$ is valid over the whole parameter domain.

Theorem 2.1 ([7]). *Suppose that $\mathbf{n}_i \in \mathbb{R}^d$, $i = 0, 1, \ldots, n$, are a set of unit vectors, $\omega_i > 0$, $\mu_i > -1$, $i = 0, 1, \ldots, n$, are real numbers and M_i, $i = 0, 1, \ldots, n$, are matrices given by Eq. (2). The matrix $M_s = \sum_{i=0}^{n} s_i M_i$, where the coefficients satisfy $s_i \geq 0$ and $\sum_{i=0}^{n} s_i > 0$, is nonsingular.*

Besides more degrees of freedom for shape editing, one main difference between matrix weighted NURBS curves and traditional NURBS curves is that the former ones have almost circular precision while the latter have only linear precision.

Proposition 2.2 ([7]). *Assume that $k > 2$ is an integer and $l = \lfloor \frac{k-1}{2} \rfloor$, where $\lfloor \rfloor$ represents the integer part of a number. Let $\lambda_{i+j} = N_{i+j,k}(t_{i+\frac{k}{2}})$, $j = \pm 1, \pm 2, \ldots, \pm l$, where $N_{i,k}(t)$ are the uniform B-splines of order k and $t_{i+\frac{k}{2}} = \frac{1}{2}(t_i + t_{i+k})$. Suppose that L is the line that passes through point P_i with direction $\mathbf{n}_i$ on a plane, and points P_{i+j} and normals $\mathbf{n}_{i+j}$, $j = \pm 1, \ldots, \pm l$ symmetrically lie on two sides of the line L on the plane. If the weights μ_{i+j}, $j = \pm 1, \pm 2, \ldots, \pm l$, are all chosen as*

$$\mu_i = \frac{\sum_{|j|=1}^{l} \lambda_{i+j} \mathbf{n}_i^T (P_i - P_{i+j})}{\sum_{|j|=1}^{l} \lambda_{i+j} \mathbf{n}_i^T \mathbf{n}_{i+j} \mathbf{n}_{i+j}^T (P_{i+j} - P_i)}, \quad (3)$$

the matrix weighted NURBS curve given by Eq. (1) with weight matrices $M_i = I + \mu_i \mathbf{n}_i \mathbf{n}_i^T$ passes through point P_i at $t = t_{i+\frac{k}{2}}$.

Proposition 2.2 states that a matrix weighted NURBS curve with local symmetric point-normal control pairs can pass through the middle control point. If all control points and control normals are uniformly sampled from a circular arc the obtained matrix weighted NURBS curve can interpolate all its control points (except for a few points near the boundaries of an open curve) when the weight matrices are computed by Eq. (2) with all $\omega_i = 1$ and μ_i given by Eq. (3). On another hand, any smooth planar curve can be approximated by smoothly connected circular arcs within arbitrary precision [28]. Therefore, matrix weighted NURBS curves constructed by Eqs. (1), (2) and (3) can approximate an arbitrary smooth planar curve very well when the control points and control normals are properly sampled from the original curve.

Though a matrix weighted NURBS curve constructed from sampled or given data can usually approximate the input points closely, the naively evaluated parameter μ_i may vary significantly and the obtained curve may not be fair when the original points are non-uniformly spaced or noisy. To construct matrix weighted NURBS curves as fair as possible, the parameter μ_i should be evaluated robustly against data noise. Practically, we modify Eq. (3) and compute μ_i as follows

$$\mu_i = \frac{\sum_{|j|=1}^{l} \lambda_{i+j} d_{ij}}{\sum_{|j|=1}^{l} \lambda_{i+j} \mathbf{n}_i^T \mathbf{n}_{i+j} d_{ij}}, \quad (4)$$
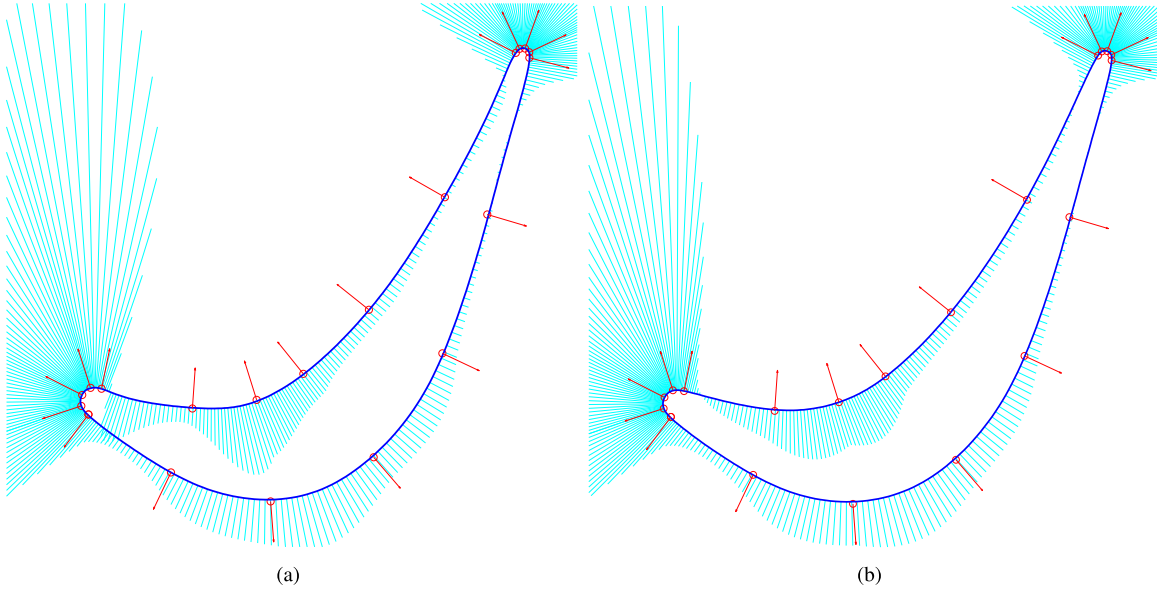
Fig. 1. Matrix weighted rational B-spline curves of degree 3 constructed from point-normal control pairs with (a) naive or (b) robust computation of the parameters μ_i within the weight matrices.

where $d_{ij} = \frac{1}{2}[\mathbf{n}_i^T(P_i - P_{i+j}) + \mathbf{n}_{i+j}^T(P_{i+j} - P_i)]$ when $\frac{1}{2}|(\mathbf{n}_i - \mathbf{n}_{i+j})^T(P_i - P_{i+j})| > \eta$ and $d_{ij} = \eta$ otherwise. The parameter η can be chosen a small value such as $\eta = 0.001$ so that μ_i can still be computed robustly even when the points and normals are sampled from a straight line. We note that μ_i computed by Eq. (4) is the same as that by Eq. (3) when the original data are uniformly sampled from a circular arc. Therefore, matrix weighted NURBS curves computed by Eqs. (1), (2) and (4) still have the almost circular precision but are less sensitive to data noise.

Fig. 1 illustrates two uniform matrix weighted rational B-spline curves of degree 3 that are constructed using the same set of control points and control normals. The weight matrices for the two curves are computed by Eq. (2) with μ_i given by Eq. (3) or (4), respectively. The curvature plots show that robust computation of the parameters μ_i within weight matrices can help to construct even higher quality curves. In the rest of the paper matrix weighted NURBS curves with point-normal control pairs will be constructed in this novel way without special declaration.

3. Matrix weighted NURBS curves with point-tangent control pairs

In addition to shape control using control points and control normals, the shapes of curves in 3D or higher dimensional spaces can be controlled more accurately using control points and tangent lines specified at the control points. In this section, we propose matrix weighted NURBS curves with point-tangent control pairs which are particularly useful for curve modeling and data fitting in 3D or higher dimensional spaces.

3.1. Construction of matrix weighted NURBS curves with point-tangent control pairs

Suppose that $P_i \in \mathbb{R}^d$, $i = 0, 1, \ldots, n$, are a set of given points and $\mathbf{t}_i \in \mathbb{R}^d$, $i = 0, 1, \ldots, n$, are a set of unit tangent vectors specified at the points. We construct a matrix weighted NURBS curve as the solution to a least squares fitting to the given points and the tangent lines that pass through the points.

Let L_i be the line that passes through point P_i with tangent direction $\mathbf{t}_i$. For an arbitrary point Q in space we assume Q_i is the

perpendicular foot of Q onto the line L_i. The vector pointing from point Q_i to Q is obtained as

$$Q - Q_i = (I - \mathbf{t}_i\mathbf{t}_i^T)(Q - P_i).$$

Denote by $A_i = I - \mathbf{t}_i\mathbf{t}_i^T$. It is verified that $A_i^T = A_i$ and $A_i^2 = A_i$. The squared distance from Q to line L_i is $(Q - Q_i)^2 = (Q - P_i)^T A_i^2 (Q - P_i)$.

Suppose that $N_{i,k}(t)$, $i = 0, 1, \ldots, n$, are the B-splines of order k defined on the knot vector $\mathbf{t} = \{t_0, t_1, \ldots, t_{n+k}\}$. We construct a matrix weighted NURBS curve that fits the given points and the tangent lines by minimizing the following functional

$$F(Q(t)) = \sum_{i=0}^{n} \omega_i N_{i,k}(t)[Q(t) - P_i]^2 \\ + \sum_{i=0}^{n} \omega_i \mu_i N_{i,k}(t)[Q(t) - P_i]^T A_i^2 [Q(t) - P_i]. \tag{5}$$

We show that with proper choices of the parameters ω_i and μ_i the functional $F(Q(t))$ is convex and has a unique minimizer. It is also shown that the obtained matrix weighted NURBS curve is valid over the whole parameter domain.

Proposition 3.1. *Suppose real numbers $\omega_i > 0$, $i = 0, 1, \ldots, n$, and $\mu_i > -1$, $i = 0, 1, \ldots, n$. The functional defined by Eq. (5) is convex and the minimizer to the functional is a valid matrix weighted NURBS curve over the parameter domain.*

Proof. To prove that the functional given by Eq. (5) is convex, we first reformulate the functional as

$$F(Q(t)) = \sum_{i=0}^{n} \omega_i N_{i,k}(t)\{[Q(t) - P_i]^2 + \mu_i[A_i(Q(t) - P_i)]^2\}.$$

Notice that

$$\begin{aligned} &[Q(t) - P_i]^2 + \mu_i[A_i(Q(t) - P_i)]^2 \\ &> [Q(t) - P_i]^2 - [A_i(Q(t) - P_i)]^2 \\ &= [Q(t) - P_i]^T(I - A_i^2)[Q(t) - P_i] \\ &= [Q(t) - P_i]^T \mathbf{t}_i\mathbf{t}_i^T[Q(t) - P_i] \\ &= \{[Q(t) - P_i]^T \mathbf{t}_i\}^2 \\ &\geq 0. \end{aligned}$$

Therefore, the functional $F(Q(t)) > 0$ for any $t \in [t_{k-1}, t_{n+1}]$, which implies that the functional $F(Q(t))$ is convex and has a unique minimizer in the domain.

To minimize the functional $F(Q(t))$, the curve $Q(t)$ is determined by solving the following equation

$$
\begin{aligned}
0 &= \frac{1}{2} \frac{\partial F(Q(t))}{\partial Q(t)} \\
&= \sum_{i=0}^{n} \omega_i N_{i,k}(t) \{ [Q(t) - P_i] + \mu_i A_i^T A_i [Q(t) - P_i] \} \\
&= \sum_{i=0}^{n} \omega_i N_{i,k}(t) \{ [Q(t) - P_i] + \mu_i A_i [Q(t) - P_i] \}.
\end{aligned}
\tag{6}
$$

The solution to Eq. (6) is

$$
Q(t) = \left[\sum_{i=0}^{n} M_i N_{i,k}(t) \right]^{-1} \sum_{i=0}^{n} M_i P_i N_{i,k}(t), \ t \in [t_{k-1}, t_{n+1}],
\tag{7}
$$

where

$$
M_i = \omega_i (I + \mu_i A_i), \quad i = 0, 1, \ldots, n.
\tag{8}
$$

We now prove that the matrix function $M(t) = \sum_{i=0}^{n} M_i N_{i,k}(t)$, $t \in [t_{k-1}, t_{n+1}]$, is nonsingular. It follows that the obtained matrix weighted NURBS curve given by Eq. (7) is valid. For ease of description, we should only prove that the matrix $M_s = \sum_{i=0}^{n} s_i M_i$ is nonsingular when $s_i \geq 0$, $i = 0, 1, \ldots, n$ and $\sum_{i=0}^{n} s_i > 0$. Substituting $A_i = I - \mathbf{t}_i \mathbf{t}_i^T$ into Eq. (8), the matrix M_s can be reformulated as $M_s = \sum_{i=0}^{n} s_i \omega_i [(1+\mu_i)I - \mu_i \mathbf{t}_i \mathbf{t}_i^T]$. Let $\bar{s}_i = s_i(1+\mu_i)$ and $\bar{\mu}_i = \frac{-\mu_i}{1+\mu_i}$. It yields that $M_s = \sum_{i=0}^{n} \bar{s}_i \omega_i (I + \bar{\mu}_i \mathbf{t}_i \mathbf{t}_i^T)$. Since $s_i \geq 0$ and $\mu_i > -1$, it follows that $\bar{s}_i \geq 0$ and $\bar{\mu}_i > -1$. Because $\sum_{i=0}^{n} s_i > 0$, we have $\sum_{i=0}^{n} \bar{s}_i > 0$. According to Theorem 2.1, the matrix $M_s = \sum_{i=0}^{n} s_i M_i$ is nonsingular. The theorem is proven. □

In a plane, the matrix weighted NURBS curves defined by point-tangent control pairs are just the curves defined by control points and normal vectors specified at the control points. But in a higher dimensional space a matrix weighted NURBS curve defined by point-tangent control pairs can be manipulated more accurately than a curve defined by point-normal control pairs.

Proposition 3.2. *Suppose that $Q(t)$ is a matrix weighted NURBS curve with point-tangent control pairs as defined by Eq. (7). The curve segment $Q(t)$, $t \in (t_i, t_{i+k})$, lies close to the line that passes through point P_i with tangent vector $\mathbf{t}_i$ when the parameter μ_i approaches infinity. If ω_i approaches infinity, the points on $Q(t)$ approach the control point P_i.*

Proof. From functional (5) we know that if μ_i approaches infinity the term $N_{i,k}(t)[A_i(Q(t) - P_i)]^2$ will approach zero for any $t \in (t_i, t_{i+k})$ when the functional $F(Q(t))$ is minimized. As $[A_i(Q(t) - P_i)]^2$ approaches zero, the curve $Q(t)$ lies close to the line that passes through point P_i with tangent $\mathbf{t}_i$. Similarly, if ω_i approaches infinity, the term $[Q(t) - P_i]^2$ with $t \in (t_i, t_{i+k})$ will approach zero and the point on curve $Q(t)$ will approach the control point P_i. □

3.2. Almost helical precision of matrix weighted NURBS curves with point-tangent control pairs

Similar to planar matrix weighted NURBS curves that have almost circular precision, matrix weighted NURBS curves with point-tangent control pairs in 3D space have almost helical precision. In the remaining part of this section we assume that $d = 3$ and show that matrix weighted rational B-spline curves can pass through their control points when the point-tangent control pairs are sampled from a cylinder helix in 3D space.

Proposition 3.3. *Suppose that $(P_i, \mathbf{t}_i)$, $i = 0, 1, \ldots, n$, are points and tangents uniformly sampled from a helix with constant curvature and torsion. Assume that $k > 2$ is an integer and $l = \lfloor \frac{k-1}{2} \rfloor$, where $\lfloor \rfloor$ represents the integer part of a number. Suppose that q is an arbitrary integer satisfying $l \leq q \leq n - l$. Let $\lambda_{q+j} = N_{q+j,k}(t_{q+\frac{k}{2}})$, $j = \pm 1, \pm 2, \ldots, \pm l$, where $N_{i,k}(t)$ are the uniform B-splines of order k and $t_{q+\frac{k}{2}} = \frac{1}{2}(t_q + t_{q+k})$. Let*

$$
\mu_q = \frac{\sum_{|j|=1}^{l} \lambda_{q+j} \mathbf{n}_q^T (P_q - P_{q+j})}{\sum_{|j|=1}^{l} \lambda_{q+j} \mathbf{n}_q^T A_{q+j} (P_{q+j} - P_q)},
\tag{9}
$$

where $A_{q+j} = I - \mathbf{t}_{q+j} \mathbf{t}_{q+j}^T$ and $\mathbf{n}_q$ is an arbitrary vector perpendicular to $\mathbf{t}_q$. If the weight matrices M_i are computed by Eq. (8) with all $\omega_i = 1$ and $\mu_{q+j} = \mu_q$, $j = \pm 1, \pm 2, \ldots, \pm l$, the matrix weighted rational B-spline curve with point-tangent control pairs $(P_i, \mathbf{t}_i)$ passes through point P_q at $t = t_{q+\frac{k}{2}}$.

Proof. Suppose that $Q(t)$ is a matrix weighted rational B-spline curve with point-tangent control pairs $(P_i, \mathbf{t}_i)$. From Eq. (6) we know that any point on the curve $Q(t)$ satisfies

$$
\sum_{i=0}^{n} \omega_i N_{i,k}(t)[Q(t) - P_i] + \sum_{i=0}^{n} \mu_i \omega_i N_{i,k}(t) A_i [Q(t) - P_i] = 0.
$$

Suppose that the matrix weighted rational B-spline curve $Q(t)$ passes through point P_q at $t = t_{q+\frac{k}{2}}$, it requires that $Q(t_{q+\frac{k}{2}}) = P_q$. Under the assumption that all $\omega_i = 1$ and μ_{q+j}, $j = \pm 1, \pm 2, \ldots, \pm l$, have the same value, namely μ_q, we have

$$
\sum_{|j|=1}^{l} \lambda_{q+j}(P_q - P_{q+j}) + \mu_q \sum_{|j|=1}^{l} \lambda_{q+j} A_{q+j}(P_q - P_{q+j}) = 0.
\tag{10}
$$

Dotting either side of Eq. (10) by $\mathbf{n}_q$, we have the expression of μ_q given by Eq. (9).

Denote by Q_{q+j} the perpendicular foot of point P_q onto the line that passes through point P_{q+j} with tangent $\mathbf{t}_{q+j}$. Based on the definition of perpendicular feet of point P_q onto lines that pass through neighboring points, we have $A_{q+j}(P_q - P_{q+j}) = P_q - Q_{q+j}$, $j = \pm 1, \pm 2, \ldots, \pm l$. Because $\lambda_{q+j} = \lambda_{q-j}$ for $j = 1, 2, \ldots, l$, Eq. (10) can be reformulated as

$$
\sum_{j=1}^{l} \lambda_{q+j}(2P_q - P_{q+j} - P_{q-j})
$$
$$
+ \mu_q \sum_{j=1}^{l} \lambda_{q+j}(2P_q - Q_{q+j} - Q_{q-j}) = 0.
$$

Under the assumption that point-tangent pairs $(P_i, \mathbf{t}_i)$ are uniformly sampled from a cylinder helix, it is verified that all vectors $2P_q - P_{q+j} - P_{q-j}$, $2P_q - Q_{q+j} - Q_{q-j}$, $j = 1, 2, \ldots, l$, are parallel to the normal vector of the helix at point P_q. Therefore, Eq. (10) holds when the parameter μ_q is given by Eq. (9). This implies that the matrix weighted rational B-spline curve with point-tangent control pairs that are sampled from a helix can interpolate its control points. The proposition is proven. □

Similar to Eq. (4) of which the parameter is evaluated for robust fitting of planar noisy data, Eq. (9) can also be modified for robust fitting of spatial data that may be non-uniformly spaced or noisy. From the proof of Proposition 3.3 we know that $A_q(P_q - P_{q+j})$ and $A_{q+j}(P_{q+j} - P_q)$ are the vector pointing from P_{q+j} to the line that passes through P_q with tangent direction $\mathbf{t}_q$ or the vector pointing from P_q to the line that passes through P_{q+j} with tangent direction $\mathbf{t}_{q+j}$. Let $V_q = \frac{A_q(P_q - P_{q+j})}{\|A_q(P_q - P_{q+j})\|}$ and $V_{q+j} = \frac{A_{q+j}(P_{q+j} - P_q)}{\|A_{q+j}(P_{q+j} - P_q)\|}$. The parameter
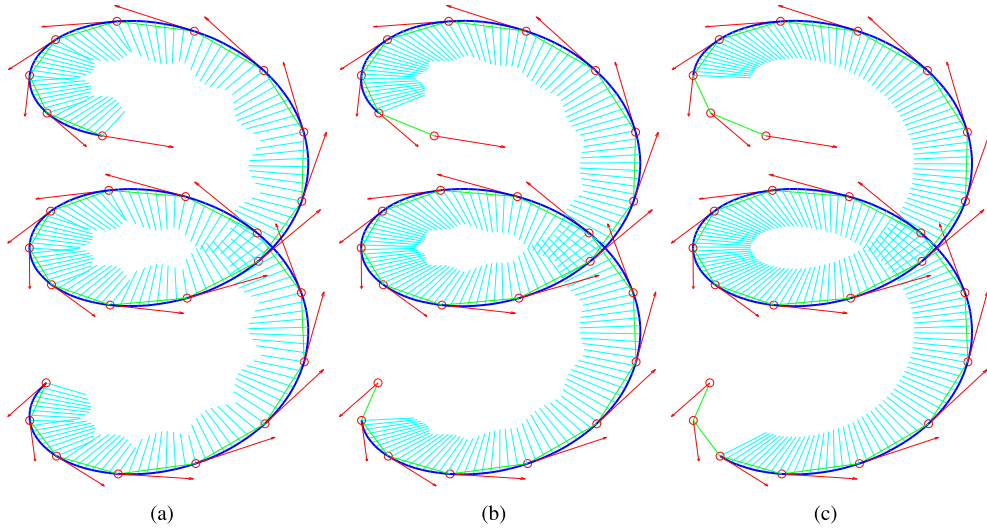
Fig. 2. Matrix weighted rational B-spline curves with control points and control tangents sampled from a helix: (a) degree 1; (b) degree 3; (c) degree 5.

μ_q for the computation of the weight matrix at point P_q is now given by

$$\mu_q = \frac{\sum_{|j|=1}^{l} \lambda_{q+j} d_{q,q+j} \eta_{q,q+j}}{\sum_{|j|=1}^{l} \lambda_{q+j} d_{q,q+j} \mathbf{n}_q^T V_{q+j}}, \tag{11}$$

where $\eta_{q,q+j} = \frac{\mathbf{n}_q^T(P_q-P_{q+j})}{V_q^T(P_q-P_{q+j})}$ and $d_{q,q+j} = \frac{1}{2}(\|A_q(P_q - P_{q+j})\| + \|A_{q+j}(P_{q+j}-P_q)\|)$. If the points and tangents are uniformly sampled from a cylinder helix with enough density, we have $A_q(P_q-P_{q+j}) = A_{q+j}(P_{q+j} - P_q) > 0$. At this time, both Eqs. (9) and (11) give the same value of the parameter. In other cases, especially when the points and tangents are non-uniformly spaced or noisy, the values computed by Eq. (11) are less sensitive to noise than those obtained by Eq. (9).

Fig. 2 illustrates an example of matrix weighted NURBS curves of various degrees with point-tangent control pairs sampled from a helix segment. Suppose that the original helix segment is given by

$$\begin{cases} x(t) = 1.5 \cos(t), \\ y(t) = 1.5 \sin(t), \qquad t \in [0, 4.5\pi]. \\ z(t) = 0.4t, \end{cases}$$

A sequence of points P_i and unit tangents $\mathbf{t}_i$ are sampled from the curve at $t_0 = 0$, $t_i = t_{i-1} + 0.18\pi$, $i = 1, 2, \ldots, 25$. We construct uniform matrix weighted rational B-spline curves of degree 1, 3 or 5 from the sampled points and tangents. In particular, the weight matrices are computed by Eq. (8) with μ_i given by Eq. (9). As a result, the obtained matrix weighted rational B-spline curves with various degrees all pass through their control points; see Fig. 2. From the figure we also know that the *curvature normal* (normal vector scaled by curvature) of the matrix weighted rational B-spline curve of degree 1 is not continuous and the curvature normal of the matrix weighted rational B-spline curve of degree 3 is continuous but not fair. The curvature normal of the matrix weighted rational B-spline curve of degree 5 resembles that of the original helix very well.

4. Algorithms for fitting or fairing Hermite-type data by matrix weighted NURBS curves

As stated in the last two sections, matrix weighted NURBS curves with point-normal or point-tangent control pairs have nice novel properties compared with conventional NURBS curves. This

makes them efficient and powerful tools for curve fitting and curve fairing. In this section we propose practical algorithms for fitting or fairing sequences of points together with normal vectors or tangents by matrix weighted NURBS curves. The algorithms are based on the following facts or observations.

- A matrix weighted NURBS curve can pass through its control points exactly when the point-normal control pairs or point-tangent control pairs are uniformly sampled from a circular arc or a helix segment.
- A smooth planar or spatial curve can be approximated with any high accuracy by smoothly connected circular arcs [28] or by a set of smoothly connected helix segments [29,30], respectively.
- Matrix weighted NURBS curves with the same set of point-tangent control pairs but with higher degrees are usually much fairer than those with lower ones.

From the first two facts we know that a smooth planar or spatial curve can be approximated efficiently by a matrix weighted NURBS curve that is constructed by properly sampled points and normals or tangents from the original curve. Usually, even higher accuracy approximating curves can be obtained when many more points and normals or tangents have been sampled from the original smooth curve. The third observation comes true because both the supports and the continuity orders of high degree B-splines are larger than those of lower ones and a matrix weighted NURBS curve is just the solution to a least squares fitting with coefficients given by the B-splines. It should be pointed out that high degree NURBS can be used to model fair curves and surfaces [31]. Besides modeling fair shapes, high degree matrix weighted NURBS curves can even be used to fair noisy data efficiently.

In the following we present algorithms for curve fitting and fairing based on point-tangent pairs. Fitting or fairing point-normal pairs can be implemented in the same way. Suppose that P_i, $i = 0, 1, \ldots, n$, are the sampled points and $\mathbf{t}_i$, $i = 0, 1, \ldots, n$, are the sampled or the estimated tangents at the points. A closed matrix weighted NURBS curve will be constructed when $P_0 = P_n$ and $\mathbf{t}_0 = \mathbf{t}_n$; otherwise, an open matrix weighted NURBS curve will be obtained. Particularly, by choosing monotone increasing knots t_i, $i = 0, 1, \ldots, n + 2k - 2$, that satisfy $t_n - t_0 = t_{n+1} - t_1 = \cdots = t_{n+2k-2} - t_{2k-2}$, a closed matrix weighted NURBS curve of order k

is obtained as

$$P(t) = \left[\sum_{i=0}^{n+k-2} M_{i\%n} N_{i,k}(t) \right]^{-1} \sum_{i=0}^{n+k-2} M_{i\%n} P_{i\%n} N_{i,k}(t),$$
$$t \in [t_{k-1}, t_{n+k-1}],$$

where M_i are the weight matrices computed by Eq. (8) with default choices of $\omega_i = 1$ and μ_i given by Eq. (11). To construct fair curves, uniform B-splines $N_{i,k}(t)$ are used as the basis functions.

An open matrix weighted NURBS curve does not interpolate its boundary control points except for $k = 2$ or multiple knots are used at the ends. To construct a uniform matrix weighted NURBS curve of order $k\,(k > 2)$ that interpolates points P_0, P_n and tangents $\mathbf{t}_0, \mathbf{t}_n$ at the points, we propose to add local symmetric points before point P_0 or after point P_n as extended control points of the fitting curve. Let $l = \lfloor (k-1)/2 \rfloor$. We add control points before P_0 by reflecting points $P_j, j = 1, \ldots, l$, with respect to the plane that passes through point P_0 with normal vector $\mathbf{t}_0$. The added points are $P_{-j} = P_j - 2a_j\mathbf{t}_0$, where $a_j = (P_j - P_0) \cdot \mathbf{t}_0, j = 1, \ldots, l$. For a space curve, the added points can be $P_{-j} = P_j - 2a_j\mathbf{t}_0 - 2b_j\mathbf{b}_0$, where $b_j = (P_j - P_0) \cdot \mathbf{b}_0, j = 1, \ldots, l$, when the binormal vector $\mathbf{b}_0$ at point P_0 is available. In the same way, we add points P_{n+j}, $j = 1, \ldots, l$, after point P_n by reflecting points P_{n-j} with respect to the planes that pass through point P_n with normal vectors $\mathbf{t}_n$ or $\mathbf{b}_n$. The tangents at the added points can be obtained by reflecting the original corresponding tangents or by estimating from the extended control polygon. Finally, a matrix weighted NURBS curve with point-tangent control pairs $(P_i, \mathbf{t}_i), i = -l, -l+1, \ldots, n+l$, is obtained as

$$P(t) = \left[\sum_{i=-l}^{n+l} M_i N_{i,k}(t) \right]^{-1} \sum_{i=-l}^{n+l} M_i P_i N_{i,k}(t). \qquad (12)$$

From Proposition 3.3 we know that a uniform matrix weighted rational B-spline curve $P(t)$ given by Eq. (12) interpolates points P_0 and P_n at the boundaries.

For convenience of implementation, the point-tangent control pairs of a fitting curve $P(t)$ are renumbered as $(P'_i, \mathbf{t}'_i), i = 0, \ldots, n'$, where $n' = n + k - 2$ when $P(t)$ is closed or $n' = n + 2l$ otherwise. The algorithm steps for fitting points and tangents by a matrix weighted NURBS curve are given as follows.

Algorithm 1. Fitting point-tangent pairs by a matrix weighted NURBS curve

---

input: point-tangents $(P_i, \mathbf{t}_i), i = 0, 1, \ldots, n$ and order k
output: a matrix weighted NURBS curve $P(t)$ of order k
  1. Compute and set the point-tangent control pairs $(P'_i, \mathbf{t}'_i)$, $i = 0, 1, \ldots, n'$;
  2. Choose the knot vector $\{t_0, t_1, \ldots, t_{n'+k}\}$;
  3. Compute parameter μ_i and matrix weight $M_i, i = 0, 1, \ldots, n'$;
  4. Construct a matrix weighted NURBS curve with control points P'_i and weights $M_i, i = 0, 1, \ldots, n'$;
  5. Output curve $P(t), t \in [t_{k-1}, t_{n'+1}]$.

Just like data fitting by B-spline curves, a matrix weighted NURBS curve that fits a sequence of points and tangents may not be fair when the points and tangents contain noise or the point-tangent pairs are sampled with highly irregular steps. To obtain a fair curve fitting the original data, the initially obtained curve should be refined. Particularly, we propose to sample points together with or without tangents from the previously obtained curves and construct new matrix weighted NURBS curves using the sampled data. Since matrix weighted NURBS curves have almost circular or helical precision, not only the repeatedly constructed curves from the resampled data become more fair but also the obtained curves still lie close to the original data after considerable number of iterations.
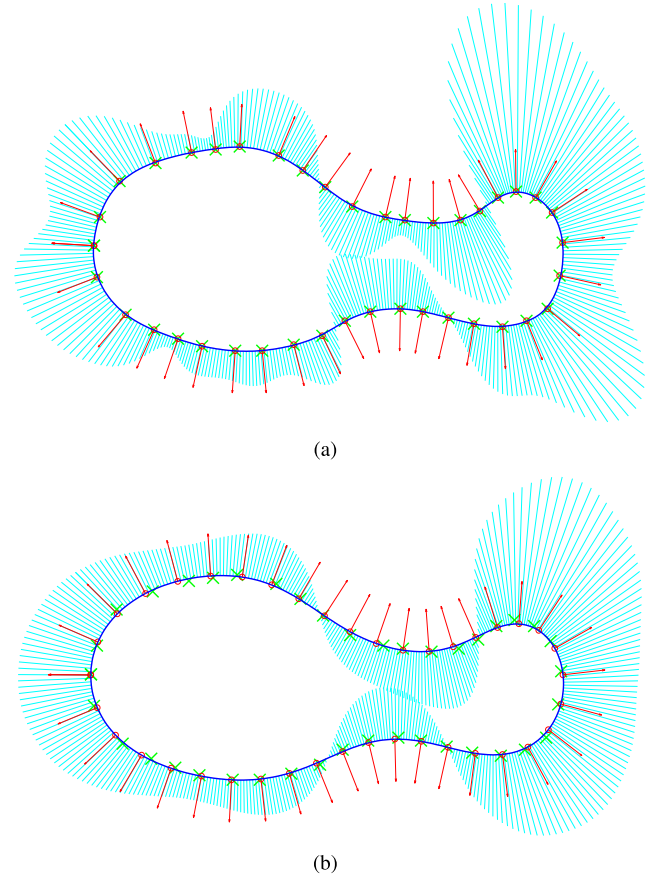

(a)

(b)

Fig. 3. Fitting and fairing noisy points by matrix weighted rational B-spline curves: (a) a quintic matrix weighted rational B-spline curve constructed from the given points and the estimated normals; (b) the fair fitting curve after 10 times of point resampling and curve refitting. The crosses '×' denote the input points and the circles '○' denote the control points of the fitting curves.

Suppose that an initial matrix weighted NURBS curve is

$$P(t) = \left[\sum_{i=0}^{m} M_i N_{i,k}(t) \right]^{-1} \sum_{i=0}^{m} M_i P_i N_{i,k}(t)$$

with knot vector $\mathbf{t} = \{t_0, t_1, \ldots, t_{m+k}\}$. We sample points from the curve at knots $t_{k-1}, t_k, \ldots, t_{m+1}$. If $P(t)$ is an open curve, the tangents and binormals at boundary points $P(t_{k-1})$ or $P(t_{m+1})$ are also computed and the sampled points are extended on both sides by adding l reflected points with respect to each boundary point. Except for points with fixed tangents, the tangents at all sampled or added points should be estimated adaptively. We compute the tangent at each intermediate point as the bisector of the turning angle at the point. The tangent at a boundary point of an open polygon is computed by the osculating arc that interpolates two ends of the boundary leg and the known tangent at another end of the leg. The tangents computed in this way are robust against data noise and can always be used for constructing fair fitting curves efficiently. The main algorithm steps for fairing Hermite-type data are as follows.
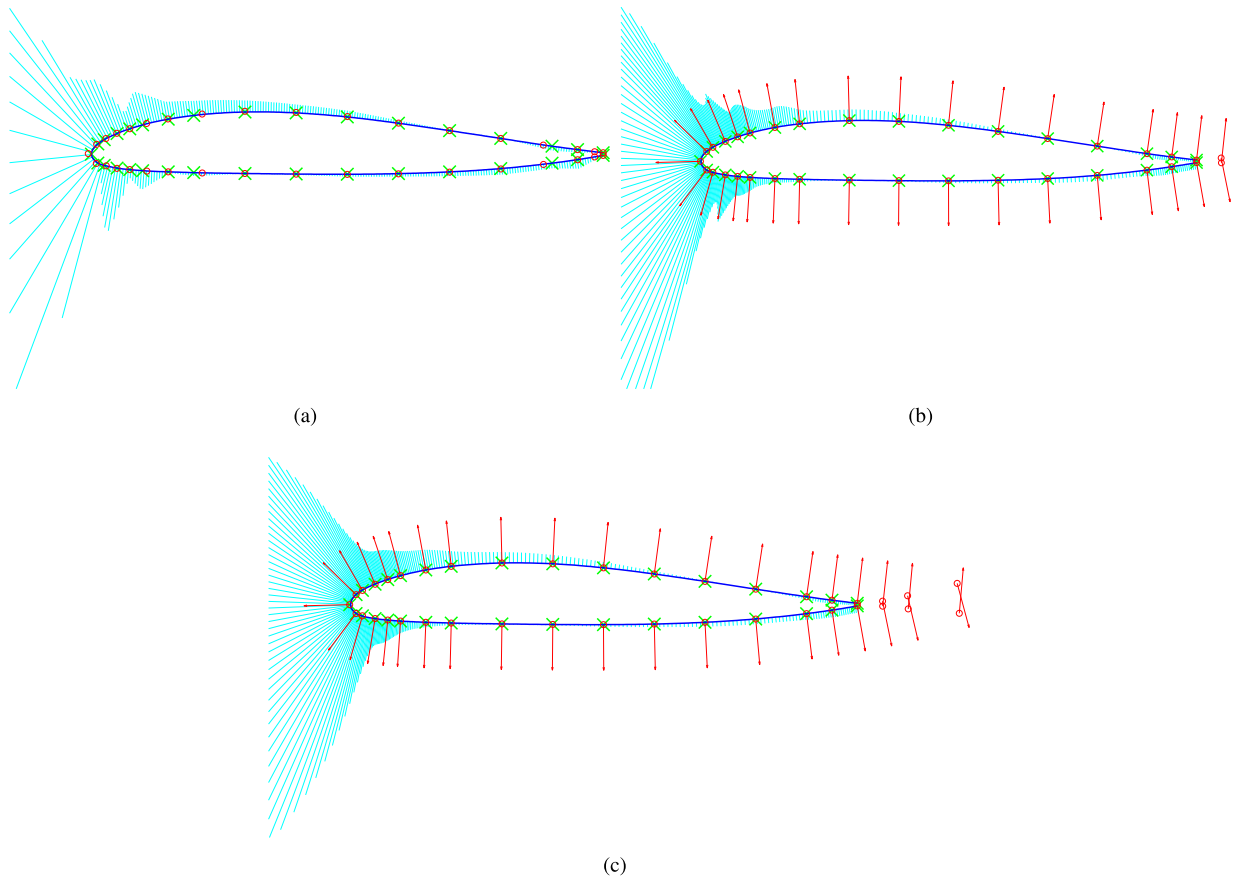
(a)

(b)

(c)

Fig. 4. Fitting a sequence of planar points by B-spline or matrix weighted NURBS curves: (a) an interpolating cubic B-spline curve; (b) matrix weighted rational B-spline curve of degree 3; (c) matrix weighted rational B-spline curve of degree 7. The crosses '×' denote the input points and the circles 'o' denote the control points of the interpolating B-spline or the fitting matrix weighted NURBS curves.

Algorithm 2. Fairing point-tangent pairs by repeated fitting of matrix weighted NURBS curves

input: points P_i, $i = 0, 1, \ldots, n$, initial fitting curve $P(t)$, and the maximum iteration number K
output: a fair matrix weighted NURBS curve $Q(t)$
1. Sample points from curve $P(t)$ at the selected knots;
2. Add local symmetric points at the boundaries;
3. Compute tangent vectors at all sampled or added points;
4. Construct a matrix weighted NURBS curve $Q(t)$ with the same degree of $P(t)$ from the points and tangents;
5. Replace $P(t)$ by $Q(t)$ and repeat steps 1–4 until the iteration number reaches K;
6. Output curve $Q(t)$.

Fig. 3(a) illustrates a uniform matrix weighted rational B-spline curve of degree 5 that is constructed from a set of noisy points and unit normals estimated at the points. After 10 times of point resampling and curve refitting by the proposed algorithm, a fair curve that lies close to the original points is obtained; see Fig. 3(b). It is also clearly illustrated that the original irregularly spaced points have become more uniform after fairing.

We note that specific requirements such as interpolation of selected non-boundary points or having specified tangent directions near selected points can be achieved by minor modification of the proposed fitting or fairing algorithms. If an intermediate point P_q is to be interpolated by the fitting curve, one can refine points $P_{q\pm j}$, $j = 1, 2, \ldots, l$, such that these points are symmetrically lying on two sides of the plane that passes through P_q with normal vector $\mathbf{t}_q$. The obtained matrix weighted NURBS curve will have approximate tangent direction $\mathbf{t}_q$ near point P_q just by increasing the value of parameter μ_q.

5. Experimental examples

In this section we present several interesting examples to show how to fit and fair Hermite-type data by matrix weighted NURBS curves in 2D or 3D space. Comparisons with B-spline curve fitting will also be given and discussed.

First, we fit points sampled from an airfoil profile by spline curves. Fig. 4(a) illustrates a non-uniform cubic B-spline curve that interpolates 33 sampled points at knots with chord-length parametrization of the points [2]. The curve is visually smooth and has continuous curvature. However, the curve is not fair since there are several unnecessary curvature undulations or extremes. By estimating normal vectors for all initial points and adding two symmetric points near the boundaries, a matrix weighted uniform rational B-spline curve of degree 3 is obtained from the given and the added points directly; see Fig. 4(b). From the figure we can see that the obtained curve fits the original points very well, but the curvature plot shows that the matrix weighted rational B-spline curve of degree 3 is still not fair enough. Fig. 4(c) shows that a higher order matrix weighted rational B-spline curve with point-normal control pairs can be more fair than lower order matrix weighted rational B-spline curves or integer B-spline curves.

Second, we present an example to show how matrix weighted rational B-spline curves can be used to fit and fair planar point sets. Fig. 5(a) illustrates a quintic B-spline curve that fits a set of points sampled from a rotor profile. From the figure we can see that the B-spline curve fits the points closely but the curvature of the curve is very sensitive to data noise. By estimating normal vectors at all initial points, a matrix weighted rational B-spline curve of degree 5 can be constructed from the points and normals
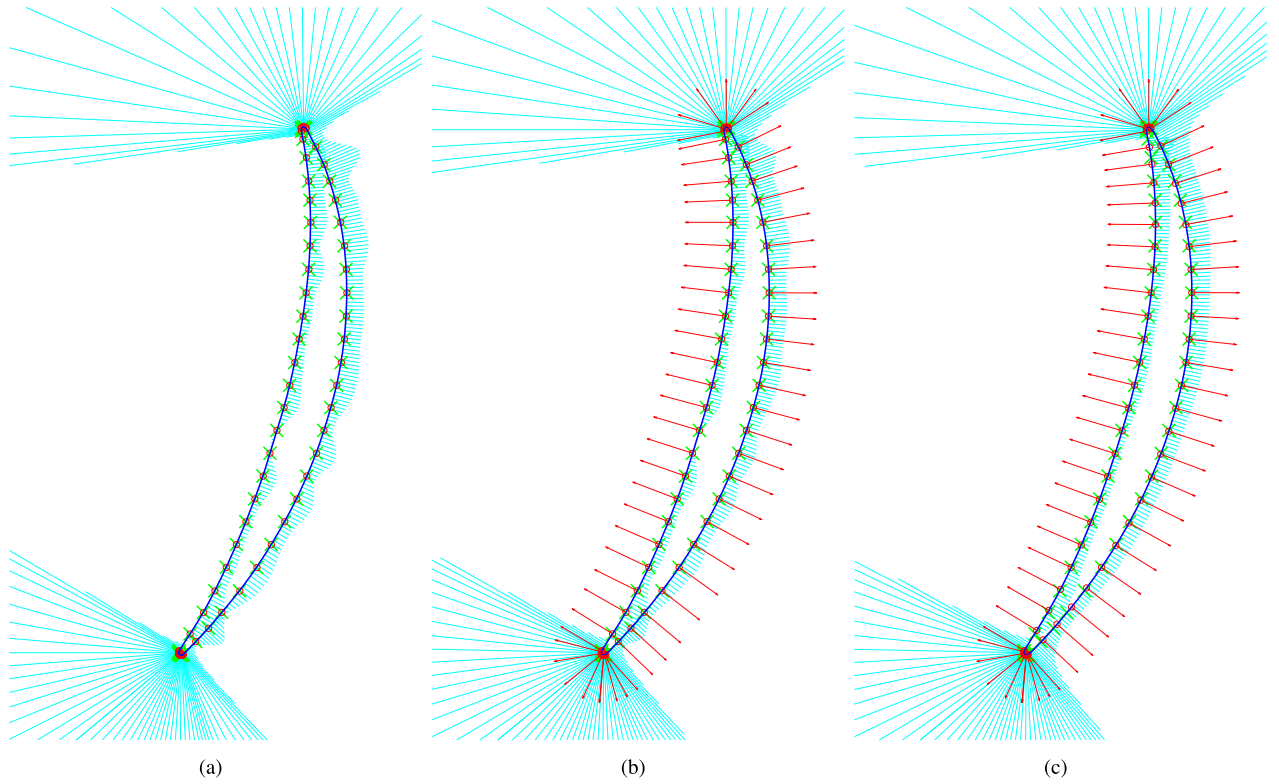
Fig. 5. Planar point set fitting or fairing by B-spline or matrix weighted NURBS curves: (a) a quintic B-spline curve fitting to the points; (b) a matrix weighted rational B-spline curve of degree 5 constructed from the given points and the estimated normals; (c) matrix weighted rational B-spline curve of degree 5 after 10 times of resampling and refitting. The crosses '×' denote the input points and the circles 'o' denote the control points of the fitting curves.
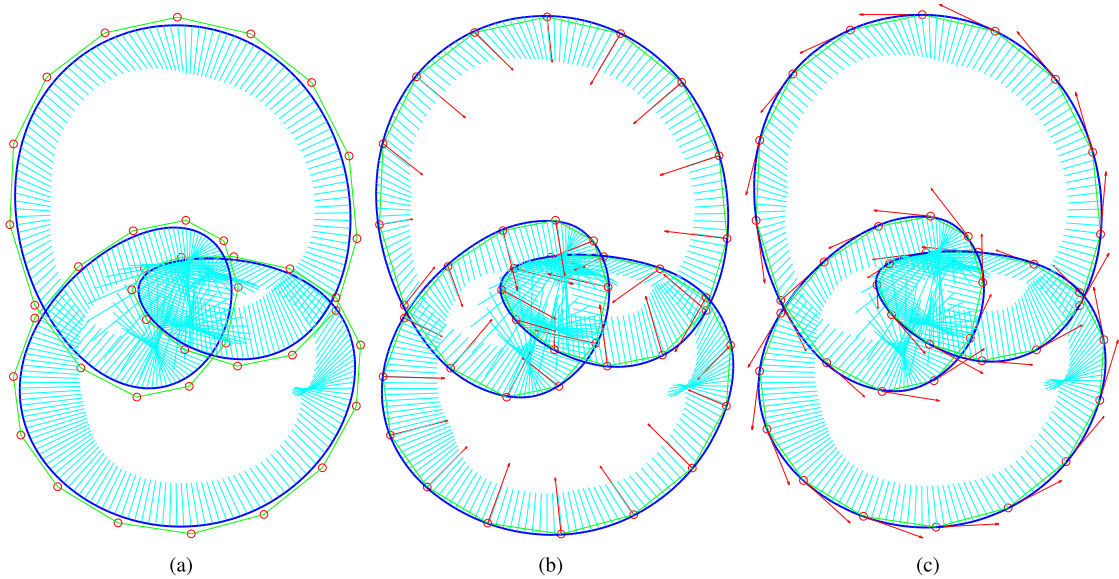

Fig. 6. Space curve modeling with the same set of control points: (a) a quintic B-spline curve; (b) a matrix weighted rational B-spline curve of degree 5 with point-normal control pairs; (c) a matrix weighted rational B-spline curve of degree 5 with point-tangent control pairs.

directly; see Fig. 5(b). Though the curvature plot shows that the matrix weighted rational B-spline curve is somewhat fairer than the B-spline curve, it still has unnecessary curvature undulations. After 10 iterations of data sampling and curve fitting by Algorithm 2, the finally obtained matrix weighted rational B-spline curve is fair enough; see Fig. 5(c). We note that the control points on the top or the bottom circular edges of the curve are kept unchanged after each time of data sampling just for the preservation of the salient features of the curve shape.

Third, we construct matrix weighted rational B-spline curves from a sequence of points, tangents or normals sampled from a closed space curve. Suppose that the original curve is given by

$$\begin{cases} x(t) = 0.15(\cos(2t) + \cos(4t)) + 1.5(\cos(t) + \cos(3t)), \\ y(t) = 0.9\sin(t) + 1.5\sin(3t), \\ z(t) = 0.6\sin(4t) - 0.3(\sin(6t) + \cos(6t)) + 0.3. \end{cases}$$

The curve $\mathbf{r}(t) = (x(t), y(t), z(t))^T$ is periodic and satisfies $\mathbf{r}(t + 2\pi) = \mathbf{r}(t)$. For a given number n, we sample n points together with
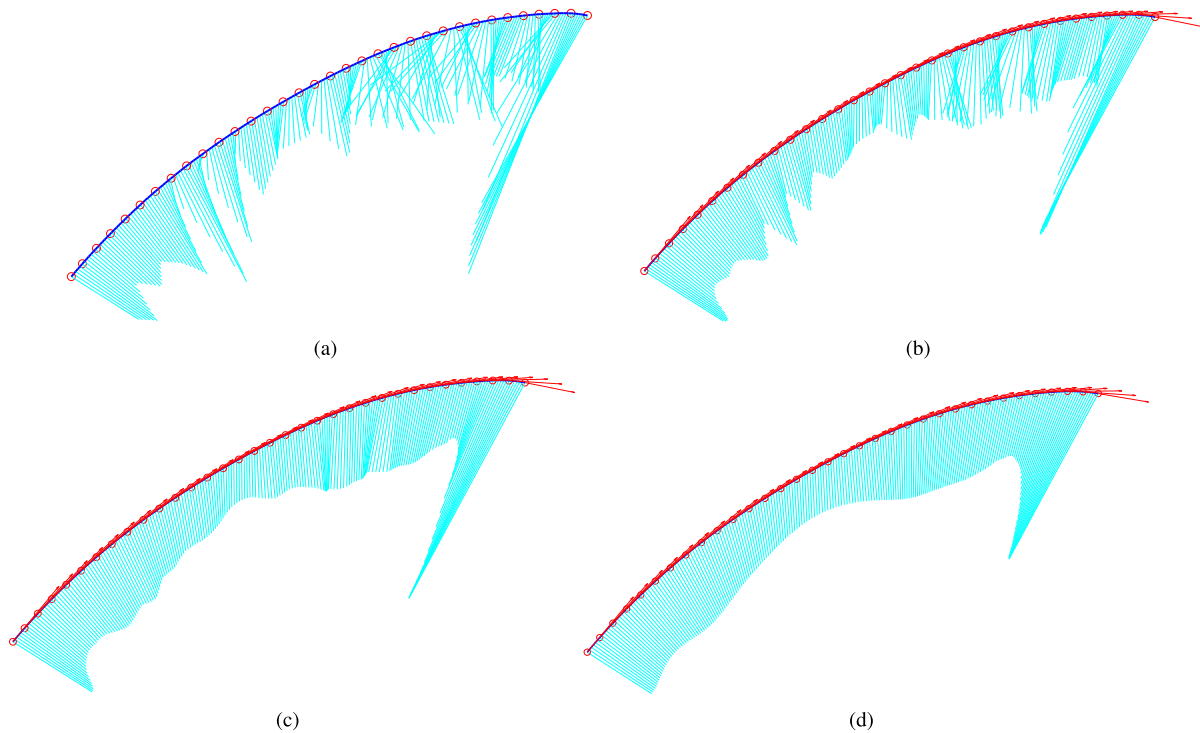
Fig. 7. Fitting and fairing Hermite-type data in 3D space: (a) a cubic B-spline curve interpolating the given points; (b) a matrix weighted rational B-spline curve of degree 3 constructed from the given points and the estimated tangents; (c) matrix weighted rational B-spline curve of degree 3 after 2 times of resampling and refitting; (d) matrix weighted rational B-spline curve of degree 3 after 20 times of resampling and refitting. The circles 'o' denote the input points or the resampled points. The control points of the interpolating B-spline curve and the added point-tangent control pairs at the boundaries of the matrix weighted rational B-spline curves are not shown for clarity.

Table 1
Average distances from control points to the quintic B-spline curve or the matrix weighted rational B-spline curves of degree 5.

# points	B-spline curve	point-normal	point-tangent
20	0.370825	0.191567	0.109114
40	0.102556	0.019438	0.010415
60	0.046716	0.003962	0.002583
80	0.026509	0.001281	0.000937
100	0.017028	0.000519	0.000414
120	0.011847	0.000248	0.000205

unit tangents and unit normal vectors at $t_0 = 0$, $t_i = t_{i-1} + \frac{2\pi}{n}$, $i = 1, 2, \ldots, n$. Then a periodic B-spline curve is constructed by using the sampled points as control points. Similarly, periodic matrix weighted rational B-spline curves with point-normal or point-tangent control pairs are also obtained. Fig. 6 illustrates a quintic B-spline curve and two matrix weighted rational B-spline curves of degree 5 with control data sampled from the original curve by choosing $n = 40$. From the figure we can see that the two obtained matrix weighted rational B-spline curves lie closely to their control points while the B-spline curve deviates from the control polygon obviously. The average distances from control points to B-spline curves or matrix weighted rational B-spline curves with different numbers of control data are summarized in Table 1. From the table we learn that matrix weighted rational B-spline curves can approximate their control points closely when the control data have been sampled properly. Moreover, a spatial matrix weighted rational B-spline curve with point-tangent control pairs can usually have even higher approximation accuracy than a matrix weighted rational B-spline curve with point-normal control pairs. In the following we present examples to show how to fit and fair measured points in 3D space by matrix weighted rational B-spline curves with point-tangent control pairs.

Fourth, we fit and fair a set of points in 3D space by cubic B-spline or matrix weighted rational B-spline curves. Fig. 7(a)

illustrates a cubic B-spline curve that interpolates the given points. The curvature normals at sampled points show that the curve is not fair. By estimating unit tangent vectors at all initial points, a matrix weighted rational B-spline curve of degree 3 is constructed directly from the points and the tangents; see Fig. 7(b) for the obtained curve. From the computed curvature normals we know that the curve is not fair, either. By applying Algorithm 2, we resample same number of points from each fitting curve and construct another matrix weighted rational B-spline curve using the sampled points and the newly estimated tangents at the points. The curves after 2 or 20 iterations of resampling and refitting are illustrated in Fig. 7(c) and (d), respectively. It is clear that much fairer curves are obtained after more times of fairing. Along with iterations of data sampling and curve refitting, the deviations from the original points to the new fitting curves may increase while the deviations from the newly sampled points to the refitting curves decrease rapidly. The maximum deviation from the original points to the initial matrix weighted rational B-spline curve is about 0.0084% of the curve length, but the deviation increases to 0.0165% and 0.0405% of the curve length after 2 or 20 times of fairing. Meanwhile, the maximum deviation from the latest set of sampling points to the final fitting curve is about 0.001% of the curve length.

Lastly, we present another example for fitting and fairing points in 3D space by B-spline or matrix weighted rational B-spline curves. The original input points are illustrated in Fig. 8(a). Differently from the last example, the points nearly lie on a straight line and they are not uniformly spaced in general. A non-uniform quintic B-spline curve can fit the points closely, but the curvature normals show that the curve is sensitive to the data noise heavily. A matrix weighted rational B-spline curve of degree 5 that fits the points and the estimated tangents at the points has similar curvature normals as the B-spline curve; see Fig. 8(b). By constructing matrix weighted rational B-spline curves from the resampled points and re-estimated tangent vectors at the points, much fairer curves can be obtained. Fig. 8(c) and (d) show the
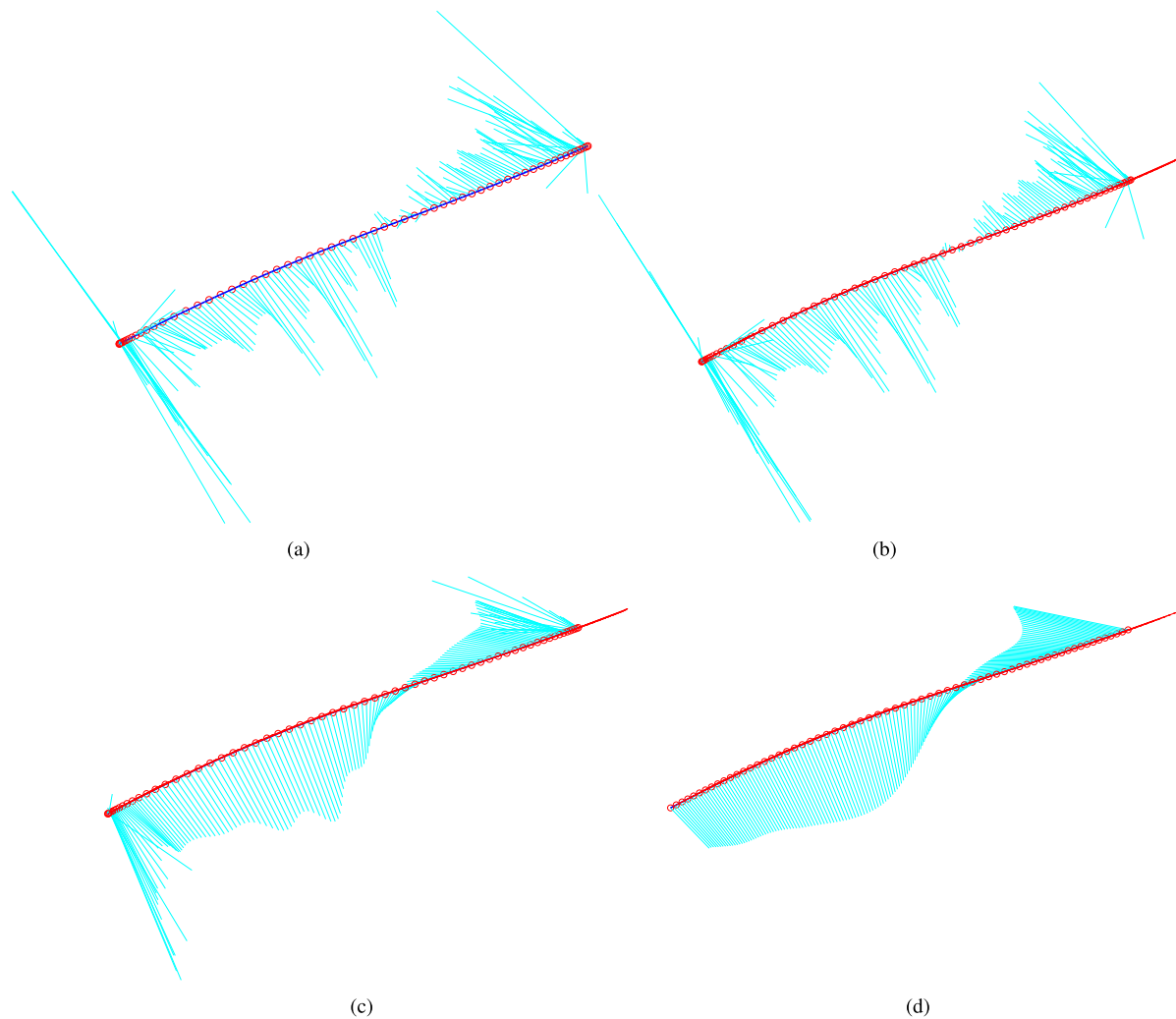
Fig. 8. Fitting and fairing Hermite-type data in 3D space: (a) a quintic B-spline curve fitting the given points; (b) a matrix weighted rational B-spline curve of degree 5 constructed from the given points and the estimated tangents; (c) matrix weighted rational B-spline curve of degree 5 after 10 times of resampling and refitting; (d) matrix weighted rational B-spline curve of degree 5 after 500 times of resampling and refitting. The circles 'o' denote the input or the resampled points.

matrix weighted rational B-spline curves of degree 5 after 10 or 500 times of point sampling and curve refitting. The maximum deviations from the original points to the initial matrix weighted rational B-spline curve or the matrix weighted rational B-spline curves after 10 or 500 iterations of refitting are 0.0029%, 0.0083% or 0.025% of the curve length, respectively.

Except for salient feature points and boundary points, in this paper we did not fair planar or spatial points with any tolerance constraint. If all or selected points can only be moved in permitted tolerances, the corresponding points sampled on a (re)fitting curve should then be re-positioned within the prescribed tolerances. In our experiments, the boundary points of open curves are kept unchanged after each time of refitting just by using symmetric control data near the boundaries.

6. Conclusions and discussions

This paper has proposed techniques to fit and fair planar or spatial Hermite-type data by matrix weighted NURBS curves. For a sequence of points together with given or estimated normals or tangents at the points, a matrix weighted NURBS curve of arbitrary degree that fits the given data is constructed by choosing the input points as control points and computing the weight matrices using the known normals or tangents. Due to almost circular or helical precision, matrix weighted NURBS curves approximate the input points closely when the control data are uniformly sampled from curves with smoothly varying tangents and curvatures. Non-uniformly spaced or noisy data can be faired efficiently by repeatedly constructing matrix weighted NURBS curves from the data and sampling new points from the curves. Direct curve construction without solving large systems is useful for speedy curve modeling and online data fitting. The faired Hermite-type data and fair fitting curves in a plane or 3D space can be used for fair shape design, tool path generation for CNC machining, etc.

A matrix weighted NURBS curve may deviate from the control points significantly when the original points are sparsely sampled from a curve with rapidly changing curvature. Sparsely spaced points can be faired under the constraint of prescribed tolerances and traditional B-spline curves can be employed to interpolate the faired points.

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