

Geometric Hermite interpolation by logarithmic arc splines

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ABSTRACT

This paper considers the problem of G^1 curve interpolation using a special type of discrete logarithmic spirals. A “logarithmic arc spline” is defined as a set of smoothly connected circular arcs. The arcs of a logarithmic arc spline have equal angles and the curvatures of the arcs form a geometric sequence. Given two points together with two unit tangents at the points, interpolation of logarithmic arc splines with a user specified winding angle is formulated into finding the positive solutions to a vector equation. A practical algorithm is developed for computing the solutions and construction of interpolating logarithmic arc splines. Compared to known methods for logarithmic spiral interpolation, the proposed method has the advantages of unbounded winding angles, simple offsets and NURBS representation.

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1. Introduction

Spirals, which have monotone curvatures, find wide applications in the fields of fair shape modeling, highway route design or artistic pattern design, etc. (Meek and Walton, 1992; Wang et al., 2004; Xu and Mould, 2009; Meek et al., 2012). Particularly, the clothoid spiral (also known as the Euler spiral) whose curvature is a linear function of its arc length, has often been used as a primary tool for curve completion or fair shape modeling (Kimia et al., 2003; Zhou et al., 2012). Another popular spiral is the logarithmic spiral whose radius of curvature is a linear function of its arc length. The study of logarithmic spirals goes back to Descartes and Jacob Bernoulli (Davis, 1993). Logarithmic spirals have many elegant properties and can be used to model fair shapes as well as natural objects (Harary and Tal, 2011).

As a generalization of Euler spirals and logarithmic spirals, Miura (2006) proposed a general equation for log-aesthetic curves. By choosing different values for a particular parameter, one can define various spirals by the equation. Except for a few special cases like circles, evaluation of log-aesthetic curves depends on numerical integration or computation of special functions (Ziatdinov et al., 2012a). If boundary points and tangents are given first, parameters for an interpolating spiral are usually determined by solving nonlinear systems or by searching strategies (Coope, 1992; Miura, 2000; Yoshida and Saito, 2006; Ziatdinov et al., 2012b).

Inspired by the fact that log-aesthetic curves are usually computed numerically or approximated by other types of curves such as polynomials or rational polynomials, one can construct interpolating spirals discretely or using polynomials directly (Baumgarten and Farin, 1997; Yoshida and Saito, 2009; Walton and Meek, 2013; Yoshida et al., 2013). Polynomials or rational polynomials that have approximate linear plots of log curvatures are quasi-log-aesthetic spirals. These spirals can be evaluated explicitly. However, these curves are not log-aesthetic spirals exactly and many quasi-log-aesthetic spirals have to be pieced together when a high accuracy of approximation is desired. Geometric Hermite interpolating curves with

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minimal energy can generate fair shapes (Yong and Cheng, 2004), but the Euler spiral and the logarithmic spiral are of special interest in shape modeling.

In this paper we consider G^1 Hermite interpolation by logarithmic arc splines. Our proposed algorithm is motivated by the equiangular property of logarithmic spirals and the high accuracy approximation of spirals by arc splines (Meek and Walton, 1999). By assuming that a logarithmic spiral is approximated by a sequence of smoothly connected circular arcs of equal angles and the curvatures or radii of curvatures of all arcs form a geometric sequence, we obtain a logarithmic arc spline. G^1 Hermite interpolation by logarithmic arc splines can be formulated as solving the free parameters from a simple equation. All solutions to the equation can be obtained using an efficient numerical method. As opposed to previous approaches which assumed bounded winding angles and unique interpolating curves, we have no such restrictions and all interpolating curves to the given boundary data can be obtained efficiently. As offsets of circular arcs are circular arcs, the offsets of logarithmic arc splines are easy to compute. Logarithmic arc splines and their offsets can be represented by NURBS or transformed into curvature continuous curves conveniently (Yang, 2004). Therefore, the proposed curve can be used as an efficient tool for shape modeling and CNC machining.

The paper is structured as follows. Section 2 briefly reviews important properties of logarithmic spirals and proposes a definition of a logarithmic arc spline. Section 3 describes basic formulations of G^1 curve interpolation by logarithmic arc splines. Theoretical analysis on the existence and algorithm steps for logarithmic arc spline interpolation are also presented. Several interesting examples are provided in Section 4, and they demonstrate the applicability of the proposed algorithm. Section 5 concludes the paper.

2. Logarithmic spiral and logarithmic arc spline

2.1. Basics of logarithmic spirals

A logarithmic spiral of which the pole lies at the origin can be represented by polar coordinates as

$$r(t) = r_0 e^{\lambda t}, \quad r_0 \in \mathbb{R}^+, \lambda \in \mathbb{R} \quad (1)$$

or, by Cartesian coordinates as

$$\mathbf{S}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = r_0 e^{\lambda t} \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}. \quad (2)$$

Particularly, $\mathbf{S}(t)$ will approach the pole when λt approaches $-\infty$.

A logarithmic spiral arc can be defined by either of the above equations when the parameter t belongs to an interval $[t_1, t_2]$. The winding angle of the logarithmic spiral arc is obtained as $\phi = t_2 - t_1$ when $\lambda > 0$ or $\phi = t_1 - t_2$ when $\lambda < 0$. If the winding angle satisfies $|\phi| \leq 2\pi$, the spiral arc is also referred as a *single-winding logarithmic spiral*; otherwise, it is a *multi-winding logarithmic spiral*.

Logarithmic spiral has several distinguished properties which make it a powerful tool for shape modeling. The clear or easily proved properties are listed with no proof.

Property 2.1. *The angle between any radial line and the tangent line that passes through the same point does not change when the point moves along the logarithmic spiral.*

This property is also known as the equiangular property, which was first observed by Rene Descartes. In particular, the angle φ between the radial line and the tangent line is computed by $\lambda = \cot \varphi$, where λ is the parameter as in Eq. (1).

Property 2.2. *Let $\mathbf{S}(t)$ be a logarithmic spiral, $k \in \mathbb{Z}^+$, the tangents at points $\mathbf{S}(t)$ or $\mathbf{S}(t + 2k\pi)$ are parallel and the angle between the tangent direction and the chord $\mathbf{S}(t + 2k\pi) - \mathbf{S}(t)$ is acute.*

Property 2.3. *Let \mathbf{P}_a and \mathbf{P}_b be the endpoints of a logarithmic spiral arc, the curvature decreasing from \mathbf{P}_a to \mathbf{P}_b and the winding angle being less than 2π . Assume α and β are the unsigned angles between $\mathbf{P}_b - \mathbf{P}_a$ and the tangent to the arc at \mathbf{P}_a or between $\mathbf{P}_b - \mathbf{P}_a$ and the tangent to the arc at \mathbf{P}_b , respectively. It follows that $\alpha > \beta$.*

Proof. Without loss of generality we assume the logarithmic spiral is represented by Eq. (2) with $\lambda > 0$, and the endpoints of a logarithmic spiral arc are $\mathbf{P}_a = \mathbf{S}(t)$ and $\mathbf{P}_b = \mathbf{S}(t + \tau)$. If the winding angle τ is less than π , the logarithmic spiral arc is convex and short and the property holds based on Vogt's theorem (Theorem 3.17 in Guggenheim, 1977).

We prove $\alpha > \beta$ for $\pi \leq \tau < 2\pi$. Since $0 < \alpha, \beta < \pi$, we should only prove $\cos \alpha < \cos \beta$. From Eq. (2), we have

$$\cos \alpha = \frac{\mathbf{S}(t + \tau) - \mathbf{S}(t)}{\|\mathbf{S}(t + \tau) - \mathbf{S}(t)\|} \cdot \frac{\mathbf{S}'(t)}{\|\mathbf{S}'(t)\|} = \frac{e^{\lambda \tau} (\lambda \cos \tau + \sin \tau) - \lambda}{\sqrt{1 + e^{2\lambda \tau} - 2e^{\lambda \tau} \cos \tau} \sqrt{1 + \lambda^2}}$$

and

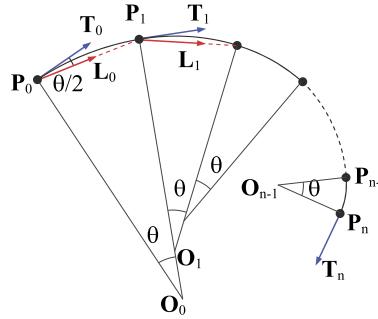


Fig. 1. A logarithmic arc spline that consists of n circular arcs.

$$\cos \beta = \frac{\mathbf{S}(t+\tau) - \mathbf{S}(t)}{\|\mathbf{S}(t+\tau) - \mathbf{S}(t)\|} \cdot \frac{\mathbf{S}'(t+\tau)}{\|\mathbf{S}'(t+\tau)\|} = \frac{\lambda e^{\lambda \tau} - \lambda \cos \tau + \sin \tau}{\sqrt{1 + e^{2\lambda \tau} - 2e^{\lambda \tau} \cos \tau} \sqrt{1 + \lambda^2}}.$$

Omitting the common denominator within $\cos \alpha$ and $\cos \beta$, the sign of $\cos \beta - \cos \alpha$ is judged by

$$\begin{aligned}
 h(t) &= \lambda e^{\lambda \tau} - \lambda \cos \tau + \sin \tau - [e^{\lambda \tau} (\lambda \cos \tau + \sin \tau) - \lambda] \\
 &= \lambda (1 + e^{\lambda \tau}) (1 - \cos \tau) + \sin \tau (1 - e^{\lambda \tau}) \\
 &= 2 \sin \frac{\tau}{2} \left[\lambda (1 + e^{\lambda \tau}) \sin \frac{\tau}{2} + (1 - e^{\lambda \tau}) \cos \frac{\tau}{2} \right].
 \end{aligned}$$

As $\lambda > 0$ and $\frac{\pi}{2} \leq \frac{\tau}{2} < \pi$, it is derived that $h(t) > 0$. This proves the property. \square

Property 2.4. If a sequence of points are sampled from a logarithmic spiral with a constant winding angle between every two neighboring samples, the curvatures at the sampled points form a geometric sequence.

It is noted that the points sampled with a constant winding angle from a logarithmic spiral also form a discrete logarithmic spiral. However, this curve is only C^0 continuous. In the following, we will present a new discrete logarithmic spiral that is C^1 continuous. A practical algorithm for Hermite interpolation by this curve will also be given.

2.2. Logarithmic arc spline

As a discrete approximation to the logarithmic spiral, a logarithmic arc spline is consisting of a sequence of smoothly connected circular arcs of which the center angles are the same and the curvatures form a geometric sequence. In particular, if all arcs are quarter-circles and the growth rate of arc radii equals to the golden ratio, the logarithmic arc spline is a discrete approximation to the golden spiral.¹ The clothoid arc spline was given in [Meek and Walton \(2004\)](#) and relaxed for interpolation of Hermite data in [Zhou et al. \(2012\)](#).

Assume that \mathbf{P}_i , $i = 0, 1, \dots, n$ are the end points or the joint points between consequent circular arcs of a logarithmic arc spline; see Fig. 1. The unit tangent at point \mathbf{P}_i is \mathbf{T}_i and the signed center angle of each arc is denoted as θ . If the rotation is counterclockwise the angle θ is positive; otherwise, it is negative. Let $l_i = \|\mathbf{P}_{i+1} - \mathbf{P}_i\|$ and $\mathbf{L}_i = (\mathbf{P}_{i+1} - \mathbf{P}_i)/l_i$ for $i = 0, 1, \dots, n - 1$. It follows that the angle between \mathbf{T}_i and \mathbf{L}_i is $\theta/2$ and the angle from \mathbf{L}_i to \mathbf{L}_{i+1} is θ . Let

$$\mathbf{M}_\theta := \mathbf{M}(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

be a rotation matrix, we have $\mathbf{L}_0 = \mathbf{M}_\theta \mathbf{T}_0$ and $\mathbf{L}_{i+1} = \mathbf{M}_\theta \mathbf{L}_i$ for $i = 0, 1, \dots, n-1$.

Assume that the (signed) radius² of the circular arc which interpolates points \mathbf{P}_i , \mathbf{P}_{i+1} and tangent \mathbf{T}_i is r_i , we have $l_i = 2r_i \sin(\theta/2)$ for $i = 0, 1, \dots, n - 1$. Under the assumption that the curvatures, or equivalently, the radii of circular arcs form a geometric sequence, we have $r_1 = \rho r_0$, $r_2 = \rho r_1 = \rho^2 r_0$, \dots , $r_{n-1} = \rho^{n-1} r_0$, where ρ is a positive constant. It is further derived that $l_1 = \rho l_0$, $l_2 = \rho^2 l_0$, \dots , $l_{n-1} = \rho^{n-1} l_0$. By accumulating the vectors between neighboring points, an arbitrary joint point \mathbf{P}_i , $0 < i \leq n$ can be computed by

$$\begin{aligned}
\mathbf{P}_i &= \mathbf{P}_0 + l_0 \mathbf{L}_0 + l_1 \mathbf{L}_1 + \dots + l_{i-1} \mathbf{L}_{i-1} \\
&= \mathbf{P}_0 + (l_0 \mathbf{I} + \rho l_0 \mathbf{M}_\theta + \dots + \rho^{i-1} l_0 \mathbf{M}_\theta^{i-1}) \mathbf{L}_0 \\
&= \mathbf{P}_0 + l_0 (\mathbf{I} - \rho \mathbf{M}_\theta)^{-1} (\mathbf{I} - \rho^i \mathbf{M}_\theta^i) \mathbf{L}_0,
\end{aligned} \tag{3}$$

¹ http://en.wikipedia.org/wiki/Golden_spiral.

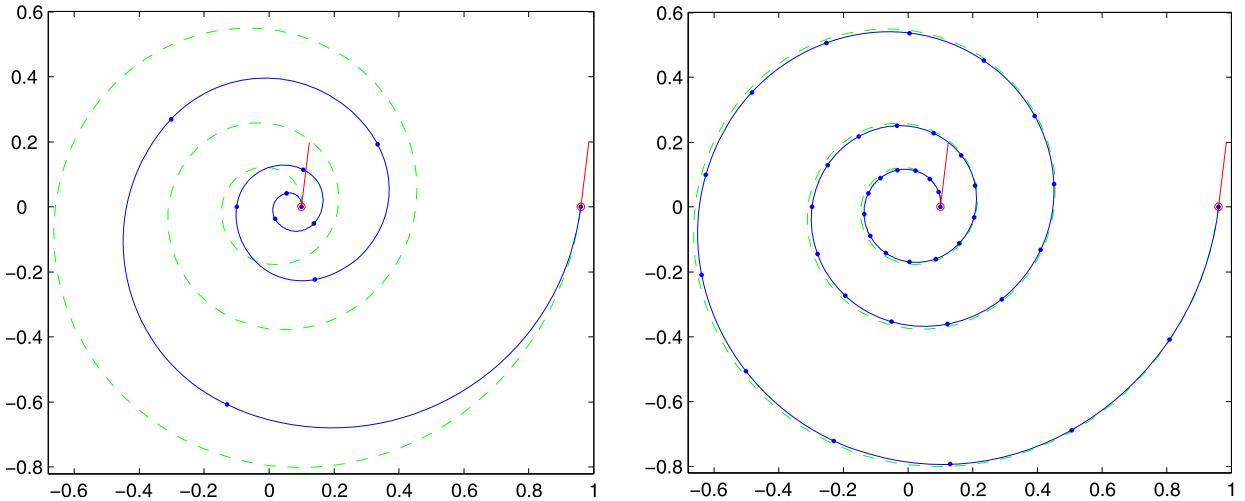


Fig. 2. Approximating a segment of logarithmic spiral (dashed) by an interpolating logarithmic arc spline (solid) that consists of 10 (left) or 40 (right) circular arcs.

Table 1

Convergence rates of approximation of a logarithmic spiral by logarithmic arc splines.

Number of arcs	Max error	$e(2n)/e(n)$	$\log_2(e(2n)/e(n))$
10	0.217324	NA	NA
20	0.051959	0.239184	-2.063804
40	0.012879	0.247869	-2.012353
80	0.003208	0.249088	-2.005275
160	0.000802	0.25	-2.0
320	0.000201	0.250623	-1.996407

where I is the 2×2 identity matrix. If the start arc and the radii ratio are known, all joint points of a logarithmic arc spline will be computed by Eq. (3).

The unit tangent vector at point \mathbf{P}_i can be computed by $\mathbf{T}_i = M_\theta^i \mathbf{T}_0 = M_{i\theta} \mathbf{T}_0$. In particular, $\mathbf{T}_n = M_{n\theta} \mathbf{T}_0$. Let $\mathbf{V}_i = M_{\pi/2} \mathbf{T}_i$, the centers of all circular arcs are obtained as $\mathbf{O}_i = \mathbf{P}_i + r_i \mathbf{V}_i$, $i = 0, 1, \dots, n-1$.

It is proved in Meek and Walton (1999) that a spiral segment can be approximated by a spiral arc spline with a high accuracy when the arc number is a large number. We check the convergence of approximating logarithmic arc splines numerically. An original logarithmic spiral is defined by $r(t) = 0.1e^{0.12t}$, $0 \leq t \leq 6\pi$. We approximate the spiral by constructing logarithmic arc splines that interpolate the boundary data of the original spiral. Details of logarithmic arc spline interpolation will be given in next section. Fig. 2 illustrates the initial spiral and an interpolating logarithmic arc spline that consists of 10 or 40 arcs. Approximation errors using various numbers of circular arcs are given in Table 1. From the table we can see that the approximation of a logarithmic spiral by an interpolating logarithmic arc spline appears to converge quadratically. However, a theoretical analysis is still not available at present.

3. G^1 curve interpolation by logarithmic arc splines

Assume \mathbf{P}_a and \mathbf{P}_b are two distinct points on a plane, \mathbf{T}_a and \mathbf{T}_b are the two corresponding unit tangents that satisfy the following two requirements. (1) Neither \mathbf{T}_a nor \mathbf{T}_b is parallel or antiparallel to $\mathbf{P}_L = \mathbf{P}_b - \mathbf{P}_a$. (2) The angle between \mathbf{T}_a and \mathbf{P}_L is acute when the vectors \mathbf{T}_a and \mathbf{T}_b are the same. We compute the winding angles of all potential logarithmic spirals that match the boundary data first and then construct a logarithmic arc spline that has a specified winding angle ϕ and interpolates the given points and the given tangents at the points.

3.1. Computing the winding angles

The logarithmic spirals interpolating points \mathbf{P}_a , \mathbf{P}_b and tangents \mathbf{T}_a , \mathbf{T}_b can be multi-winding spirals of which the magnitudes of winding angles are larger than 2π or just a single-winding spiral of which the absolute winding angle is no more than 2π . We refer the winding angle corresponding to the single-winding interpolating spiral as *the minimum winding angle*. The winding angles corresponding to multi-winding interpolating spirals can then be obtained by adding/subtracting $2k\pi$, where $k \in \mathbb{Z}^+$, from the minimum winding angle.

The computation of the minimum winding angle ϕ_{\min} depends on whether the tangents \mathbf{T}_a and \mathbf{T}_b lie on one or two sides of the line through \mathbf{P}_a and \mathbf{P}_b . Let $\mathbf{U} = \mathbf{P}_L / \|\mathbf{P}_L\|$, the unsigned angles between \mathbf{T}_a and \mathbf{U} or between \mathbf{T}_b and \mathbf{U} can

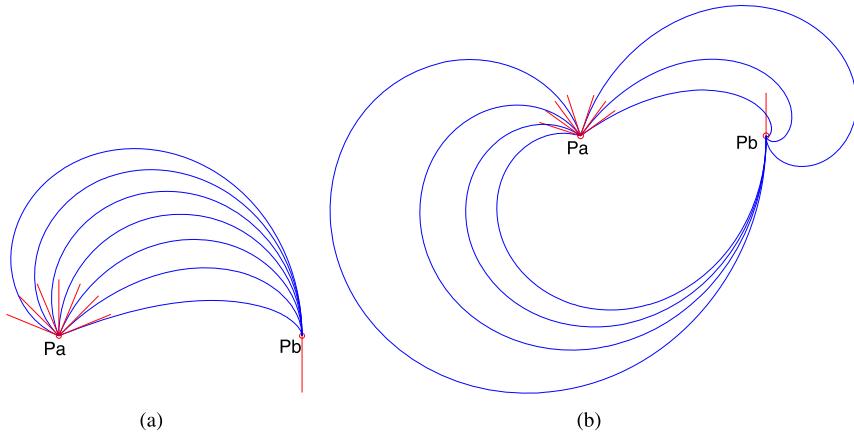


Fig. 3. Curve interpolation by single-winding logarithmic spirals: (a) logarithmic spirals that interpolate boundary tangents lying at two sides of a chord; (b) logarithmic spirals that interpolate boundary tangents lying on one side of a chord.

be computed by $\alpha = \cos^{-1}(\mathbf{T}_a \cdot \mathbf{U})$ and $\beta = \cos^{-1}(\mathbf{T}_b \cdot \mathbf{U})$, respectively. If vectors \mathbf{T}_a and \mathbf{T}_b lie on two sides of chord $\mathbf{P}_a\mathbf{P}_b$, the interpolating single-winding logarithmic spiral lies on one side of the chord and the minimum winding angle satisfies $|\phi_{\min}| = \alpha + \beta$. Moreover, the winding direction of the tangent along the spiral from \mathbf{P}_a to \mathbf{P}_b coincides with the direction from \mathbf{T}_a to \mathbf{U} (with angle less than π). In this case, the (signed) minimum winding angle ϕ_{\min} is computed by

$$\phi_{\min} = \begin{cases} +(\alpha + \beta), & \text{if } \mathbf{T}_a \wedge \mathbf{U} > 0 \\ -(\alpha + \beta), & \text{otherwise} \end{cases}$$

where $\mathbf{T}_a \wedge \mathbf{U}$ represents the scalar cross product of two planar vectors. Fig. 3(a) illustrates several single-winding logarithmic spirals lying on one side of the given chord.

If the two given tangents lie on one side of the chord, the interpolating single-winding logarithmic spiral will wind around one end vertex and cross the line through \mathbf{P}_a and \mathbf{P}_b . Based on Property 2.3, the interpolating logarithmic spiral will wind around \mathbf{P}_a when $\alpha > \beta$ or wind around \mathbf{P}_b when $\alpha < \beta$. Particularly, the winding direction of the tangent along the spiral from \mathbf{P}_a to \mathbf{P}_b coincides with the direction from \mathbf{U} to \mathbf{T}_a when the interpolating spiral winds around \mathbf{P}_a . Similarly, the winding direction of the tangent coincides with the direction from \mathbf{T}_b to \mathbf{U} when the spiral winds around \mathbf{P}_b . Combining these two cases together we have

$$\phi_{\min} = \begin{cases} 2\pi - |\alpha - \beta|, & \text{if } (\alpha - \beta)(\mathbf{U} \wedge \mathbf{T}_a) > 0 \\ |\alpha - \beta| - 2\pi, & \text{if } (\alpha - \beta)(\mathbf{U} \wedge \mathbf{T}_a) < 0 \end{cases}$$

Fig. 3(b) illustrates several single-winding logarithmic spirals that wind around one of the two end vertices of the chord.

If $\mathbf{T}_a = \mathbf{T}_b$ and $\alpha = \beta < \frac{\pi}{2}$, an interpolating logarithmic spiral can wind around either of the two vertices. If we choose $\phi_{\min} = 2\pi$, the interpolating spiral winds around vertex \mathbf{P}_a when $\mathbf{U} \wedge \mathbf{T}_a > 0$ or winds around vertex \mathbf{P}_b when $\mathbf{U} \wedge \mathbf{T}_a < 0$. If we choose $\phi_{\min} = -2\pi$ the interpolating spiral will wind around vertex \mathbf{P}_b or \mathbf{P}_a , respectively in the mentioned two cases.

Let ϕ_a and ϕ_b be the winding angles of the tangent direction along the interpolating single-winding spiral from \mathbf{T}_a to \mathbf{U} or from \mathbf{U} to \mathbf{T}_b , respectively. These two angles have the same sign with ϕ_{\min} and satisfy $\phi_{\min} = \phi_a + \phi_b$. Particularly, the angles also satisfy $|\phi_a| = \alpha$ and $|\phi_b| = \beta$ when \mathbf{T}_a and \mathbf{T}_b lie on two sides of the chord $\mathbf{P}_a\mathbf{P}_b$. If \mathbf{T}_a and \mathbf{T}_b lie on the same side of the chord, the angles satisfy $|\phi_a| = 2\pi - \alpha$ and $|\phi_b| = \beta$ when $\alpha > \beta$ or $|\phi_a| = \alpha$ and $|\phi_b| = 2\pi - \beta$ otherwise. We note that ϕ_a and ϕ_b are not essential for the construction of interpolating curves, but they will be used for the analysis of solutions to the interpolation problem.

When the minimum winding angle has been obtained, the winding angles for multi-winding interpolating logarithmic spirals can be defined by

$$\phi = \begin{cases} \phi_{\min} + 2k\pi, & \text{if } \phi_{\min} > 0 \\ \phi_{\min} - 2k\pi, & \text{otherwise} \end{cases} \quad (4)$$

where k is an arbitrary positive integer.

3.2. Formulation of logarithmic arc spline interpolation

For given boundary data and a specified winding angle ϕ which satisfies Eq. (4), we construct an interpolating logarithmic arc spline consisting of n segments of circular arcs. From the definition of logarithmic arc spline the center angle of each circular arc can be chosen as $\theta = \frac{\phi}{n}$. The unit vector corresponding to the chord of the first circular arc is obtained as $\mathbf{M}_{\theta/2}\mathbf{T}_a$, where $\mathbf{M}_{\theta/2}$ is the rotation matrix as defined in Section 2. When a logarithmic arc spline interpolates points \mathbf{P}_a and

\mathbf{P}_b at two ends, it implies that $\mathbf{P}_0 = \mathbf{P}_a$ and $\mathbf{P}_n = \mathbf{P}_b$ with \mathbf{P}_i , $i = 1, 2, \dots, n-1$ as the intermediate joint points of the arc spline. From Eq. (3) we have

$$\mathbf{P}_b - \mathbf{P}_a = l_0(\mathbf{I} - \rho \mathbf{M}_\theta)^{-1}(\mathbf{I} - \rho^n \mathbf{M}_\phi)\mathbf{L}_0, \quad (5)$$

where l_0 is the chord length of the first arc and ρ is the ratio between consequent arc radii. This equation can be reformulated as

$$l_0(\mathbf{I} - \rho^n \mathbf{M}_\phi)\mathbf{L}_0 + \rho \mathbf{M}_\theta \mathbf{P}_L = \mathbf{P}_L. \quad (6)$$

To solve the unknowns ρ and l_0 , we derive two scalar equations from Eq. (6). Crossing both sides of Eq. (6) with $(\mathbf{I} - \rho^n \mathbf{M}_\phi)\mathbf{L}_0$, we have

$$f(\rho) = a\rho^{n+1} + b\rho^n + c\rho + d = 0, \quad (7)$$

where $a = (\mathbf{M}_\phi \mathbf{L}_0) \wedge (\mathbf{M}_\theta \mathbf{P}_L)$, $b = -(\mathbf{M}_\phi \mathbf{L}_0) \wedge \mathbf{P}_L$, $c = -\mathbf{L}_0 \wedge (\mathbf{M}_\theta \mathbf{P}_L)$ and $d = \mathbf{L}_0 \wedge \mathbf{P}_L$. Crossing both sides of Eq. (6) with $\mathbf{M}_\theta \mathbf{P}_L$, we have another scalar equation

$$l_0(\mathbf{I} - \rho^n \mathbf{M}_\phi)\mathbf{L}_0 \wedge (\mathbf{M}_\theta \mathbf{P}_L) = \mathbf{P}_L \wedge (\mathbf{M}_\theta \mathbf{P}_L). \quad (8)$$

Provided ρ has already been obtained, l_0 can be solved out from Eq. (8) immediately. It yields

$$l_0 = \frac{\mathbf{P}_L \wedge (\mathbf{M}_\theta \mathbf{P}_L)}{(\mathbf{I} - \rho^n \mathbf{M}_\phi)\mathbf{L}_0 \wedge (\mathbf{M}_\theta \mathbf{P}_L)}. \quad (9)$$

If there exist positive solutions to Eq. (7) and the corresponding l_0 are also positive, we can then construct logarithmic arc splines interpolating the given boundary data by the method stated in Section 2.2.

As $\mathbf{M}_\phi = \mathbf{M}_{\phi_{\min}}$ and $\mathbf{A} \wedge \mathbf{B} = \|\mathbf{A}\| \|\mathbf{B}\| \sin \varphi$, where φ is the signed angle from \mathbf{A} to \mathbf{B} , the coefficients within Eq. (7) can be reformulated as

$$a = l_e \sin(\phi_b - \theta/2); \quad b = -l_e \sin(\phi_b + \theta/2); \quad c = l_e \sin(\phi_a + \theta/2); \quad d = -l_e \sin(\phi_a - \theta/2),$$

where $l_e = \|\mathbf{P}_L\|$. Before proving the existence of interpolating logarithmic arc splines we prove the existence of positive roots to Eq. (7).

Theorem 3.1. Assume that \mathbf{T}_a and \mathbf{T}_b are the specified unit tangents at two ends of chord $\mathbf{P}_a \mathbf{P}_b$. The angles α , β and ϕ are computed or specified as in Section 3.1. If we choose the total number of arcs satisfying $n > \frac{|\phi|}{2 \min\{\alpha, \pi - \alpha, \beta, \pi - \beta\}}$ and compute $\theta = \frac{\phi}{n}$, the following results hold.

1. If \mathbf{T}_a and \mathbf{T}_b lie on two sides of chord $\mathbf{P}_a \mathbf{P}_b$, there exist one or three positive roots to Eq. (7).
2. If \mathbf{T}_a and \mathbf{T}_b lie on the same side of chord $\mathbf{P}_a \mathbf{P}_b$, there exist two positive roots to Eq. (7).

Proof. From the computation of α and β we know that $0 < \alpha, \beta < \pi$. By choosing $n > \frac{|\phi|}{2 \min\{\alpha, \pi - \alpha, \beta, \pi - \beta\}}$ and computing $\theta = \frac{\phi}{n}$ we know that $|\frac{\theta}{2}| < \min\{\alpha, \pi - \alpha, \beta, \pi - \beta\} \leq \frac{\pi}{2}$. Thus, $\sin \theta$ has the same sign as θ . Based on the relationship between α , β and ϕ_a , ϕ_b we conclude that the signs of $\sin(\phi_a \pm \theta/2)$ or $\sin(\phi_b \pm \theta/2)$ are the same as that of $\sin(\phi_a)$ or $\sin(\phi_b)$, respectively.

We next consider the solutions to Eq. (7) under the case that \mathbf{T}_a and \mathbf{T}_b lie on two sides of chord $\mathbf{P}_a \mathbf{P}_b$. In this case the winding angles ϕ_a and ϕ_b satisfy $\max\{|\phi_a|, |\phi_b|\} < \pi$. Therefore, the coefficients a and d of Eq. (7) have different signs. Since $f(0) = d$ and $\lim_{\rho \rightarrow +\infty} f(\rho) = \lim_{\rho \rightarrow +\infty} a\rho^n$, the equation $f(\rho) = 0$ has at least one root in $(0, +\infty)$. From the expressions of the coefficients of Eq. (7) we know that the signs of a , b , c and d change three times. Based on Descartes' rule of signs we know that Eq. (7) can have one or three positive roots.

Lastly, we prove the theorem for the case that \mathbf{T}_a and \mathbf{T}_b lie on the same side of chord $\mathbf{P}_a \mathbf{P}_b$. From Section 3.1, we know that $|\phi_a| < \pi < |\phi_b| < 2\pi$ or $|\phi_b| < \pi < |\phi_a| < 2\pi$. In this case the coefficients a , b , c and d satisfy $ad > 0$, $bc > 0$ but $ab < 0$ and $ac < 0$. Therefore, Eq. (7) has no more than two positive roots according to Descartes' rule of signs. Without loss of generality, we assume $a > 0$ and then $f(0)$ and $\lim_{\rho \rightarrow +\infty} f(\rho)$ are positive, too. Rewrite Eq. (7) as

$$f(\rho) = a \left(\rho^n + \frac{c}{a} \right) \left(\rho + \frac{b}{a} \right) + \frac{1}{a} (ad - bc).$$

If $|\phi_{\min}| < 2\pi$, we have $f(\sqrt[n]{\frac{-c}{a}}) = \frac{1}{a} (ad - bc) = \frac{l_0^2}{a} \sin(\phi_{\min}) \sin \theta < 0$. In this case Eq. (7) has exactly two positive roots. If $|\phi_{\min}| = 2\pi$, it is verified that $\frac{c}{a} = -1$ and Eq. (7) reduces to $f(\rho) = a(\rho^n - 1)(\rho + \frac{b}{a})$. So, the positive roots to the equation are $\rho_1 = 1$ and $\rho_2 = -\frac{b}{a}$. \square

Besides the positive ρ we should also check whether the corresponding l_0 is positive or not to construct an interpolating logarithmic arc spline from the given boundary data. By using the same technique as formulating a , b , c and d , Eq. (9) can be reformulated as

$$l_0 = l_e \frac{\sin \theta}{\sin(\phi_a + \theta/2) + \rho^n \sin(\phi_b - \theta/2)}. \quad (10)$$

As $\sin \theta$, $\sin(\phi_a + \theta/2)$ and $\sin(\phi_b - \theta/2)$ have the same sign when \mathbf{T}_a and \mathbf{T}_b lie on two sides of the chord $\mathbf{P}_a\mathbf{P}_b$, from Eq. (10) we have $l_0 > 0$ when $\rho > 0$. Then the logarithmic arc spline computed by each positive root to $f(\rho) = 0$ and the corresponding l_0 interpolates the given boundary data.

From the proof of Theorem 3.1 we know that the two positive roots to Eq. (7) satisfy $\rho_1 < \bar{\rho} = \sqrt[n]{-c/a} < \rho_2$ when $\mathbf{T}_a \neq \mathbf{T}_b$ lie on one side of the chord $\mathbf{P}_a\mathbf{P}_b$. As $\bar{\rho} \approx 1$ when n is a large number the two roots always satisfy $\rho_1 < 1$ and $\rho_2 > 1$. From Eq. (10) we have $l_0 \approx l_e \frac{\sin \theta}{\sin(\phi_a + \theta/2)}$ if $\rho_1^n \ll 1$ and $l_0 \approx l_e \frac{\sin \theta}{\rho_2^n \sin(\phi_b - \theta/2)}$ when $\rho_2^n \gg 1$. By choosing $\rho = \rho_1$ when $\alpha < \beta$ or choosing $\rho = \rho_2$ when $\alpha > \beta$ we all have $l_0 > 0$. The chosen parameter ρ and the computed chord length l_0 can then be used for the construction of an interpolating logarithmic arc spline. If the two given tangents \mathbf{T}_a and \mathbf{T}_b are equal, the positive root to Eq. (7) is chosen as $\rho = -\frac{b}{a}$. Substitute it into Eq. (10) one can get $l_0 > 0$, which will be used for the construction of an interpolating logarithmic arc spline.

3.3. The interpolation algorithm

In this subsection we first discuss how to obtain all positive roots to Eq. (7) and then we present the algorithm summary of geometric Hermite interpolation by logarithmic arc splines.

Except for the special solution $\rho = -\frac{b}{a}$ for the case that the given tangents $\mathbf{T}_a = \mathbf{T}_b$ and the angle between \mathbf{T}_a and \mathbf{P}_L is acute, the solutions to Eq. (7) should be solved numerically. The solutions to equation $f(\rho) = 0$ can be obtained by Newton's method when $\mathbf{T}_a \neq \mathbf{T}_b$. Starting from an initial value ρ_0 , the approximate solutions can be updated as follows

$$\rho_{i+1} = \rho_i - \frac{f(\rho_i)}{f'(\rho_i)}.$$

The iteration process continues until $|\rho_{i+1} - \rho_i| < \varepsilon$, where ε is a tolerance given by users. In our experiments, we choose $\varepsilon = 1 \times 10^{-10}$. In case the iteration process does not converge after a certain number (such as 100) of iterations, the initial ρ_0 should be chosen other values.

To estimate a proper initial value for solving Eq. (7) by Newton's algorithm we reformulate the equation as

$$\rho = -\frac{b\rho^n + d}{a\rho^n + c}.$$

If a solution to Eq. (7) is much less than 1, it can then be estimated as $\rho = -\frac{b\rho^n + d}{a\rho^n + c} \approx -\frac{d}{c}$. Similarly, a solution much greater than 1 can be estimated as $\rho = -\frac{b\rho^n + d}{a\rho^n + c} \approx -\frac{b}{a}$. If the given tangents \mathbf{T}_a and \mathbf{T}_b lie on two sides of the chord $\mathbf{P}_a\mathbf{P}_b$, or equivalently $ac > 0$, Eq. (7) can have three solutions. The initial values for the solutions are then chosen as $\rho_0 = 1$, $\rho_0 = -\frac{d}{c}$ or $\rho_0 = -\frac{b}{a}$, respectively. Note that the three initial values may lead to the same final solution because Eq. (7) can have just one positive solution.

From Theorem 3.1 we know that there exist two positive roots to Eq. (7) when the coefficients satisfy $ac < 0$. Particularly, the two solutions lie in intervals $(0, \bar{\rho})$ and $(\bar{\rho}, +\infty)$, respectively. So, the initial value for the solution should lie in the corresponding interval too. To find a solution in interval $(0, \bar{\rho})$, the initial value can be chosen as

$$\rho_0 = \begin{cases} -\frac{d}{c}, & \text{if } -\frac{d}{c} < \bar{\rho} \\ \bar{\rho} - 0.01, & \text{otherwise} \end{cases}$$

Similarly, the initial value for finding a solution in interval $(\bar{\rho}, +\infty)$ can be chosen as

$$\rho_0 = \begin{cases} -\frac{b}{a}, & \text{if } -\frac{b}{a} > \bar{\rho} \\ \bar{\rho} + 0.01, & \text{otherwise} \end{cases}$$

To sum up, we outline the algorithm steps for logarithmic arc spline interpolation (see Algorithm 1).

Fig. 4 illustrates an example of logarithmic arc spline interpolation that has one or three solutions. Two unit tangents at two end points lie on two sides of the chord and the angles between the tangents with $\mathbf{P}_b - \mathbf{P}_a$ are $\alpha = 0.45\pi$ and $\beta = 0.46\pi$, respectively. By choosing $\phi = \alpha + \beta$ and $n = 10$ we obtain a single-winding logarithmic arc spline consisting of 10 arcs; see Fig. 4(a) for the interpolating curve. If we choose $\phi = \alpha + \beta + 8\pi$ and $n = 50$ we have three positive solutions to Eq. (7). Based on each solution and the corresponding l_0 an interpolating logarithmic arc spline consisting of 50 arcs is obtained; see Figs. 4(b)–(d) for the three interpolating curves.

Fig. 5 illustrates an example of logarithmic arc spline interpolation when the specified tangents lie on one side of the chord. The tangents at two ends of the chord $\mathbf{P}_a\mathbf{P}_b$ are specified such that the angles between $\mathbf{P}_b - \mathbf{P}_a$ and the two tangents

Algorithm 1 Logarithmic arc spline interpolation.**input:** Points P_a, P_b , tangents T_a, T_b and winding number k **output:** a sequence of circular arcs

1. Compute the angles $\alpha, \beta, \phi_{\min}$ and ϕ from the input data as described in Section 3.1;
2. Choose an integer n satisfying $n > \frac{|\phi|}{2\min\{|\alpha, \pi - \alpha, \beta, \pi - \beta|\}}$ and compute $\theta = \frac{\phi}{n}$;
3. Compute the coefficients a, b, c and d for Eq. (7);
4. if $(ac > 0)$ Compute all positive solutions to Eq. (7) using Newton's method;
5. if $((ac < 0) \& (T_a \neq T_b))$ do
 - if $(\alpha < \beta)$ Compute the root $\rho = \rho_1 \in (0, \bar{\rho})$;
 - else Compute the root $\rho = \rho_2 \in (\bar{\rho}, +\infty)$;
6. if $((T_a = T_b) \& (\alpha < \frac{\pi}{2}))$ Compute $\rho = -\frac{b}{a}$ as the root to Eq. (7);
7. Compute l_0 based on a selected root ρ by Eq. (9);
8. Compute (P_i, T_i) explicitly based on the obtained ρ and l_0 as described in Section 2;
9. Compute arc radii and centers based on the joint points and tangents.

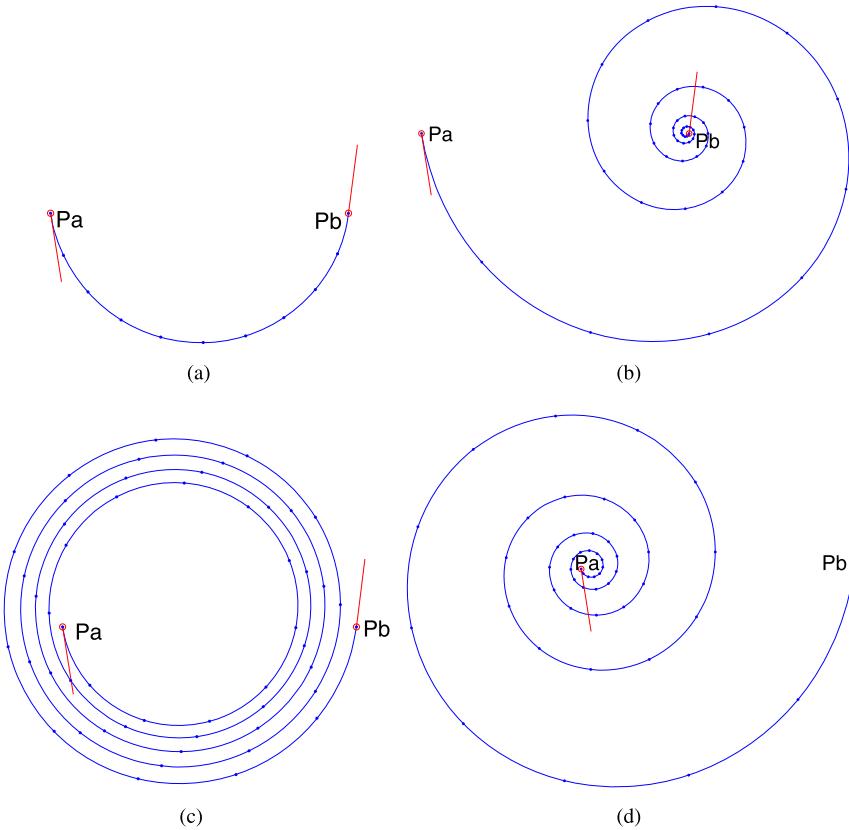


Fig. 4. Logarithmic arc spline interpolation: (a) the interpolating single-winding logarithmic arc spline with $\rho = 0.994329$; (b)–(d) the interpolating multi-winding logarithmic arc splines with $\rho = 0.914623$, $\rho = 1.008966$ or $\rho = 1.069486$, respectively.

are $\alpha = \frac{\pi}{2}$ and $\beta = \frac{2\pi}{3}$, respectively; see Fig. 5(a). By choosing $\phi = \phi_{\min} = -\frac{11}{6}\pi$ and $n = 10$ the two positive solutions to the equation $f(\rho) = 0$ are obtained as $\rho_1 = 0.886635$ and $\rho_2 = 1.429398$. By choosing $\rho = \rho_1$ we obtain a logarithmic arc spline interpolating the given boundary data. If we choose $\rho = \rho_2$, the obtained l_0 is negative and a logarithmic arc spline interpolating the opposite directions of the specified end tangents is obtained. When we choose $\phi = \phi_{\min} - 4\pi$ and $n = 20$ we obtain a multi-winding logarithmic arc spline interpolating the Hermite data; see Fig. 5(b).

An example for logarithmic arc spline interpolation to Hermite data with $T_a = T_b$ is given in Fig. 2. A discrete logarithmic spiral consisting 10 or 40 circular arcs spline has been used to interpolate the boundary data of logarithmic spiral arc with winding angle 6π .

In Kurnosenko (2010) a spiral computed by inversions of logarithmic spiral has been used to interpolate G^2 Hermite data with a big winding angle. As this spiral is in fact a rational function composed of logarithmic spiral and Möbius map, it is not exactly the logarithmic spiral generally. Our proposed logarithmic arc spline can approximate a continuous logarithmic spiral with a high accuracy and maintains many elegant properties of logarithmic spirals. Due to its flexibility and simplicity for computation, logarithmic arc splines can be used in fields of design and modeling.

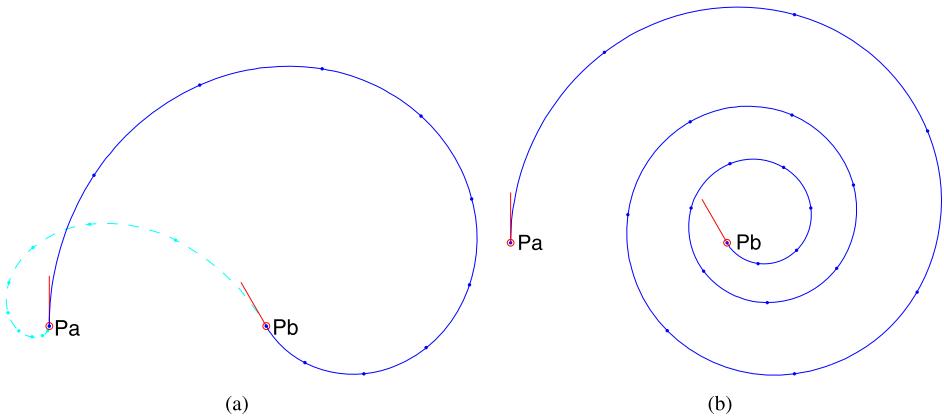


Fig. 5. Logarithmic arc spline interpolation: (a) the logarithmic arc spline (solid) interpolating the specified tangents and the logarithmic arc spline (dashed) interpolating the opposite directions of the given tangents; (b) an interpolating curve with a large winding angle.

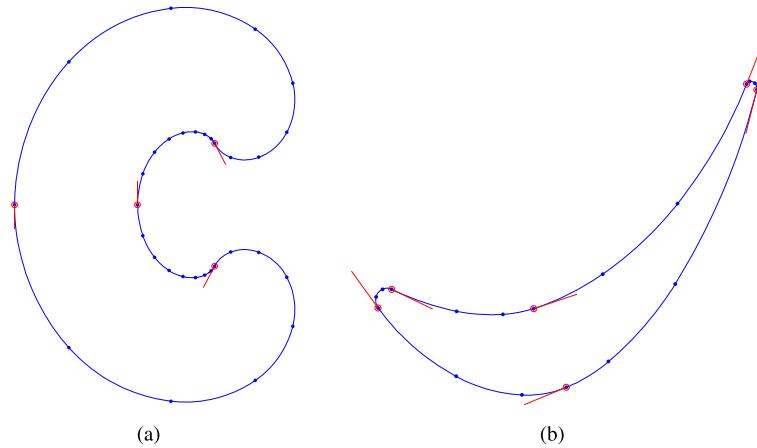


Fig. 6. Shape modeling by logarithmic arc spline interpolation: (a) model a “C”-shape curve by interpolating 4 pairs Hermite data; (b) model a blade profile by interpolating 6 pairs of Hermite data.

4. Applications

In this section we present a few more examples to show the applicability of logarithmic arc spline interpolation for various shape modeling purposes.

First, we present examples to model a “C”-shape curve and a blade profile by logarithmic arc spline interpolation. To model a “C”-like shape users can just input four points plus four tangents; see Fig. 6(a). From the figure we can see that the end tangents at two pairs of neighboring points lie on one side of the chord while the tangents for the rest two pairs of neighboring points lie on two sides of the corresponding chord. By choosing $\phi = \phi_{\min}$ and $n = 8$ for each pair of Hermite data, a G^1 smooth curve consisting of 32 circular arcs is obtained. By the same technique we model a blade profile by logarithmic arc spline interpolation. Starting from 6 input points together with 6 unit tangents at the points (Fig. 6(b)), the minimum winding angles for the curves interpolating each pair of consequent points and tangents are first computed. After then, an interpolating arc spline consisting of total 18 circular arcs is obtained by interpolating 6 pairs of Hermite data.

Next, we present an example of curve interpolation using single-winding as well as multi-winding logarithmic arc splines. A sequence of initial points and tangents are given in Fig. 7(a). For each pair of consequent points together with the unit tangents at the points the angles between the tangents and the chord are $\alpha = \frac{3}{8}\pi$, $\beta = \frac{\pi}{2}$ or $\alpha = \frac{\pi}{2}$, $\beta = \frac{3}{8}\pi$. By choosing $\phi = \phi_{\min} = -\frac{7\pi}{8}$ and $n = 5$ for constructing each logarithmic arc spline, a smooth curve consisting of total 80 circular arcs is obtained; see Fig. 7(b) for the interpolating curve. When we choose the winding angle $\phi = -\frac{7\pi}{8} - 2\pi$ and $n = 20$ for the construction of each interpolating logarithmic arc spline, the corresponding equation $f(\rho) = 0$ has just one positive root. As a result, a multi-winding logarithmic arc spline is obtained to interpolate each pair of Hermite data; see Fig. 7(c) for the interpolating arc splines or Fig. 7(d) for the final smooth curve.

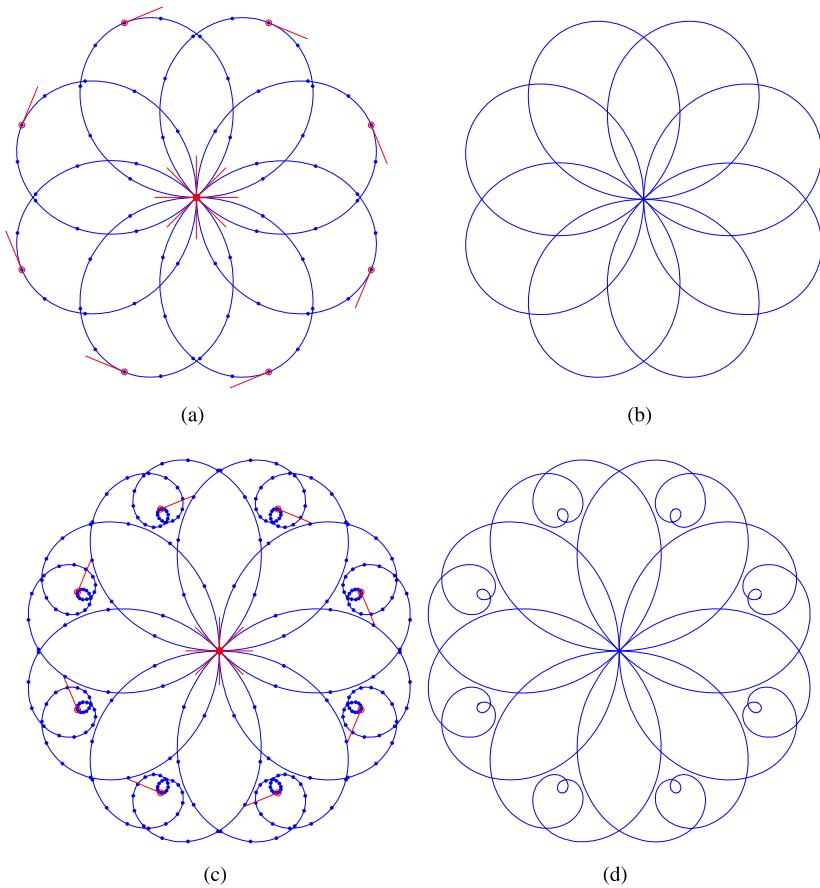


Fig. 7. Curve interpolation by logarithmic arc splines: (a) the input Hermite data and the interpolating single-winding logarithmic arc splines; (b) the interpolating logarithmic arc splines without the input data; (c) the Hermite data and the interpolating multi-winding logarithmic arc splines; (d) the interpolating curve without the tangents.

5. Conclusion

In this paper we have introduced logarithmic arc splines which are tangent smooth and have similar properties of logarithmic spirals. Logarithmic arc splines can approximate logarithmic spirals with high accuracies when the numbers of arcs are sufficiently large. Given two points together with two unit tangents at the points, a practical algorithm is developed to construct interpolating logarithmic arc splines with a specified winding angle. By formulating the interpolation problem as solving a vector equation, all solutions to the equation and all logarithmic arc splines that interpolate the boundary data can be obtained. As compared with previous logarithmic spiral interpolation schemes, the proposed method can be used to construct interpolating spirals with unbounded winding angles. It also benefits that the obtained curves are compatible with NURBS and the offsets are simple to compute.

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