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Geometric Hermite interpolation by a family of intrinsically defined planar curves

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ABSTRACT

This paper proposes techniques of interpolation of intrinsically defined planar curves to Hermite data. In particular, a family of planar curves corresponding to which the curvature radius functions are polynomials in terms of the tangent angle are used for the purpose. The Cartesian coordinates, the arc lengths and the offsets of this type of curves can be explicitly obtained provided that the curvature functions are known. For given G^1 or G^2 boundary data with or without prescribed arc lengths the free parameters within the curvature functions can be obtained just by solving a linear system. By choosing low order polynomials for representing the curvature radius functions, the interpolating curves can be spirals that have monotone curvatures or fair curves with small numbers of curvature extremes. Several examples of shape design or curve approximation using the proposed method are presented.

1. Introduction

Constructing fair curves that interpolate given boundary data is a fundamental task in shape design and geometric processing. Usually, G^1 Hermite interpolation is to find a planar or spatial curve between two given points, matching the corresponding unit tangent vectors at the points. If the G^1 interpolating curve also matches the given curvatures at the two ends, a G^2 Hermite interpolating curve is obtained.

A lot of work have been given in the literature to deal with the problem of G^1 or G^2 Hermite interpolation. Due to their popularity, polynomial or rational polynomial curves have been frequently used for geometric Hermite interpolation. Besides the interpolation constraints, the additional degrees of freedom of a polynomial curve can usually be determined by minimizing a bending energy function [1–5]. Alternatively, the free parameters within a polynomial or a rational curve should be determined by the criterion that the curve is a spiral [6–8] or the interpolating curve can have circle or conic precision [9,10]. When a uniform B-spline curve is used for interpolation of a sequence of sampled points, tangents and curvatures, the control points of the interpolating curve should be uniformly spaced such that the obtained curve is as fair as possible [11]. As Pythagorean Hodograph (PH) curves have rational offsets, this kind of curves has usually been used for interpolation of Hermite data for NC machining [12–15].

For applications such as fair shape design, highway route design and description of trajectories of mobile robots, various types of spirals that have monotone curvatures have been used for interpolation of G^1 Hermite data [16–20] or G^2 Hermite data [21–24]. Levien and Séquin have used Euler spiral for shape design [25] and argued some properties of a "best" interpolating spline [26]. Due to their nice properties, spiral fat arcs can even be used to find the intersections between curves [27]. Miura et al. derived a general equation of aesthetic curves, and presented their radius of curvature in terms of arc lengths [28,29]. In order to control the aesthetic curves interactively, Yoshida et al. proposed to represent the aesthetic curves with respect to the tangent angle [30]. Meek et al. derived the condition and proposed a method for planar two-point G^1 Hermite interpolation with log-aesthetic spirals [19]. For convenience of evaluation, Ziatdinov et al. derived the analytic parametric equations of log-aesthetic curves in terms of incomplete gamma functions [31]. A general family of fair curves called superspirals of which the radius of curvature is represented by Gauss hypergeometric function is given in [32]. Though superspirals have elegant properties, lack of simple analytic representation may restrict the practical applications.

Our proposed curve interpolation schemes are motivated by analytic representation of spirals and curvature plot driven curve design [33,34]. We expect that fair curves are explicitly defined by prescribed curvature functions and the obtained curves can interpolate given boundary data by choosing proper free parameters

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within the curvature functions. Based on the fact that the Cartesian coordinates of a spiral can be obtained by integrals of the radius of curvature we define a family of curves by choosing the curvature radius as polynomials. This is based on the following two observations. First, any continuous curvature radius function can be approximated by polynomials according to the Weierstrass approximation theorem. Second, the integrals that define the Cartesian coordinates of the curves can be explicitly evaluated. As a result, if the G^1 or G^2 boundary data are given, the coefficients of the polynomials can be directly derived from the interpolation equations and the interpolating curve can be explicitly obtained.

The paper is organized as follows. In Section 2, we define a family of integral curves of which the radius of curvature is represented by polynomials. Section 3 describes the planar two-point G^1 Hermite interpolation by a single integral curve with or without arc length constraint. Section 4 describes the planar two-point G^2 Hermite interpolation by a single integral curve with non-vanishing curvature radius. Section 5 is devoted to interpolation of the two-point Hermite data by two smooth-connected regular integral curves. How to interpolate a sequence of points with a G^1 or G^2 spline curve is discussed in Section 6. In Section 7, several interesting examples are given and Section 8 concludes the paper.

2. A family of intrinsically defined planar curves

In this section we revisit the definition of planar curves using curvature radius functions in terms of the tangent angle. Some nice properties of this kind of curves will be discussed. In particular, explicit formulae for computing the Cartesian coordinates will be given when the radius of curvatures are represented by polynomials.

Suppose that θ is the angle between the tangent direction of a local convex curve $\mathbf{r}(\theta)$ and the positive direction of x-axis. Then the Cartesian coordinates of the curve can be computed by the intrinsic equation [35]

$$\mathbf{r}(\theta) = \begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} = \begin{pmatrix} \displaystyle\int_0^\theta \rho(t)\cos t dt \\ \displaystyle\int_0^\theta \rho(t)\sin t dt \end{pmatrix} \tag{1}$$

where $\rho(t)$ is the radius of curvature of the curve. From the equation we know that the curve $\mathbf{r}(\theta)$ passes through the origin at $\theta = 0$ and is tangent to the x-axis at the origin.

With simple evaluation we have

$$\mathbf{r}'(\theta) = \begin{pmatrix} x'(\theta) \\ y'(\theta) \end{pmatrix} = \begin{pmatrix} \rho(\theta)\cos\theta \\ \rho(\theta)\sin\theta \end{pmatrix}.$$

Then the length of the derivative is $|\mathbf{r}'(\theta)| = \rho(\theta)$ and the unit normals to the curve are $\pm(-\sin\theta, \cos\theta)^T$. If the function $\rho(\theta)$ does not vanish on the whole domain, the obtained curve $\mathbf{r}(\theta)$ is regular. Since a regular integral curve defined by Eq. (1) is locally convex, it is known that any regular curve segment $\mathbf{r}(\theta), \theta_1 \le \theta \le \theta_2$ lies in the triangle formed by $\mathbf{r}(\theta_1)$, $\mathbf{r}(\theta_2)$ and the intersection point of tangent lines passing through $\mathbf{r}(\theta_1)$ or $\mathbf{r}(\theta_2)$ when $|\theta_2 - \theta_1| < \frac{\pi}{2}$. A potential application of this property is to find the intersections between two integral curves.

The arc length of the curve defined by Eq. (1) can be obtained as

$$L(\theta) = \int_0^\theta \rho(t)dt. \tag{2}$$

Choosing the unit normal to the curve as $\mathbf{n}(\theta) = (-\sin\theta, \cos\theta)^T$, the offset curve to $\mathbf{r}(\theta)$ with a sighed distance h is obtained as

$$\mathbf{r}_{offset}(\theta) = \mathbf{r}(\theta) + h\mathbf{n}(\theta). \tag{3}$$

From Eq. (1) we know that if $\rho(t)$ is chosen some elementary functions, e.g., polynomials, exponential functions or trigonometric functions, etc., the Cartesian coordinates of the curve can be explicitly obtained. Consequently, the arc length and the offset of the curve can be explicitly obtained too. For ease of computation we choose $\rho(t)$ as polynomials in this paper. As discussed later, the curves defined in this way can be used for geometric Hermite interpolation directly.

It is clear that the integral curve $\mathbf{r}(\theta)$ is a circle or a circular arc when $\rho(t)$ equals a constant. To introduce more degrees of freedom for designing the curvature profiles or for interpolation of Hermite data we choose polynomials other than constants to represent the curvature radius of an integral curve.

Firstly, we assume that the radius of curvature is formulated as a linear function

$$\rho(\theta) = c\theta + d, \quad \theta \in [0, \phi]. \tag{4}$$

Substituting Eq. (4) into Eq. (1), we obtain the Cartesian coordinates of the curve as follows

$$\begin{cases} x(\theta) = c(\theta\sin\theta + \cos\theta - 1) + d\sin\theta \\ y(\theta) = c(-\theta\cos\theta + \sin\theta) + d(1 - \cos\theta). \end{cases} \tag{5}$$

We note that the curve $\mathbf{r}(\theta)$ just represents a circle involute when the parameter d is chosen zero [35]. By choosing proper values for both free parameters c and d, the curve $\mathbf{r}(\theta)$ will be used for interpolation of G^1 Hermite data.

Secondly, we derive the explicit representation of curve $\mathbf{r}(\theta)$ by assuming that the radius of curvature is a cubic function

$$\rho(\theta) = a\theta^3 + b\theta^2 + c\theta + d, \quad \theta \in [0, \phi]. \tag{6}$$

Substituting Eq. (6) into Eq. (1), it yields

$$\begin{cases} x(\theta) = (a\theta^3 + b\theta^2 + c\theta + d - 6a\theta - 2b)\sin\theta \\ \qquad + (3a\theta^2 + 2b\theta + c - 6a)\cos\theta + 6a - c \\ y(\theta) = (6a\theta + 2b - a\theta^3 - b\theta^2 - c\theta - d)\cos\theta \\ \qquad + (3a\theta^2 + 2b\theta + c - 6a)\sin\theta - 2b + d. \end{cases} \tag{7}$$

When all the coefficients a, b, c and d are free, the integral curve $\mathbf{r}(\theta)$ can be used for interpolation of G^2 Hermite data in a specially chosen coordinate system. If the coefficient a is set zero, the Cartesian coordinates by Eq. (7) are the coordinates of curve $\mathbf{r}(\theta)$ with a quadratic integral kernel $\rho(\theta)$. This kind of curves will be used for G^1 Hermite interpolation that has a prescribed arc length.

Fig. 1 illustrates an integral curve that is defined by a linear curvature radius function. An integral curve together with a cubic curvature radius function is illustrated in Fig. 2.

We note that PH curves have also explicit representations of arc lengths and offsets [36], but the curves defined by Eq. (1) no longer lie in the polynomial space and their offsets are not rational curves anymore. If the curvature radius function $\rho(\theta)$ is of degree n, the intrinsically defined curves and their offsets just lie in the linear space $\Omega = span\{1, \sin\theta, \cos\theta, \theta\sin\theta, \ldots, \theta^n\cos\theta\}$. Within the new space typical curves such as circle involutes can be represented exactly by the presented forms.

3. G^1 Hermite interpolation

In this section we propose algorithms for constructing integral curves as defined in Section 2 to interpolate given G^1 Hermite data. Suppose that $\{P_1, T_1; P_2, T_2\}$ are the given boundary points and the unit tangents at the points. We will construct an interpolating curve by choosing a linear curvature radius function. If the arc length of the interpolating curve has also been given, a quadratic curvature radius function will be used for the construction of the interpolating curve.

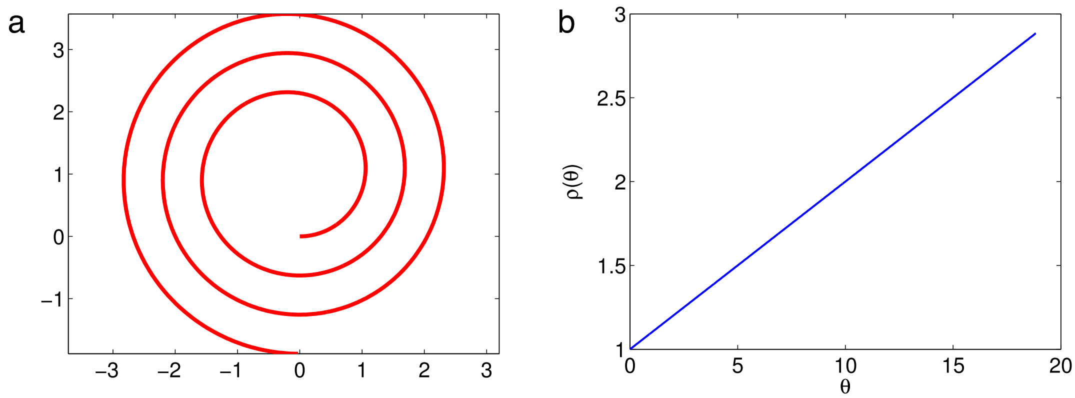

Fig. 1. (a) A curve defined by a linear curvature radius function; (b) the plot of the curvature radius function $\rho(\theta) = 0.1\theta + 1,\ \theta \in [0, 6\pi]$.

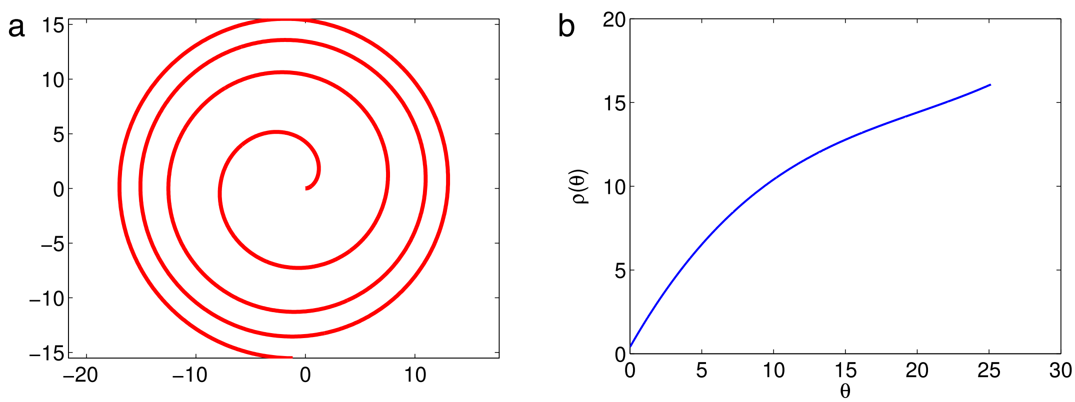

Fig. 2. (a) A curve defined by a cubic curvature radius function; (b) the plot of the curvature radius function $\rho(\theta) = 0.001\theta^3 - 0.06\theta^2 + 1.5\theta + 0.4,\ \theta \in [0, 8\pi]$.

For convenience of description we assume that P_1 is located at the origin and T_1 is parallel to the positive direction of x-axis. Furthermore, we assume that the curvature or the curvature radius of the interpolating curve is positive. If the boundary data do not satisfy these assumptions, they should be transformed into another coordinate system by translation, rotation or reflection operations and then the interpolating curve will be transformed into the original coordinate system.

3.1. G^1 Hermite interpolating curves that have linear curvature radius functions

As $\mathbf{r}(0) = P_1$ and ϕ is the angle between T_1 and T_2, the G^1 interpolating curve should satisfy $\mathbf{r}(\phi) = P_2$. By choosing $\rho(\theta) = c\theta + d$ and based on Eq. (5) we have

$$\begin{cases} c(\cos\phi + \phi\sin\phi - 1) + d\sin\phi &= x_2, \\ c(\sin\phi - \phi\cos\phi) + d(1 - \cos\phi) &= y_2. \end{cases}$$

Solving the linear system, the free parameters c and d are obtained as

$$c = \frac{x_2(1-\cos\phi) - y_2\sin\phi}{2\cos\phi + \phi\sin\phi - 2}, \quad d = \frac{x_2(\phi\cos\phi - \sin\phi) + y_2(\cos\phi + \phi\sin\phi - 1)}{2\cos\phi + \phi\sin\phi - 2}. \tag{8}$$

From its definition we know that the obtained curve $\mathbf{r}(\theta)$ is regular over the whole parameter domain $[0, \phi]$ if and only if the parameters c and d are valid and the linear function $\rho(\theta) = c\theta + d$ does not vanish over the domain. Concretely, if the denominator of Eq. (8) does not vanish and the linear function $\rho(\theta)$ satisfies $\rho(0)\rho(\phi) = d(c\phi + d) > 0$ the obtained curve is regular.

Proposition 1. *Suppose that P_1 lies at the origin, T_1 is parallel to the positive direction of the x-axis at the origin and ϕ is the signed angle from T_1 to T_2. If the boundary data satisfy*

$$\begin{cases} 2\cos\phi + \phi\sin\phi - 2 \neq 0 \\ (A_1x_2 + B_1y_2)(A_2x_2 + B_2y_2) > 0 \end{cases}$$

where $A_1 = \phi\cos\phi - \sin\phi$, $B_1 = \cos\phi + \phi\sin\phi - 1$, $A_2 = \phi - \sin\phi$ and $B_2 = \cos\phi - 1$, the G^1 Hermite interpolating curve given by Eqs. (5) and (8) is regular.

Let $g(\phi) = 2\cos\phi + \phi\sin\phi - 2$. It can be verified that $g(\phi) < 0$ when the angle ϕ satisfies $0 < \phi < 2\pi$. If $\phi > 0$, it is known that the tangent direction of a regular integral curve rotates counter-clockwise when the angle θ increases. Therefore, we have $\rho(\theta) > 0$ for $0 \le \theta \le \phi$. Because $\rho(0) = (A_1x_2 + B_1y_2)/g(\phi)$ and $\rho(\phi) = (A_2x_2 + B_2y_2)/g(\phi)$, we conclude that the G^1 interpolating curve is regular when both $A_1x_2 + B_1y_2 < 0$ and $A_2x_2 + B_2y_2 < 0$ hold.

Two points P_1, P_2 and two unit tangents T_1, T_2 are given as illustrated in Fig. 3. The minimum angle between T_1 and T_2 is $\pi/3$. By choosing $\phi = \pi/3$, an interpolating curve with increasing curvature radius is obtained; see Fig. 3(a) and (b) for the curve and the plot of the curvature radius function. Use the same geometric Hermite data but choose $\phi = 7\pi/3$, an interpolating curve with decreasing curvature radius is obtained; see Fig. 3(c) and (d).

3.2. G^1 Hermite interpolation with prescribed arc lengths

For the given G^1 Hermite data together with a prescribed arc length, an interpolating curve can be obtained by choosing the curvature radius function as $\rho(\theta) = b\theta^2 + c\theta + d$. Suppose that the data are properly positioned, the angle between T_1 and T_2 is ϕ and the prescribed arc length is L, the interpolating curve $\mathbf{r}(\theta)$ should satisfy $\mathbf{r}(\phi) = P_2$ and $\int_0^\phi \rho(\theta)d\theta = L$. By choosing $a = 0$ for Eq. (7) and by simple computation we have

$$\begin{cases} B_x b + C_x c + D_x d = x_2 \\ B_y b + C_y c + D_y d = y_2 \\ \dfrac{\phi^3}{3} b + \dfrac{\phi^2}{2} c + \phi d = L \end{cases} \tag{9}$$

where

$$\begin{cases} B_x = \phi^2 \sin\phi + 2\phi\cos\phi - 2\sin\phi \\ C_x = \phi\sin\phi + \cos\phi - 1 \\ D_x = \sin\phi \\ B_y = -\phi^2\cos\phi + 2\phi\sin\phi + 2\cos\phi - 2 \\ C_y = -\phi\cos\phi + \sin\phi \\ D_y = 1 - \cos\phi. \end{cases}$$

Solving the linear system (9), we have

$$\begin{aligned} b &= \frac{\hat{b}_1 x_2 + \hat{b}_2 y_2 + \hat{b}_3 L}{\phi f}, \\ c &= \frac{\hat{c}_1 x_2 + \hat{c}_2 y_2 + \hat{c}_3 L}{f}, \\ d &= \frac{L}{\phi} - \frac{1}{3}\phi^2 b - \frac{1}{2}\phi c, \end{aligned} \tag{10}$$

where

$$\begin{cases} \hat{b}_1 = -\frac{1}{2}\phi^2\cos\phi + \phi\sin\phi - \frac{1}{2}\phi^2 \\ \hat{b}_2 = -\frac{1}{2}\phi^2\sin\phi - \phi\cos\phi + \phi \\ \hat{b}_3 = \phi\sin\phi + 2\cos\phi - 2 \\ \hat{c}_1 = \frac{2}{3}\phi^2\cos\phi - 2\phi\sin\phi - 2\cos\phi + \frac{1}{3}\phi^2 + 2 \\ \hat{c}_2 = \frac{2}{3}\phi^2\sin\phi + 2\phi\cos\phi - 2\sin\phi \\ \hat{c}_3 = -\phi\sin\phi - 2\cos\phi + 2 \end{cases}$$

and $f = 4\cos\phi + 4\phi\sin\phi - 4 - \frac{4}{3}\phi^2\cos\phi - \frac{1}{6}\phi^3\sin\phi - \frac{2}{3}\phi^2$. If f does not vanish, the parameters b, c and d are valid and a G^1 interpolating curve is obtained by Eqs. (7) and (10).

To judge whether the interpolating curve is regular or not over the whole domain we should then judge whether the inequality $\rho(\theta) > 0$ holds for all $\theta \in [0, \phi]$. Let $t = \frac{\theta}{\phi}$ and suppose that b, c, d are given by Eq. (10). The quadratic curvature radius function $\rho(\theta)$ can be represented as follows

$$\begin{aligned} \rho(\theta) &= b\theta^2 + c\theta + d \\ &= l_0 B_0^2(t) + l_1 B_1^2(t) + l_2 B_2^2(t), \quad t \in [0, 1] \end{aligned}$$

where $l_0 = d$, $l_1 = d + \frac{1}{2}c\phi$, $l_2 = d + c\phi + b\phi^2$ and $B_i^2(t) = \frac{2!}{i!(2-i)!}t^i(1-t)^{2-i}$, $i = 0, 1, 2$ are the Bernstein basis functions. Combining this fact with Eq. (10) we know that l_0, l_1 and l_2 are just functions of x_2, y_2, ϕ and L. In particular, l_0, l_1 and l_2 are linear functions of L when x_2, y_2 and ϕ are fixed. Since $\rho(\theta) \geq \min\{l_0, l_1, l_2\}$ for $0 \leq \theta \leq \phi$, the inequality $\rho(\theta) > 0$ holds over the whole domain when all l_0, l_1 and l_2 are greater than zero.

Proposition 2. *Suppose that $\{P_1, T_1; P_2, T_2\}$ are the given Hermite data, where P_1 lies at the origin, T_1 is parallel to the positive direction of the x-axis. Assume that ϕ is the signed angle from T_1 to T_2 and L is the prescribed arc length. If the given data satisfy the following inequality system*

$$\begin{cases} l_0(x_2, y_2, \phi, L) > 0 \\ l_1(x_2, y_2, \phi, L) > 0 \\ l_2(x_2, y_2, \phi, L) > 0 \end{cases}$$

the interpolating curve obtained by Eqs. (7) and (10) is regular.

We note that the conditions addressed in Proposition 2 are conservative for checking whether an interpolating curve is regular or not. An interpolating curve may still be regular even if the mentioned inequality system does not hold. We construct interpolating curves under the assumption that the boundary conditions satisfy the inequality system. From Proposition 2 we also know that a feasible arc length L can be chosen in the interval given by the inequalities $l_i(L) > 0, i = 1, 2, 3$ when the two-point G^1 Hermite data have already been given.

Given $\{P_1, T_1; P_2, T_2\}$ as shown in Fig. 4(a) together with a prescribed arc length L, an interpolating curve with one curvature extremum is obtained; see Fig. 4 for the curve and the plot of the curvature radius.

4. G^2 Hermite interpolation

Planar two-point G^2 Hermite data consist of two points, two unit tangent vectors and two curvature radii given at the points, which are denoted by $\{P_1, T_1, r_1; P_2, T_2, r_2\}$. We derive formulae for constructing G^2 interpolating curves from the given boundary data. Conditions under which the G^2 interpolating curves are guaranteed to be regular or being spirals will also be discussed.

4.1. *Constructing G^2 interpolating curves*

As in the previous section we assume that the G^1 Hermite data are properly positioned. To guarantee that a G^1 interpolating curve also interpolates two given curvature radii at the ends, we choose the curvature radius function as a cubic function, as given by Eq. (6). If the parameters a, b, c and d of $\rho(t)$ are known, the G^2 interpolating curve $\mathbf{r}(\theta)$ is computed by Eq. (7).

The free parameters of a cubic function $\rho(t)$ should be determined by the following interpolation conditions

$$\begin{cases} \mathbf{r}(\phi) = P_2, \\ \rho(0) = r_1, \\ \rho(\phi) = r_2. \end{cases}$$

By substituting Eqs. (6) and (7), the interpolating equations can be reformulated as

$$\begin{cases} A_x a + B_x b + C_x c + D_x d = x_2, \\ A_y a + B_y b + C_y c + D_y d = y_2, \\ d = r_1, \\ \phi^3 a + \phi^2 b + \phi c + d = r_2, \end{cases} \tag{11}$$

where

$$A_x = \phi^3\sin\phi + 3\phi^2\cos\phi - 6\phi\sin\phi - 6\cos\phi + 6,$$
$$A_y = -\phi^3\cos\phi + 3\phi^2\sin\phi + 6\phi\cos\phi - 6\sin\phi$$

and the remaining coefficients are as defined in system (9).

Denote $g = (-8\phi^2+24)\cos\phi + (-\phi^3+24\phi)\sin\phi - 4\phi^2 - 24$. The solutions to the linear system (11) are as follows,

$$\begin{aligned} a &= \frac{a_1 r_1 + a_2 r_2 + a_3}{\phi g}, \\ b &= \frac{b_1 r_1 + b_2 r_2 + b_3}{g}, \\ c &= \frac{r_2 - r_1}{\phi} - \phi^2 a - \phi b, \\ d &= r_1, \end{aligned} \tag{12}$$

where

$$\begin{cases} a_1 = \phi^2\cos\phi - 4\cos\phi - 4\phi\sin\phi + \phi^2 + 4 \\ a_2 = -\phi^2\cos\phi + 4\cos\phi - \phi^2 - 4 + 4\phi\sin\phi \\ a_3 = x_2(2\phi\cos\phi + \phi^2\sin\phi - 2\phi) \\ \qquad + y_2(-\phi^2\cos\phi + 2\phi\sin\phi - \phi^2) \\ b_1 = -2\phi^2\cos\phi + 12\cos\phi + 9\phi\sin\phi - \phi^2 - 12 \\ b_2 = \phi^2\cos\phi + 2\phi^2 - 3\phi\sin\phi \\ b_3 = x_2(-6\phi\cos\phi - 2\phi^2\sin\phi + 6\sin\phi) \\ \qquad + y_2(2\phi^2\cos\phi - 6\cos\phi - 6\phi\sin\phi + \phi^2 + 6). \end{cases}$$

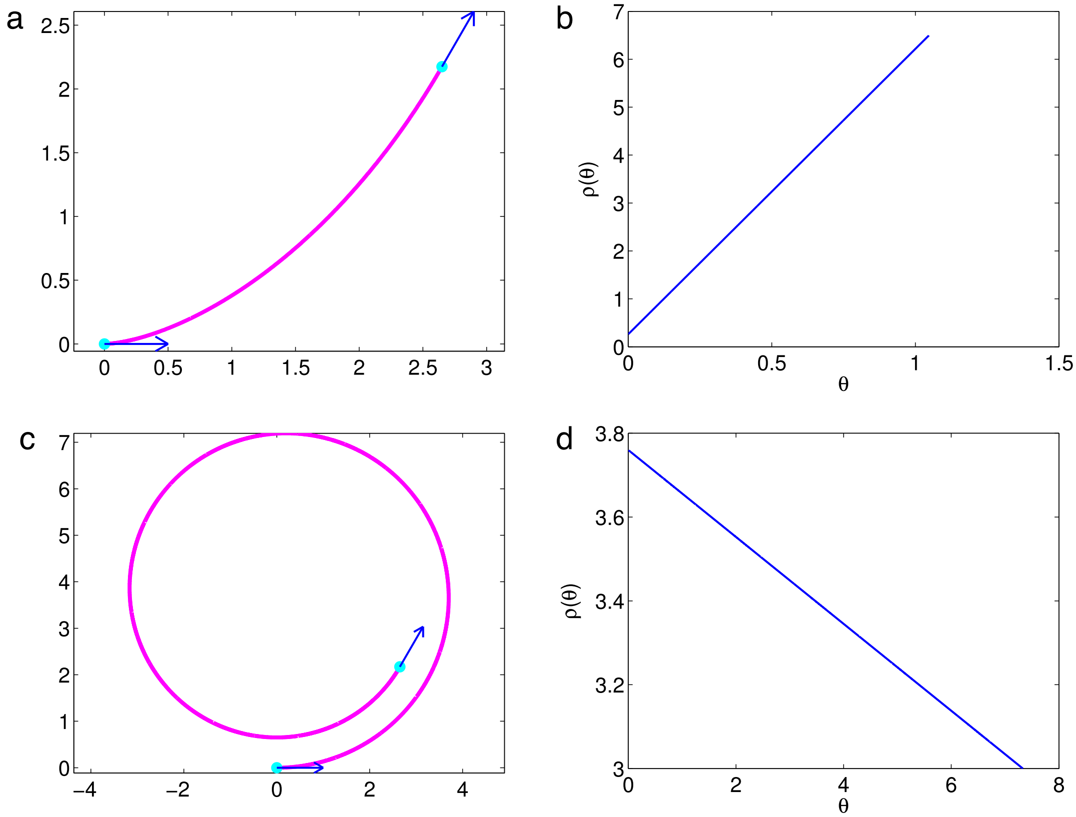

Fig. 3. G^1 Hermite interpolation: (a) the interpolating curve with $\phi = \pi/3, P_2 = (2.65, 2.17)$; (b) the curvature radius plot of the curve in (a); (c) the interpolating curve with $\phi = 7\pi/3, P_2 = (2.65, 2.17)$; (d) the curvature radius plot of the curve in (c).

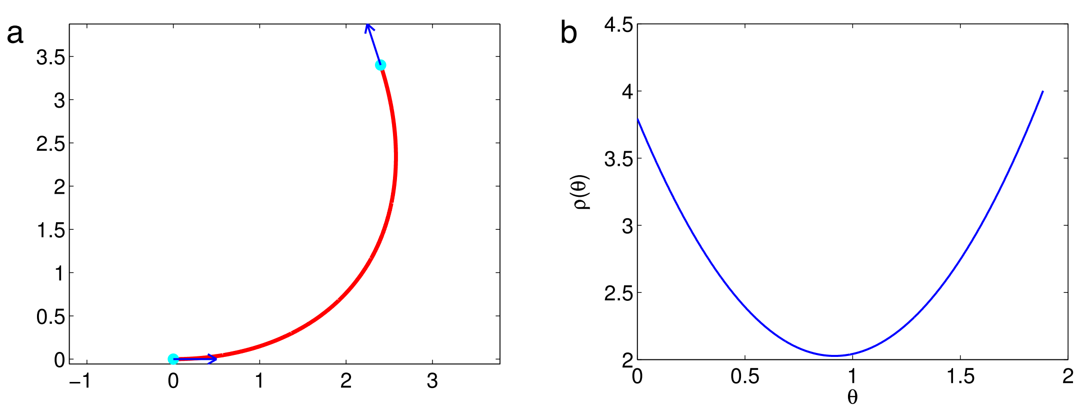

Fig. 4. G^1 Hermite interpolation with a prescribed arc length: (a) the curve that interpolates the Hermite data with $\phi = 0.6\pi, P_2 = (2.4, 3.4)$ and preserves the given arc length $L = 5$; (b) the plot of curvature radius of the interpolating curve.

When g does not vanish, the parameters a, b, c and d are valid and a G^2 interpolating curve is obtained by Eq. (7).

4.2. An admissible input of the curvature radii at two ends

To guarantee that the interpolating curves obtained by Eqs. (7) and (12) are regular, the input curvature radii at two ends should satisfy some conditions when the two-point G^1 Hermite data are already known. The conditions under which the interpolating curves are spirals will also be discussed.

Let $t = \frac{\theta}{\phi}$. The cubic curvature radius function $\rho(\theta) = a\theta^3 + b\theta^2 + c\theta + d, \theta \in [0, \phi]$ can be represented by Bernstein basis functions as follows,

$$\rho(\theta) = \rho_0 B_0^3(t) + \rho_1 B_1^3(t) + \rho_2 B_2^3(t) + \rho_3 B_3^3(t), \quad t \in [0, 1] \quad (13)$$

where

$$\begin{cases} \rho_0 = d, \\ \rho_1 = d + \frac{1}{3}c\phi, \\ \rho_2 = d + \frac{2}{3}c\phi + \frac{1}{3}b\phi^2, \\ \rho_3 = d + c\phi + b\phi^2 + a\phi^3. \end{cases}$$

If ρ_0, ρ_1, ρ_2 and ρ_3 are all positive numbers, the function satisfies $\rho(\theta) > 0$ for all $\theta \in [0, \phi]$ and the interpolating curve is regular over the whole domain.

It is obvious that $\rho(0) = \rho_0 = r_1$, $\rho(\phi) = \rho_3 = r_2$. In order to judge whether or not the curvature radii r_1 and r_2 are properly input, we reformulate ρ_1 and ρ_2 as the functions with respect to r_1, r_2. We have

$$\rho_1 = \left[\frac{2}{3} - \frac{\phi^2}{3g}(a_1+b_1)\right] r_1 + \left[\frac{1}{3} - \frac{\phi^2}{3g}(a_2+b_2)\right] r_2 - \frac{\phi^2}{3g}(a_3+b_3), \tag{14}$$

$$\rho_2 = \left[\frac{1}{3} - \frac{\phi^2}{3g}(2a_1+b_1)\right] r_1 + \left[\frac{2}{3} - \frac{\phi^2}{3g}(2a_2+b_2)\right] r_2 - \frac{\phi^2}{3g}(2a_3+b_3). \tag{15}$$

So, if the input curvature radii satisfy $r_1 > 0, r_2 > 0, \rho_1(r_1, r_2) > 0$ and $\rho_2(r_1, r_2) > 0$, the interpolating curve is regular.

Similarly, the derivative of the curvature radius function is reformulated as

$$\begin{aligned}\rho'(\theta) &= 3a\theta^2 + 2b\theta + c \\ &= d_0B_0^2(t) + d_1B_1^2(t) + d_2B_2^2(t), \quad t \in [0, 1]\end{aligned}$$

where

$$\begin{cases} d_0 = c, \\ d_1 = c + b\phi, \\ d_2 = c + 2b\phi + 3a\phi^2. \end{cases}$$

Once again, we reformulate d_0, d_1, d_2 as the functions with respect to r_1 and r_2. It yields that

$$d_0 = \left[-\frac{1}{\phi} - \frac{\phi}{g}(a_1+b_1)\right] r_1 + \left[\frac{1}{\phi} - \frac{\phi}{g}(a_2+b_2)\right] r_2 - \frac{\phi}{g}(a_3+b_3),$$

$$d_1 = \left[-\frac{1}{\phi} - \frac{\phi}{g}a_1\right] r_1 + \left[\frac{1}{\phi} - \frac{\phi}{g}a_2\right] r_2 - \frac{\phi}{g}a_3,$$

$$d_2 = \left[-\frac{1}{\phi} + \frac{\phi}{g}(2a_1+b_1)\right] r_1 + \left[\frac{1}{\phi} + \frac{\phi}{g}(2a_2+b_2)\right] r_2 + \frac{\phi}{g}(2a_3+b_3).$$

If both r_1 and r_2 are positive and $d_i(r_1, r_2), i = 0, 1, 2$ have the same sign, then the curvature radius function $\rho(\theta)$ is monotone and has no zeros within the interval $[0, \phi]$. Therefore, the interpolating curve is a spiral.

Summarizing the above analysis we have the following proposition.

Proposition 3. *Suppose that the Hermite data $\{P_1, T_1; P_2, T_2\}$ are as given in Proposition 1 and the G^1 interpolating curve is regular. We construct a G^2 interpolating curve by using given curvature radii r_1 and r_2 at two ends.*

(1) *If r_1 and r_2 satisfy*

$$\begin{cases} r_1, r_2 > 0 \\ \rho_1(r_1, r_2) > 0 \\ \rho_2(r_1, r_2) > 0 \end{cases}$$

the G^2 interpolating curve is regular over the domain $[0, \phi]$.

(2) *If r_1 and r_2 satisfy*

$$\begin{cases} r_2 > r_1 > 0 \\ d_0(r_1, r_2) > 0 \\ d_1(r_1, r_2) > 0 \\ d_2(r_1, r_2) > 0 \end{cases}$$

or

$$\begin{cases} r_1 > r_2 > 0 \\ d_0(r_1, r_2) < 0 \\ d_1(r_1, r_2) < 0 \\ d_2(r_1, r_2) < 0 \end{cases}$$

the G^2 interpolating curve is a spiral segment.

Proposition 3 can be used to check whether an interpolating curve is regular or whether it is a spiral. Based on the inequalities given in the proposition, one can also choose proper values for r_1 and r_2 such that the interpolating curve is guaranteed to be regular or to be a spiral when the two-point G^1 Hermite data are already known.

Fig. 5 illustrates an example of G^2 Hermite interpolation by the proposed technique. The two boundary points and the unit tangents at the points are the same as the example illustrated in Fig. 3(a). The allowable region for the choices of boundary curvature radii for the construction of regular G^2 interpolating curves is illustrated in Fig. 5(a). By choosing $r_1 = 2$ and $r_2 = 9$, a G^2 interpolating curve is obtained. See Fig. 5(b) for the interpolating curve and Fig. 5(c) for the plot of curvature radius of the curve. The region for the choices of r_1 and r_2 for the construction of G^2 interpolating spirals is illustrated in Fig. 6(a). By choosing $r_1 = 0.7$ and $r_2 = 6.7$, a G^2 interpolating curve with monotone curvature radius is obtained; see Figs. 6(b) and 6(c) for the interpolating curve and the curvature radius plot of the curve, respectively.

5. Two-point Hermite interpolation by a pair of smooth-connected curves

From Sections 3 and 4 we know that a G^1 or G^2 Hermite interpolating curve is regular only when the boundary data are properly given. For arbitrary given boundary data the interpolating curves may not be regular over the whole domain. If a single curve interpolating the two-point Hermite data is singular, we can insert one new point together with a properly chosen tangent direction or curvature radius at the point and construct a pair of regular integral curves to interpolate the given and the added Hermite data.

Similarly to the assumptions given in Proposition 1, we suppose that $\{P_0, T_0; P_2, T_2\}$ are the given Hermite data, where P_0 lies at the origin, T_0 is parallel to the positive direction of the x-axis. Denote ϕ_2 as the angle from T_0 to T_2. A new point P_1 and a new tangent direction T_1 will be inserted when the conditions given in Proposition 1 are not satisfied. In fact, the insertion point and tangent always exist when neither T_0 nor T_2 is parallel to the vector $P_2 - P_0$. This is because that the proposed curves are just the generalizations of circular arcs and at least there always exist a family of biarc curves interpolating the boundary data [37,38].

Besides the joint points of two tangent continuous circular arcs, the insertion points can even be chosen in a wider region. We demonstrate this assertion by choosing insertion points when the tangents T_0, T_2 lie on two sides of the line P_0P_2, under which a C-shaped interpolating biarc curve exists. If the tangents T_0 and T_2 lie on the same side of the line P_0P_2, we construct an S-shaped biarc curve directly. Similarly to the tangent selection at a joint point for a C-shaped biarc curve, we first choose the tangent direction T_1 as the unit vector parallel to the line P_0P_2. Then we choose the joint point P_1 in an admissible region.

Let ϕ_1 be the angle from T_0 to T_1. Then the angle from T_1 to T_2 is $\phi_2 - \phi_1$. Denote $P_i = (x_i, y_i)^T$, $i = 1, 2$. We choose P_1 based on the assumption that the G^1 interpolating curves from P_0 to P_1 or from P_1 to P_2 are both regular. To apply Proposition 1 for the second interpolating curve we transform the data $\{P_1, T_1; P_2, T_2\}$ to $\{\bar{P}_1, \bar{T}_1; \bar{P}_2, \bar{T}_2\}$ by translation and rotation such that $\bar{P}_1$ lies at the origin and $\bar{T}_1$ is parallel to the positive direction of the x-axis. Thus, the angle from $\bar{T}_1$ to $\bar{T}_2$ is $\phi_2 - \phi_1$ and the coordinates of new point $\bar{P}_2 = (\bar{x}, \bar{y})^T$ are obtained as

$$\begin{pmatrix}\bar{x} \\ \bar{y}\end{pmatrix} = \begin{pmatrix}\cos\phi_1 & \sin\phi_1 \\ -\sin\phi_1 & \cos\phi_1\end{pmatrix} \begin{pmatrix}x_2 - x_1 \\ y_2 - y_1\end{pmatrix}.$$

By applying Proposition 1 to both Hermite data $\{P_0, T_0; P_1, T_1\}$ and $\{\bar{P}_1, \bar{T}_1; \bar{P}_2, \bar{T}_2\}$, we obtain inequalities about the free coordinates of point P_1 based on the assumption that the interpolating curves

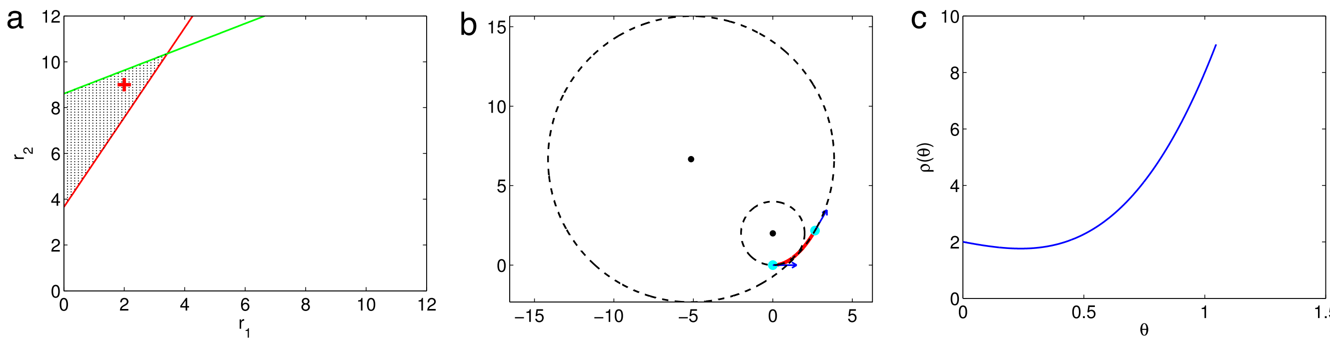

Fig. 5. G^2 interpolation by a C-shaped curve: (a) the admissible region for G^2 Hermite interpolation with respect to radii r_1 and r_2; (b) $P_1 = (0, 0), P_2 = (2.65, 2.17), \phi = \pi/3, r_1 = 2, r_2 = 9$; (c) the plot of radius of curvature.

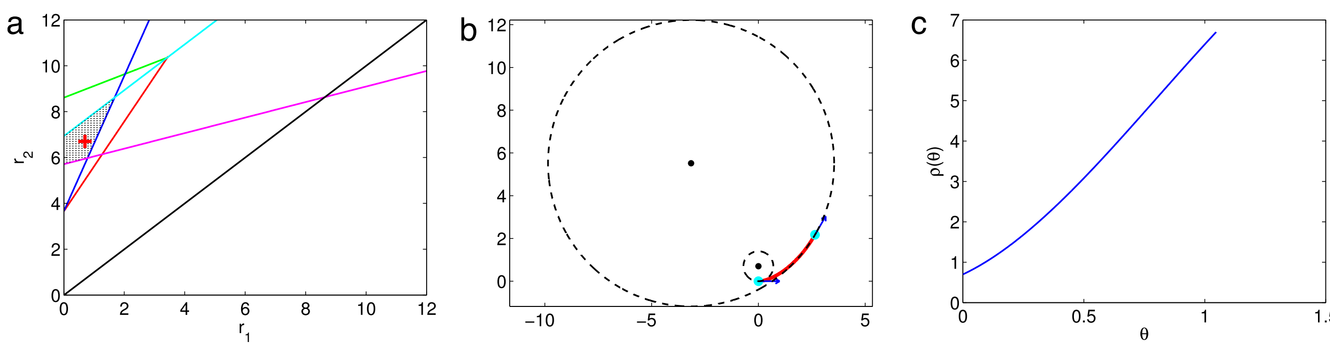

Fig. 6. G^2 interpolation by a spiral segment: (a) the admissible region for G^2 Hermite interpolation with respect to radii r_1 and r_2; (b) $P_1 = (0, 0), P_2 = (2.65, 2.17), \phi = \pi/3, r_1 = 0.7, r_2 = 6.7$; (c) the plot of radius of curvature.

are regular. Suppose that $A_1(\phi)$, $A_2(\phi)$, $B_1(\phi)$ and $B_2(\phi)$ are the functions of ϕ as defined in Proposition 1 and let $\bar{\phi} = \phi_2 - \phi_1$. The joint point P_1 should be chosen satisfying the following inequalities

$$\begin{cases} R_{11}(x_1, y_1) \cdot R_{12}(x_1, y_1) > 0 \\ R_{21}(x_1, y_1) \cdot R_{22}(x_1, y_1) > 0 \\ 2\cos\phi_1 + \phi_1 \sin\phi_1 - 2 \neq 0 \\ 2\cos\bar{\phi} + \bar{\phi}\sin\bar{\phi} - 2 \neq 0 \end{cases} \tag{16}$$

where

$$R_{11}(x_1, y_1) = A_1(\phi_1)x_1 + B_1(\phi_1)y_1$$
$$R_{12}(x_1, y_1) = A_2(\phi_1)x_1 + B_2(\phi_1)y_1$$
$$R_{21}(x_1, y_1) = A_1(\bar{\phi})[\cos\phi_1(x_2 - x_1) + \sin\phi_1(y_2 - y_1)] + B_1(\bar{\phi})[\cos\phi_1(y_2 - y_1) - \sin\phi_1(x_2 - x_1)]$$
$$R_{22}(x_1, y_1) = A_2(\bar{\phi})[\cos\phi_1(x_2 - x_1) + \sin\phi_1(y_2 - y_1)] + B_2(\bar{\phi})[\cos\phi_1(y_2 - y_1) - \sin\phi_1(x_2 - x_1)].$$

We explain the choice of the joint point P_1 by assuming that two C-shaped integral curves with positive curvature radius are constructed to interpolate the Hermite data. Practically, it can be further assumed that $0 < \phi_1, \bar{\phi} < 2\pi$. Similarly to Proposition 1, regular interpolating curves can be obtained by choosing point P_1 satisfying the inequalities $R_{ij}(x_1, y_1) < 0$, $i, j = 1, 2$. We note that the admissible region defined by these inequalities is not empty. Besides the joint points of interpolating biarc curves, one can choose some other insertion points in the admissible region for the construction of fair interpolating curves.

Now, we consider how to interpolate G^2 Hermite data $\{P_0, T_0, r_0; P_2, T_2, r_2\}$ by a pair of smooth-connected integral curves when one interpolating curve is not regular. If there is no single regular curve interpolating the given boundary data according to Proposition 3, we can insert one new point P_1 together with a unit tangent vector T_1 and a properly chosen radius r_1 at the point and construct two integral curves to interpolate the Hermite data.

To make the task even simpler, we first choose the joint point P_1 and the unit tangent vector T_1 in the same way as for G^1 Hermite interpolation. After that, we choose a radius r_1 at the joint point P_1 based on the criterion that both curves interpolating Hermite data $\{P_0, T_0, r_0; P_1, T_1, r_1\}$ or $\{P_1, T_1, r_1; P_2, T_2, r_2\}$ are regular.

To apply Proposition 3 for choosing values for r_1, the transformed Hermite data $\{\bar{P}_1, \bar{T}_1, r_1; \bar{P}_2, \bar{T}_2, r_2\}$ will be used instead of $\{P_1, T_1, r_1; P_2, T_2, r_2\}$. Suppose that the cubic curvature radius functions of the curves that interpolate $\{P_0, T_0, r_0; P_1, T_1, r_1\}$ or $\{\bar{P}_1, \bar{T}_1, r_1; \bar{P}_2, \bar{T}_2, r_2\}$ have been formulated as $\rho(t) = \sum_{i=0}^{3} \rho_i B_i^3(t)$ or $\bar{\rho}(t) = \sum_{i=0}^{3} \bar{\rho}_i B_i^3(t)$, respectively. From Eq. (13) we know that $\rho_0 = r_0, \rho_3 = \bar{\rho}_0 = r_1$ and $\bar{\rho}_3 = r_2$. Similarly to the derivation of Eqs. (14) and (15), the remaining coefficients are obtained as follows

$$\rho_1 = \left[\frac{1}{3} - \frac{\phi_1^2}{3g}(a_2 + b_2)\right] r_1 + \left[\frac{2}{3} - \frac{\phi_1^2}{3g}(a_1 + b_1)\right] r_0 - \frac{\phi_1^2}{3g}(a_3 + b_3)$$

$$\rho_2 = \left[\frac{2}{3} - \frac{\phi_1^2}{3g}(2a_2 + b_2)\right] r_1 + \left[\frac{1}{3} - \frac{\phi_1^2}{3g}(2a_1 + b_1)\right] r_0 - \frac{\phi_1^2}{3g}(2a_3 + b_3)$$

$$\bar{\rho}_1 = \left[\frac{2}{3} - \frac{\bar{\phi}^2}{3\bar{g}}(\bar{a}_1 + \bar{b}_1)\right] r_1 + \left[\frac{1}{3} - \frac{\bar{\phi}^2}{3\bar{g}}(\bar{a}_2 + \bar{b}_2)\right] r_2 - \frac{\bar{\phi}^2}{3\bar{g}}(\bar{a}_3 + \bar{b}_3)$$

$$\bar{\rho}_2 = \left[\frac{1}{3} - \frac{\bar{\phi}^2}{3\bar{g}}(2\bar{a}_1 + \bar{b}_1)\right] r_1 + \left[\frac{2}{3} - \frac{\bar{\phi}^2}{3\bar{g}}(2\bar{a}_2 + \bar{b}_2)\right] r_2 - \frac{\bar{\phi}^2}{3\bar{g}}(2\bar{a}_3 + \bar{b}_3)$$

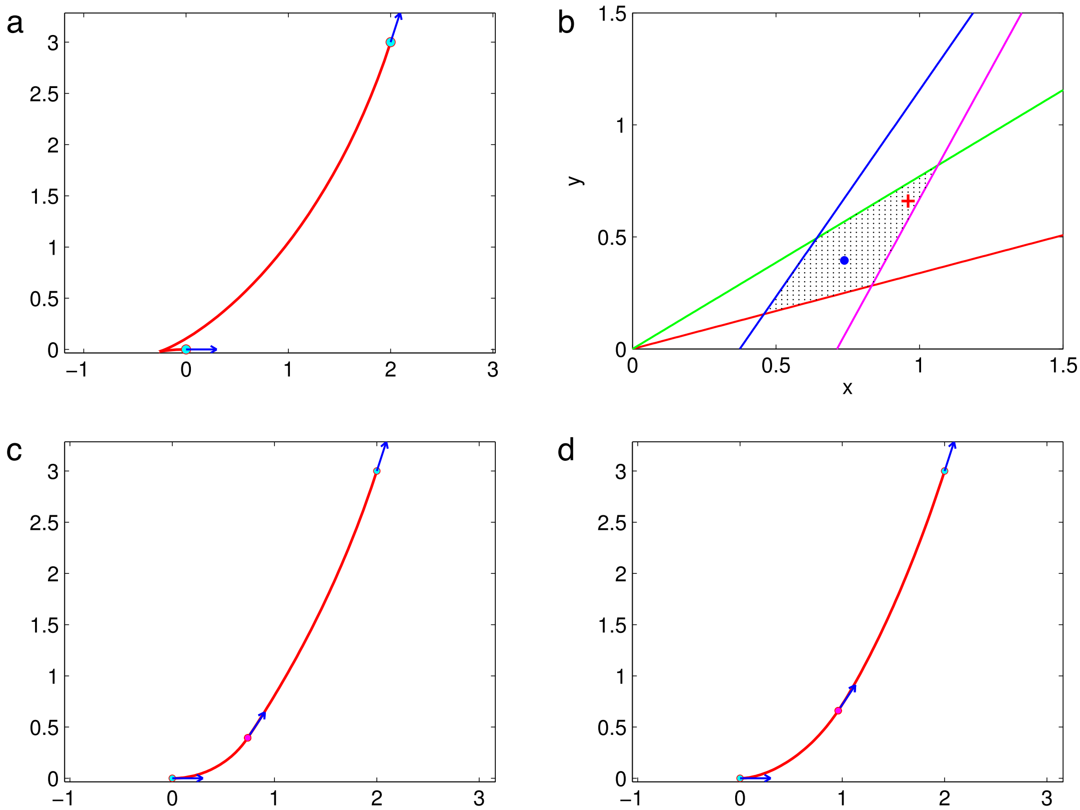

Fig. 7. G^1 Hermite interpolation: (a) a single integral curve that interpolates the Hermite data, where $P_0 = (0, 0)$, $P_2 = (2, 3)$ and $\phi_2 = 0.4\pi$; (b) the admissible region for choosing an inserted point at where the two interpolating curves have a common predefined tangent direction; (c) an interpolating biarc curve, with joint point $P_1 = (0.7382, 0.3951)$ which corresponds to the solid circle in (b); (d) the interpolating integral curves with joint point $P_1 = (0.96, 0.66)$ which corresponds to the cross point in (b).

where the coefficients g, a_1, a_2, a_3, b_1, b_2, b_3 or $\bar{g}$, $\bar{a}_1$, $\bar{a}_2$, $\bar{a}_3$, $\bar{b}_1$, $\bar{b}_2$, $\bar{b}_3$ are as defined in Eq. (12) by using new angles between the boundary tangents and the coordinates of new end points. Since r_0 and r_2 are known positive values, the coefficients ρ_1, ρ_2, $\bar{\rho}_1$ and $\bar{\rho}_2$ stated above are just linear functions of the free variable r_1. From Proposition 3, if we choose values for r_1 satisfying $r_1 > 0$, $\rho_1(r_1) > 0$, $\rho_2(r_1) > 0$, $\bar{\rho}_1(r_1) > 0$ and $\bar{\rho}_2(r_1) > 0$, two regular integral curves interpolating the given or the chosen Hermite data will be obtained.

At present, it is not guaranteed that the inequalities $\rho_i(r_1) > 0$, $\bar{\rho}_i(r_1) > 0$, $i = 1, 2$ definitely have a positive solution. If the inequalities have no positive solution, the original Hermite data can still be interpolated by a single or two G^1 continuous regular curves.

Fig. 7 illustrates an example of G^1 Hermite interpolation by a pair of regular integral curves while a single interpolating curve is singular. The initial Hermite data and a single interpolating curve are shown in Fig. 7(a). By choosing the tangent direction as the unit vector parallel to the chord P_0P_2, the admissible region for the choice of P_1 is obtained; see Fig. 7(b). Fig. 7(c) shows an interpolating biarc curve and Fig. 7(d) gives a pair of G^1 continuous integral curves that also interpolate the original Hermite data by choosing a different joint point from that of the biarc curve. Using the same boundary points and tangents as in Fig. 7 but choosing two additional curvature radii at the points we obtain a pair of G^2 Hermite data. Though a single curve that interpolates the G^2 Hermite data exists, it is not regular; see Fig. 8(a). To interpolate the G^2 Hermite data by a pair of G^2 continuous regular integral curves, we insert the same joint point and tangent as in Fig. 7(d) and choose a curvature radius in the admissible interval as illustrated in Fig. 8(b). The interpolating regular curves are shown in Fig. 8(c) and their curvature radii are plotted in Fig. 8(d).

6. Constructing an interpolating spline curve

Based on the Hermite interpolation schemes stated above we present here a practical algorithm for constructing G^1 or G^2 interpolating spline curves from a sequence of given points. Suppose that $\{Q_i\}_{i=0}^n$ are a sequence of given points in $\mathbb{R}^2$ and no two neighboring points are the same. Assuming $\{\mathbf{d}_i\}_{i=0}^n$ are the unit tangent vectors corresponding to the points and $\{r_i\}_{i=0}^n$ are the curvature radii, we construct a G^1 or G^2 interpolating spline curve by constructing Hermite interpolating curves to every two neighboring points together with the tangents or curvature radii at the points.

Assume that $\mathbf{u} \times \mathbf{v}$ represents the scalar product of two planar vectors. Let $\Delta Q_i = Q_{i+1} - Q_i$ for $i = 0, 1, \ldots, n-1$. If $\mathbf{d}_i \times \Delta Q_i$ and $\Delta Q_i \times \mathbf{d}_{i+1}$ have the same sign, the tangents $\mathbf{d}_i$ and $\mathbf{d}_{i+1}$ lie on two sides of the line Q_iQ_{i+1} and a local convex interpolating curve will be constructed to interpolate the Hermite data. Otherwise, the tangents $\mathbf{d}_i$ and $\mathbf{d}_{i+1}$ lie on the same side of the line Q_iQ_{i+1} and a new point together with a unit tangent should be inserted between the two points or the two-point Hermite data are interpolated by a biarc curve directly. Even if every two neighboring points together with the tangents at the points permit a local convex interpolating curve, some joint points may still be treated as inflection points. If $\mathbf{d}_{i-1} \times \Delta Q_{i-1}$ and $\mathbf{d}_i \times \Delta Q_i$ have different signs, Q_i is considered as an inflection point. The interpolating spline curve has only G^1 continuity at the inflection points.

We present algorithm steps for constructing a G^1 interpolating spline curve. The G^2 interpolating spline curve (if it exists) can be constructed in a similar way. The G^1 interpolating spline curve can be obtained by interpolating every two consecutive points. For points Q_i and Q_{i+1} together with unit tangents $\mathbf{d}_i$ and $\mathbf{d}_{i+1}$ the Hermite interpolating curve is constructed as follows:

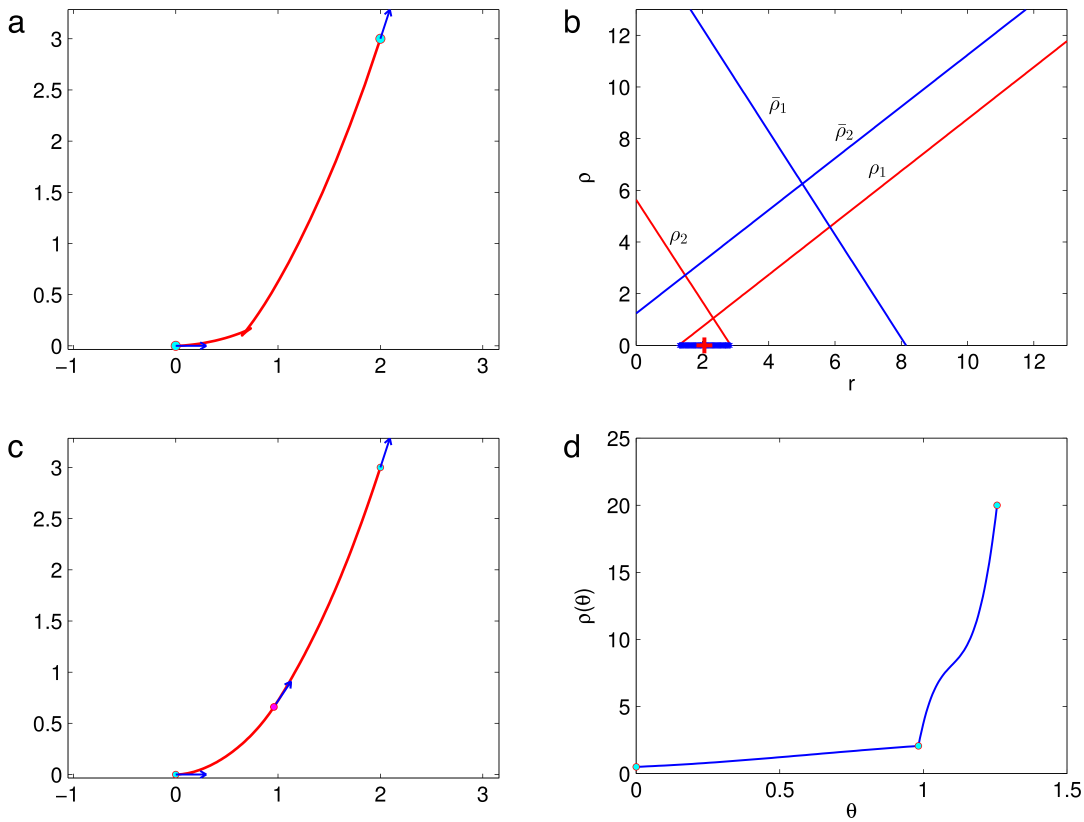

Fig. 8. G^2 Hermite interpolation: (a) a single integral curve that interpolates G^2 Hermite data with $P_0 = (0, 0), P_2 = (2, 3), \phi_2 = 0.4\pi, r_0 = 0.5, r_2 = 20$; (b) the admissible interval (the thick interval on the horizontal axis) for the choice of curvature radius at a predefined joint point; (c) the interpolating G^2 continuous curves; (d) the plot of radius of curvature for the interpolating curves.

(1) Transform the coordinate system such that the transformed boundary data match the basic assumptions.
(2) Compute the angle ϕ between the two unit tangents.
(3) Solve the free parameters for the curvature radius function.
(4) Check whether the interpolating curve is regular or not.
  (4.1) If there exists a regular interpolating curve, compute the curve and transform it to the original coordinate system.
  (4.2) Otherwise, interpolate the Hermite data by a pair of regular integral curves.

7. Examples

In this section we present a few more examples to demonstrate the applications of the proposed curve interpolation technique in geometric modeling. The comparison with PH curve interpolation and the approximation rates to known smooth curves will also be given or discussed.

First, we model the profile of character 'δ' by interpolating a sequence of sampled points with PH spline curves [39] or the Hermite interpolating curves presented in this paper. As there exist several inflection points within the sampled points, the interpolating PH spline curves are constructed independently for each set of convex data. Consequently, the final PH spline curve is piecewise G^2 continuous and has only G^1 continuity at the inflection points; see Fig. 9(a) for the interpolating PH spline curve and Fig. 9(c) for the curvature plot of the interpolating curve. We construct smooth interpolating curves by assigning a unit tangent vector to every sampled point. For each pair of neighboring points together with unit tangents at the points two curvature radii are first estimated and an interpolating curve is constructed by employing the technique presented in Section 4. If a point is not an inflection point a unique curvature radius will be used for the construction of two interpolating curves. Two different curvature radii will be assigned to an inflection point when two interpolating curves are constructed on two sides of the point. With proper choices of tangents and curvature radii at the sampled points, some interpolating curves can be circular arcs and the whole interpolating curve can reach G^2 continuity except at the inflection points at where the continuity is G^1; see Fig. 9(b) for the presented interpolating curve and Fig. 9(d) for the curvature plot of the curve.

Second, we approximate an analytic curve by a piecewise G^2 interpolating curve. The original curve is given by

$$x(t) = 0.1\cos(2t) + \cos(t) + \cos(3t) + 0.1\cos(4t),$$
$$y(t) = 0.6\sin(t) + \sin(3t)$$

with $t \in [0, 2\pi]$. To approximate the curve by n pieces of interpolating curves we first sample a sequence of points at $t_i = \frac{2\pi i}{n}$, $i = 0, \ldots, n$. The tangents and curvatures at the sampled points are also computed. Then, we construct interpolating curves for each pair of consequent sampled points together with unit tangents and curvature radii at the points by using the technique proposed in Section 4. As a result, a G^2 continuous spline curve is obtained that passes through all the sampled points. Fig. 10(a) illustrates the interpolating curves together with the curvature profile when n is chosen 20. The approximation deviation from the original curve is shown in Fig. 10(b). We have also constructed G^2 continuous curves interpolating several different sets of sampled data. The maximum approximation errors and the maximum absolute curvature differences are given in Table 1. From the table we can see that both the approximation errors and the curvature differences decrease quickly when more sampled points are interpolated.

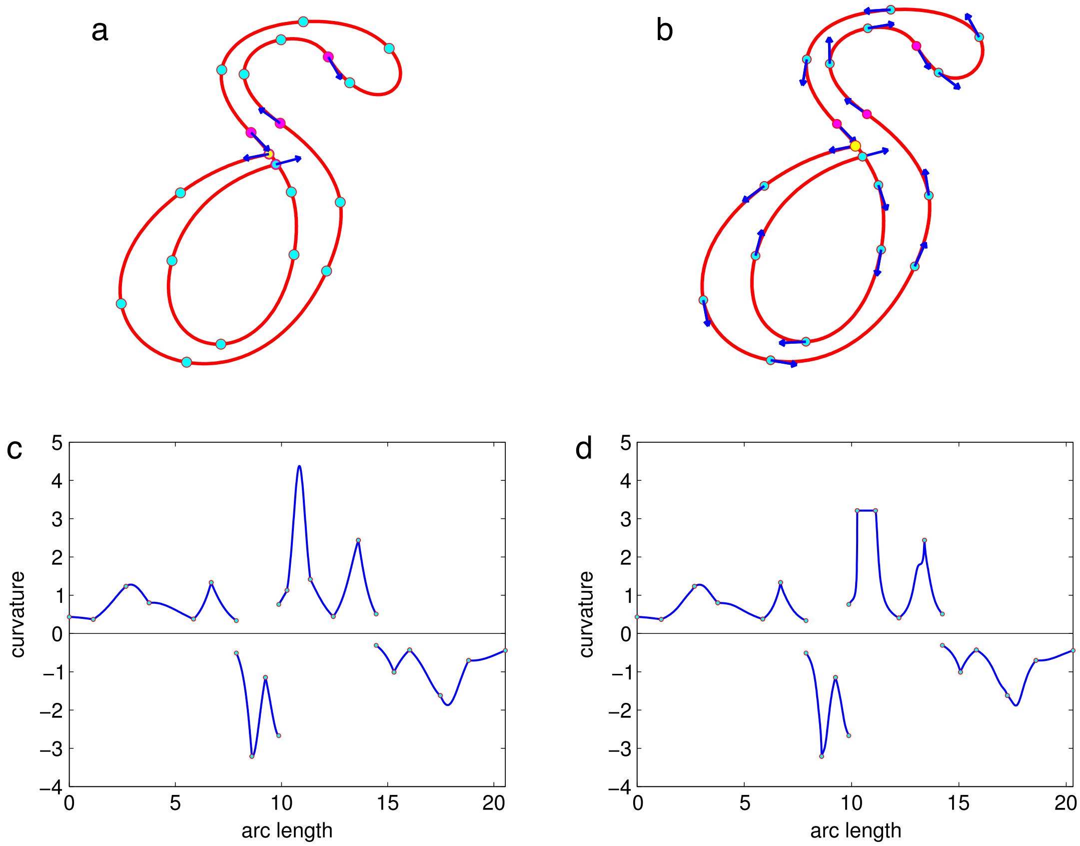

Fig. 9. Piecewise G^2 interpolating curves, where the magenta dots are the inflection points and the yellow point is the starting point: (a) a cubic PH spline curve; (b) the curve obtained by the presented method; (c) the curvature plot of the PH spline curve; (d) the curvature plot of the interpolating curve in (b). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

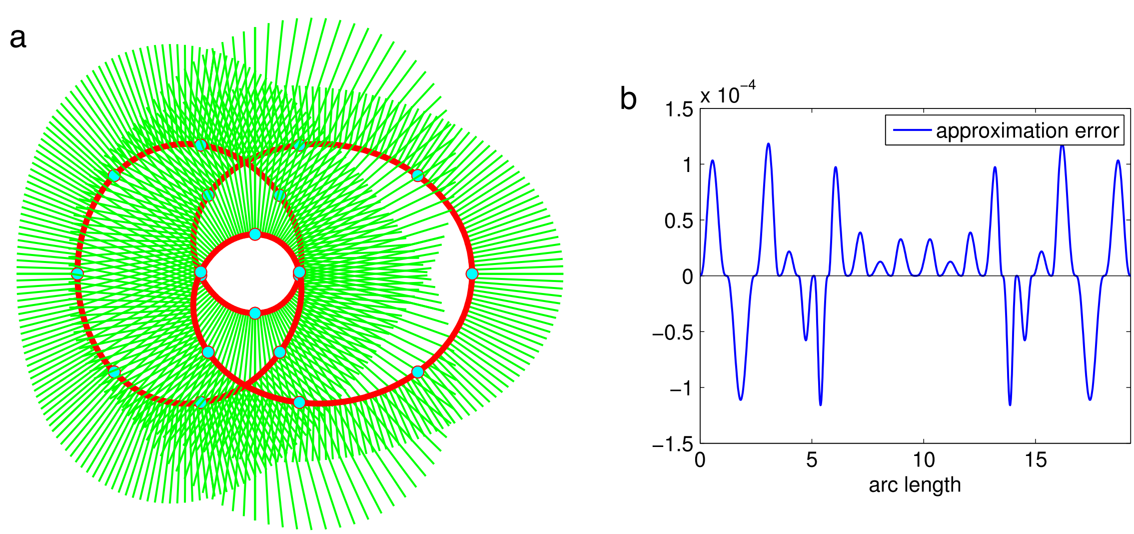

Fig. 10. (a) The curves interpolating the sampled data and the curvature profile of the interpolating curves; (b) the approximation error along the curve.

Table 1
The maximum approximation errors and the maximum absolute curvature differences for different sets of interpolating curves.

#segments	Max approx. error	Max curvature difference
20	1.18E−4	7.93E−3
40	4.25E−6	7.32E−4
80	7.14E−8	7.86E−5
160	7.95E−9	2.09E−5

Last, we present two examples of rotational surface modeling by constructing profile curves using the proposed curve interpolation techniques. To model a vase shape the profile is constructed by the proposed G^1 Hermite interpolation method with given points (1.2, 6), (1.1, 4.5), (2.1, 2), (1.3, 0.4), (0, 0) together with unit tangent vectors (−0.643, −0.766), (0.866, −0.5), (−0.087, −0.996), (−0.766, −0.643), (−1, 0) at the points. See Fig. 11(a) for the profile and Fig. 11(b) for the obtained surface. A glass cup is also modeled by rotating a profile that is initially defined in the *xy*-plane. Based on the

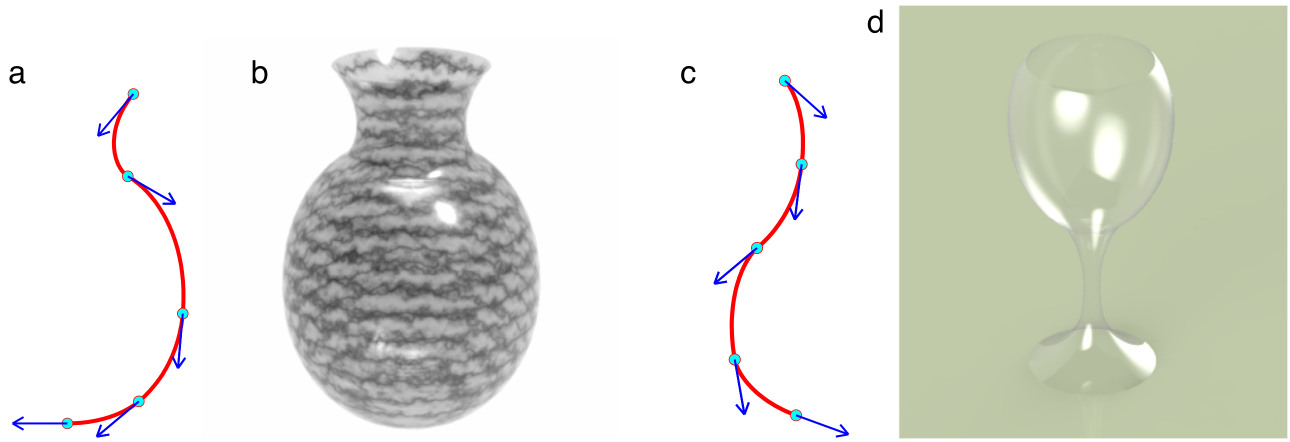

Fig. 11. (a) The vase profile modeled by interpolation of given Hermite data; (b) the vase shape modeled by the constructed profile; (c) the glass cup profile modeled by interpolation of given Hermite data; (d) the glass cup modeled by the constructed profile.

given points (1.1, 6), (1.4, 4.5), (0.6, 3), (0.2, 1), (1.3, 0) and the corresponding unit tangents (0.743, −0.669), (−0.122, −0.992), (−0.766, −0.643), (0.174, −0.985), (0.939, −0.342), a planar profile is first constructed and then the glass cup is obtained by rotating the profile around the y-axis. Figs. 11(c) and 11(d) illustrate the profile and the surface, respectively.

8. Conclusions and discussions

This paper has proposed to define and control the shapes of a family of planar curves via predefined curvature radius functions. The defined curves together with their arc lengths and the offsets can be explicitly computed when the curvature radius is given by a wide class of elementary functions. By choosing the curvature radius functions as polynomials, the curvature radius functions can be obtained by solving linear systems under the constraint of interpolating given boundary data or having prescribed arc lengths. Particularly, G^1 Hermite interpolation with or without the constraint of arc length and G^2 Hermite interpolation by the proposed curves have been given. The potential applications of the proposed technique include fair shape design as well as CNC machining tool path generation, etc.

In the same way as the proposed technique in the paper, one can construct G^2 interpolating curves with arc length constraint or G^3 continuous interpolating curves. To make the curvature based integral curves more useful in practical modeling or CNC machining systems, efficient evaluation or trimming algorithms should be developed, which are left as interesting future work.

Acknowledgments

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