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Geometric interpolation by PH curves with quadratic or quartic rational normals[☆,☆☆]

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ABSTRACT

Pythagorean-hodograph (PH) curves have nice properties and have found important applications in geometric modeling and CNC machining. While the unit normals of PH curves of degree n are generally rational curves of degree $n-1$, this paper investigates PH curves of arbitrary degrees but with only quadratic rational unit normals when the curves are convex or with quartic rational unit normals when the curves have single inflection points. PH curves with quadratic or quartic rational normals have simple Gauss maps and hodographs of the curves are given by low degree tangent vector fields together with simple real scaling functions. Practical algorithms for interpolation of point-normal pairs or point-normal-curvature pairs together with unit normals at selected parameter coordinates or at inflection points by the investigated PH curves without or with the constraint of arc lengths have been given. The parameters for defining the interpolating PH curves are either obtained directly from the input data or by solving simple linear systems. This method of PH curve interpolation has unique solutions and the shapes of the interpolating PH curves are controlled well by the interpolated data. Even though the interpolating curves may have cusps, the regularity of the PH curves can be checked easily based on the signs of the real scaling functions within the hodographs.

1. Introduction

Polynomial Pythagorean-hodograph (PH) curves were introduced by Farouki and Sakkalis in 1990 [1]. PH curves have polynomial arc lengths and rational offsets and they have found important applications in geometric modeling and CNC machining [2]. Inspired by its original definition, PH curves have been studied and generalized extensively in the past few decades. Particularly, PH space curves, PH rational curves, PH spline curves or PH curves in non-polynomial spaces, etc., have been developed [3–8]. More theories and techniques about PH curves and their applications can be found in the book [9] or a recent survey paper [10].

Though PH curves are generally represented as Bézier curves, it needs more assumptions or constraints for defining PH curves. Roughly, the methods for defining or constructing PH curves can be classified into two categories. The first kind of methods are by integrating given tangent vector fields or fitting known normal vector fields. If the norms of the tangent vector fields are polynomials or the normal vectors are unit, the obtained curves or surfaces are PH curves or Pythagorean normal surfaces [1,11,12]. Another kind of methods for constructing PH curves are by using properly defined control polygons for Bézier curves [13–18]. The control polygon based techniques can even be used to define PH spline curves or for recognizing PH curves from Bézier curves [19,20].

Interpolation of Hermite data by single or multi-connected PH curves is an efficient way for approximating general smooth curves or fitting discrete data and it has been used in various applications [21–26]. Besides interpolating the boundary data, a PH curve may also have a prescribed arc length [27]. It should be noted that Hermite interpolation by PH curves may have multiple solutions and one has to choose the best one based on properly defined fairness measures [28]. Alternatively, one can use additional degrees of freedom for shape control or optimization of an interpolating PH curve [29]. When a sequence of points are interpolated by a spline of PH curves, the tangents or curvatures at the joint points will be determined by solving nonlinear systems [30]. If only locally convex curves are desired, one may use intrinsically defined curves or specially defined arc splines for interpolation [31,32].

Our approach for defining planar PH curves is based on the integral of scaled low degree tangent vector fields. Particularly, we assume a convex PH curve has quadratic rational unit normals

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while a PH curve with a single inflection point has quartic rational unit normals obtained by reparameterizing quadratic rational unit normals. This also implies that the angle between any two normals on a PH curve is less than π. PH curves with low degree rational normals have simple Gauss maps and the basic shapes of the PH curves can be characterized by the normals very well. The hodographs of a family of PH curves with the same rational unit normals are obtained as the scaling tangent vector fields perpendicular to the normals. Based on the scaling functions of hodographs, the obtained PH curves have additional degrees of freedom for shape editing or data interpolation. The regularity of the PH curves can be checked easily based on the signs of the scaling functions.

We apply PH curves with quadratic or quartic rational unit normals for geometric interpolation. Given two distinct points, two unit normals at the points and another unit normal at a selected parameter coordinate or at the inflection point, a G^1 interpolating PH curve will be constructed. If the arc length or curvatures at two end points are also given, a G^1 interpolating PH curve with a prescribed arc length or a G^2 interpolating PH curve will be obtained. For each type of geometric interpolation we first construct a quadratic or quartic rational Bézier curve to interpolate the three unit normals. The free parameters of the scaling functions within the hodographs are then obtained by solving simple linear systems that are developed from the interpolation conditions of end points, arc lengths or end curvatures. The interpolating PH curves are unique and usually have please-looking shapes when the input data are properly given.

The paper is organized as follows. In Section 2, we present definition of PH curves with low order rational normals. Section 3 describes the technique of geometric interpolation by planar PH curves with quadratic rational unit normals. Methods for geometric interpolation by inflectional PH curves with quartic rational unit normals are given in Section 4. Examples of geometric interpolation by PH curves with quadratic or quartic rational normals are presented in Section 5. Section 6 concludes the paper with a brief summary and discussion.

2. PH curves with low order rational normals

Suppose $P(t) = (x(t), y(t))$ is a planar polynomial curve. If the derivative of the curve satisfies $x'^2(t)+y'^2(t) = \sigma^2(t)$, where $\sigma(t)$ is a polynomial, the curve $P(t)$ is referred a PH curve [1]. Then, the unit normal vector along a PH curve can be obtained as

$$\mathbf{n}(t) = \left(\frac{-y'(t)}{\sigma(t)}, \frac{x'(t)}{\sigma(t)}\right)$$

and the offset to curve $P(t)$ with a signed distance d can be represented by a rational curve

$$P_o(t) = P(t) + d\mathbf{n}(t).$$

Generally, $\mathbf{n}(t)$ is a rational curve of degree $n-1$ when the curve $P(t)$ is a polynomial curve of degree n. On another hand, if the hodograph $P'(t) = (x'(t), y'(t))$ is known, the curve $P(t)$ can be obtained by integral of $P'(t)$ directly.

2.1. PH curves with prescribed rational normals

In this paper we are interested in a special kind of PH curves of which the normal vectors are just of low order rational curves. Particularly, we use low order rational normals to characterize the basic shapes of PH curves and use degrees of freedom of the real scaling functions within the hodographs for modeling and interpolation.

Suppose $\mathbf{n}(t) = \left(\frac{-\eta(t)}{\omega(t)}, \frac{\xi(t)}{\omega(t)}\right)$, $0 \le t \le 1$, represents a circular arc of radius 1. Let $U(t) = (\xi(t), \eta(t))$. It yields that $U(t) \cdot \mathbf{n}(t) \equiv 0$. A family of PH curves with prescribed unit normal $\mathbf{n}(t)$ are obtained as

$$P(t) = \int_0^t \rho(\tau)U(\tau)d\tau + P_0, \quad (1)$$

where $\rho(t)$ is an arbitrary real polynomial and P_0 is a chosen point on the plane.

Because $\xi^2(t) + \eta^2(t) = \omega^2(t)$, we have $\|U(t)\| = \omega(t)$, where $\|\cdot\|$ means the Euclidean norm of a vector. The norm of the derivative of curve given by Eq. (1) is $\|P'(t)\| = \omega(t)|\rho(t)|$. As explained later, a PH curve constructed by Eq. (1) is regular only when the sign of $\rho(t)$ does not change and the normal vectors are properly given. Particularly, if $\rho(t) > 0$ for $0 \le t \le 1$, $U(t)$ coincides with the tangent direction of $P(t)$ and the arc length of the curve can be computed by

$$L(t) = \int_0^t \omega(\tau)\rho(\tau)d\tau. \quad (2)$$

With simple computation, the curvature of the curve $P(t)$ can be obtained as

$$k(t) = \frac{P'(t) \wedge P''(t)}{\|P'(t)\|^3} = \frac{U(t) \wedge U'(t)}{\omega^3(t)\rho(t)},$$

where $P'(t) \wedge P''(t)$ represents the scalar cross product of two planar vectors. Equivalently, the function $\rho(t)$ can be computed when the normal vectors and the curvatures of a PH curve are known. It yields

$$\rho(t) = \frac{U(t) \wedge U'(t)}{\omega^3(t)k(t)}. \quad (3)$$

If $\rho(t)$ is negative, the derivative of $P(t)$ has opposite direction with $U(t)$. Then, the unsigned arc length and curvatures of $P(t)$ can be computed just by replacing $\rho(t)$ with $-\rho(t)$ within above equations. From the computation of curvatures, we have the following proposition.

Proposition 1. *If $\rho(t) \neq 0$, for $0 \le t \le 1$, then the PH curve given by Eq. (1) is non-singular.*

2.2. PH curves with quadratic or quartic rational normals

The lowest degree rational curves to represent the normal vector field $\mathbf{n}(t)$ of a PH curve $P(t)$ are quadratic rational Bézier curves. Then, the scaled tangent vector field $U(t)$ is a quadratic Bézier curve $U_{quad}(t) = \sum_{i=0}^{2} U_i B_{i,2}(t)$, where $B_{i,2}(t)$, $i = 0, 1, 2$, are Bernstein basis functions. A regular PH curve with quadratic rational unit normals is convex. As explained in Section 4, the unit normals $\mathbf{n}(t)$ can be quartic rational Bézier curves just by reparameterizing initial quadratic rational normals. Quartic rational unit normals can be used to construct PH curves with inflections. Consequently, the vector field $U(t)$ becomes $U_{quar}(t) = \sum_{i=0}^{4} U_i B_{i,4}(t)$.

When the normal vector field $\mathbf{n}(t)$ or the scaled tangent vector field $U(t)$ has already been defined or computed, we construct PH curves by Eq. (1) using $U(t)$ and a properly chosen real function $\rho(t)$. The degrees of freedom within $\rho(t)$ permit shape adjustment or data interpolation by the obtained PH curves. Particularly, $\rho_I(t) = \sum_{j=0}^{1} \rho_j B_{j,1}(t)$, $\rho_{II}(t) = \sum_{j=0}^{2} \rho_j B_{j,2}(t)$ and $\rho_{III}(t) = \sum_{j=0}^{3} \rho_j B_{j,3}(t)$ will be used to construct PH curves for G^1 or G^2 interpolation without or with constraint of arc lengths. We compute hodographs for PH curves by choosing different $U(t)$ and $\rho(t)$ as follows.

- **Choose** $U(t) = U_{quad}(t)$ **and** $\rho(t) = \rho_I(t)$. The product of $U_{quad}(t)$ and $\rho_I(t)$ is obtained as

$$\begin{aligned} H_3(t) &= U_{quad}(t)\rho_I(t) \\ &= \textstyle\sum_{k=0}^{3} D_k B_{k,3}(t), \end{aligned} \quad (4)$$

where

$$\begin{cases} D_0 &= \rho_0 U_0 \\ D_1 &= \frac{1}{3}(\rho_1 U_0 + 2\rho_0 U_1) \\ D_2 &= \frac{1}{3}(2\rho_1 U_1 + \rho_0 U_2) \\ D_3 &= \rho_1 U_2. \end{cases}$$

• **Choose** $U(t) = U_{quad}(t)$ **and** $\rho(t) = \rho_{II}(t)$. The product of $U_{quad}(t)$ and $\rho_{II}(t)$ is

$$\begin{aligned} H_4(t) &= U_{quad}(t)\rho_{II}(t) \\ &= \textstyle\sum_{k=0}^{4} D_k B_{k,4}(t), \end{aligned} \tag{5}$$

where

$$\begin{cases} D_0 &= \rho_0 U_0 \\ D_1 &= \frac{1}{2}(\rho_1 U_0 + \rho_0 U_1) \\ D_2 &= \frac{1}{6}(\rho_2 U_0 + 4\rho_1 U_1 + \rho_0 U_2) \\ D_3 &= \frac{1}{2}(\rho_2 U_1 + \rho_1 U_2) \\ D_4 &= \rho_2 U_2. \end{cases}$$

• **Choose** $U(t) = U_{quad}(t)$ **and** $\rho(t) = \rho_{III}(t)$. The product of $U_{quad}(t)$ and $\rho_{III}(t)$ is

$$\begin{aligned} H_5(t) &= U_{quad}(t)\rho_{III}(t) \\ &= \textstyle\sum_{k=0}^{5} D_k B_{k,5}(t), \end{aligned} \tag{6}$$

where

$$\begin{cases} D_0 &= \rho_0 U_0 \\ D_1 &= \frac{1}{5}(3\rho_1 U_0 + 2\rho_0 U_1) \\ D_2 &= \frac{1}{10}(3\rho_2 U_0 + 6\rho_1 U_1 + \rho_0 U_2) \\ D_3 &= \frac{1}{10}(\rho_3 U_0 + 6\rho_2 U_1 + 3\rho_1 U_2) \\ D_4 &= \frac{1}{5}(2\rho_3 U_1 + 3\rho_2 U_2) \\ D_5 &= \rho_3 U_2. \end{cases}$$

• **Choose** $U(t) = U_{quar}(t)$ **and** $\rho(t) = \rho_I(t)$. The product of $U_{quar}(t)$ and $\rho_I(t)$ is

$$\begin{aligned} \bar{H}_5(t) &= U_{quar}(t)\rho_I(t) \\ &= \textstyle\sum_{k=0}^{5} \bar{D}_k B_{k,5}(t), \end{aligned} \tag{7}$$

where

$$\begin{cases} \bar{D}_0 &= \rho_0 U_0 \\ \bar{D}_1 &= \frac{1}{5}(\rho_1 U_0 + 4\rho_0 U_1) \\ \bar{D}_2 &= \frac{1}{5}(2\rho_1 U_1 + 3\rho_0 U_2) \\ \bar{D}_3 &= \frac{1}{5}(3\rho_1 U_2 + 2\rho_0 U_3) \\ \bar{D}_4 &= \frac{1}{5}(4\rho_1 U_3 + \rho_0 U_4) \\ \bar{D}_5 &= \rho_1 U_4. \end{cases}$$

• **Choose** $U(t) = U_{quar}(t)$ **and** $\rho(t) = \rho_{II}(t)$. The product of $U_{quar}(t)$ and $\rho_{II}(t)$ is

$$\begin{aligned} \bar{H}_6(t) &= U_{quar}(t)\rho_{II}(t) \\ &= \textstyle\sum_{k=0}^{6} \bar{D}_k B_{k,6}(t), \end{aligned} \tag{8}$$

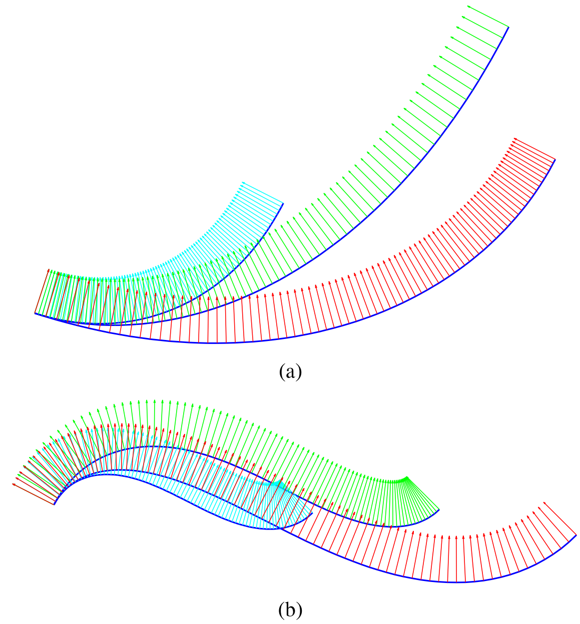

Fig. 1. Families of PH curves with the same (a) quadratic or (b) quartic rational unit normals.

where

$$\begin{cases} \bar{D}_0 &= \rho_0 U_0 \\ \bar{D}_1 &= \frac{1}{3}(\rho_1 U_0 + 2\rho_0 U_1) \\ \bar{D}_2 &= \frac{1}{15}(\rho_2 U_0 + 8\rho_1 U_1 + 6\rho_0 U_2) \\ \bar{D}_3 &= \frac{1}{5}(\rho_2 U_1 + 3\rho_1 U_2 + \rho_0 U_3) \\ \bar{D}_4 &= \frac{1}{15}(6\rho_2 U_2 + 8\rho_1 U_3 + \rho_0 U_4) \\ \bar{D}_5 &= \frac{1}{3}(2\rho_2 U_3 + \rho_1 U_4) \\ \bar{D}_6 &= \rho_2 U_4. \end{cases}$$

• **Choose** $U(t) = U_{quar}(t)$ **and** $\rho(t) = \rho_{III}(t)$. The product of $U_{quar}(t)$ and $\rho_{III}(t)$ is

$$\begin{aligned} \bar{H}_7(t) &= U_{quar}(t)\rho_{III}(t) \\ &= \textstyle\sum_{k=0}^{7} \bar{D}_k B_{k,7}(t), \end{aligned} \tag{9}$$

where

$$\begin{cases} \bar{D}_0 &= \rho_0 U_0 \\ \bar{D}_1 &= \frac{1}{7}(3\rho_1 U_0 + 4\rho_0 U_1) \\ \bar{D}_2 &= \frac{1}{7}(\rho_2 U_0 + 4\rho_1 U_1 + 2\rho_0 U_2) \\ \bar{D}_3 &= \frac{1}{35}(\rho_3 U_0 + 12\rho_2 U_1 + 18\rho_1 U_2 + 4\rho_0 U_3) \\ \bar{D}_4 &= \frac{1}{35}(4\rho_3 U_1 + 18\rho_2 U_2 + 12\rho_1 U_3 + \rho_0 U_4) \\ \bar{D}_5 &= \frac{1}{7}(2\rho_3 U_2 + 4\rho_2 U_3 + \rho_1 U_4) \\ \bar{D}_6 &= \frac{1}{7}(4\rho_3 U_3 + 3\rho_2 U_4) \\ \bar{D}_7 &= \rho_3 U_4. \end{cases}$$

When the product of selected $U(t)$ and $\rho(t)$ has been formulated as a Bézier curve, i.e., $\rho(t)U(t) = \sum_{i=0}^{k} D_i B_{i,k}(t)$, the PH curve $P(t)$ is obtained by integral (1) as

$$P(t) = \sum_{i=0}^{k+1} P_i B_{i,k+1}(t), \tag{10}$$

where $P_{i+1} = P_i + \frac{1}{k+1} D_i$, $i = 0, 1, \ldots, k$.

Fig. 1 illustrates two families of PH curves with same quadratic or quartic rational unit normals. Particularly, the normal vectors in Fig. 1(a) are computed by a quadratic rational Bézier curve that satisfies $\mathbf{n}(0) = (\cos(0.4\pi), \sin(0.4\pi))$, $\mathbf{n}(\frac{2}{9}) = (\cos(0.5\pi), \sin(0.5\pi))$ and $\mathbf{n}(1) = (\cos(0.85\pi), \sin(0.85\pi))$. The PH curves with cyan, green or red normals are obtained by employing Eq. (4) and Eq. (10) and choosing $\rho(t) = 1$, $\rho(t) = 1 + 2t$ or $\rho(t) = 3 - 2t$. Given $\mathbf{n}_0 = (\cos(0.85\pi), \sin(0.85\pi))$, $\mathbf{n}_1 = (\cos(0.35\pi), \sin(0.35\pi))$ and $\mathbf{n}_2 = (\cos(0.75\pi), \sin(0.75\pi))$, the PH curves with cyan, green or red normals in Fig. 1(b), all having normal vectors $\mathbf{n}_0$, $\mathbf{n}_2$ at two ends and $\mathbf{n}_1$ at the inflection points, are obtained from Eq. (7) and Eq. (10) by choosing $\rho(t) = 1$, $\rho(t) = 2 - t$ or $\rho(t) = 1 + 2t$, respectively. Quadratic or quartic rational unit normal interpolation will be explained in the following two sections. It is noticed that each family of PH curves have basically similar shapes but the curves also differ a lot when different real functions $\rho(t)$ are used for defining the hodographs.

3. Geometric interpolation by PH curves with quadratic rational unit normals

Assume P_A and P_B are two distinct points sampled from a convex curve and $\mathbf{n}_0$ and $\mathbf{n}_2$ are two unit normals at the two points. Assume that the angle between $\mathbf{n}_0$ and $\mathbf{n}_2$ is less than π and $\mathbf{n}_1$ is another unit normal lying between $\mathbf{n}_0$ and $\mathbf{n}_2$ with a given parameter $0 < t_0 < 1$, we construct a convex PH curve with quadratic rational normals interpolating the points and normals. If the arc length L_{arc} or the curvatures k_A and k_B at the two boundary points are also given, a G^1 interpolating PH curve with prescribed arc length or a G^2 interpolating PH curve will be constructed.

3.1. Quadratic rational unit normal interpolation

As a first step of the proposed PH curve interpolation, we construct a quadratic rational curve $\mathbf{n}(t)$ satisfying $\|\mathbf{n}(t)\| \equiv 1$ as well as $\mathbf{n}(0) = \mathbf{n}_0$, $\mathbf{n}(t_0) = \mathbf{n}_1$ and $\mathbf{n}(1) = \mathbf{n}_2$.

Before constructing an interpolating rational curve $\mathbf{n}(t)$ directly, we first construct a quadratic rational Bézier curve $\mathbf{v}(s)$ to interpolate points $\mathbf{n}_0$ and $\mathbf{n}_2$ at the boundaries; see Fig. 2. Let $\mathbf{v}_0 = \mathbf{n}_0$, $\mathbf{v}_2 = \mathbf{n}_2$, $\mu = \frac{\|\mathbf{n}_0+\mathbf{n}_2\|}{2}$ and $\mathbf{v}_1 = \frac{1}{\mu}\frac{\mathbf{n}_0+\mathbf{n}_2}{\|\mathbf{n}_0+\mathbf{n}_2\|}$. A quadratic rational Bézier curve is constructed as

$$\mathbf{v}(s) = \frac{\mathbf{v}_0 B_{0,2}(s) + \mu\mathbf{v}_1 B_{1,2}(s) + \mathbf{v}_2 B_{2,2}(s)}{B_{0,2}(s) + \mu B_{1,2}(s) + B_{2,2}(s)}, \quad s \in [0, 1].$$

Because $\mathbf{v}(s)$ just represents a segment of circular arc, we have $\mathbf{v}(0) = \mathbf{n}_0$, $\mathbf{v}(1) = \mathbf{n}_2$ and $\|\mathbf{v}(s)\| \equiv 1$. We then construct a quadratic rational Bézier curve $\mathbf{n}(t)$ by a Mobius transformation of the parameter of $\mathbf{v}(s)$. Let $s = s(t) = \frac{\gamma t}{\gamma t+(1-t)}$, where $\gamma > 0$ is a free parameter. Substituting $s = s(t)$ into $\mathbf{v}(s)$, we have

$$\mathbf{n}(t) = \mathbf{v}(s(t)) = \frac{\mathbf{v}_0 B_{0,2}(t) + \gamma\mu\mathbf{v}_1 B_{1,2}(t) + \gamma^2\mathbf{v}_2 B_{2,2}(t)}{B_{0,2}(t) + \gamma\mu B_{1,2}(t) + \gamma^2 B_{2,2}(t)}, \quad t \in [0, 1]. \tag{11}$$

Same as $\mathbf{v}(s)$, the normal field $\mathbf{n}(t)$ also satisfies $\mathbf{n}(0) = \mathbf{n}_0$, $\mathbf{n}(1) = \mathbf{n}_2$ and $\|\mathbf{n}(t)\| \equiv 1$.

The parameter γ within Eq. (11) is chosen based on the interpolation condition $\mathbf{n}(t_0) = \mathbf{n}_1$. Let $V(t) = \mathbf{v}_0 B_{0,2}(t) + \gamma\mu\mathbf{v}_1 B_{1,2}(t) + \gamma^2\mathbf{v}_2 B_{2,2}(t)$. Since $V(t_0)||\mathbf{n}_1$, we have

$$V(t_0) \wedge \mathbf{n}_1 = a_0\gamma^2 + b_0\gamma + c_0 = 0,$$

where $a_0 = B_{2,2}(t_0)\mathbf{v}_2 \wedge \mathbf{n}_1$, $b_0 = \mu B_{1,2}(t_0)\mathbf{v}_1 \wedge \mathbf{n}_1$ and $c_0 = B_{0,2}(t_0)\mathbf{v}_0 \wedge \mathbf{n}_1$. When $\mathbf{n}_1$ lies between $\mathbf{n}_0$ and $\mathbf{n}_2$, it yields that $a_0c_0 < 0$. Therefore, the above equation has two real solutions

$$\gamma_{1,2} = \frac{-b_0 \pm \sqrt{b_0^2 - 4a_0c_0}}{2a_0}.$$

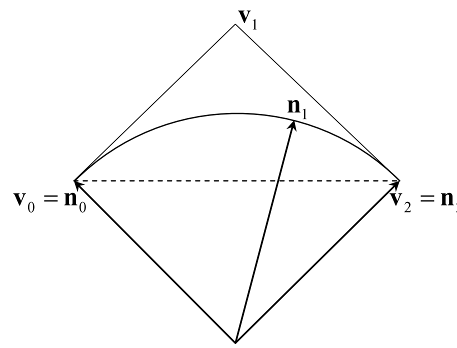

Fig. 2. Quadratic rational unit normal interpolation.

Since $\gamma_1\gamma_2 = \frac{c_0}{a_0} < 0$, the unique positive solution γ_1 or γ_2 will be chosen as the parameter for the construction of interpolating normal field.

Assume $\mathbf{v} = (\mathbf{v}_x\ \mathbf{v}_y)$ and let

$$\mathrm{M}_\theta(\mathbf{v}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{v}_x \\ \mathbf{v}_y \end{pmatrix}$$

be the vector rotated from $\mathbf{v}$ by angle θ. The quadratic tangent vector field $U(t)$ perpendicular to $\mathbf{n}(t)$ is obtained as $U(t) = \sum_{i=0}^{2} U_i B_{i,2}(t)$, where $U_0 = \mathrm{M}_{-\frac{\pi}{2}}(\mathbf{v}_0)$, $U_1 = \mathrm{M}_{-\frac{\pi}{2}}(\gamma\mu\mathbf{v}_1)$ and $U_2 = \mathrm{M}_{-\frac{\pi}{2}}(\gamma^2\mathbf{v}_2)$. The denominator of $\mathbf{n}(t)$ is represented as $\omega(t) = B_{0,2}(t) + \gamma\mu B_{1,2}(t) + \gamma^2 B_{2,2}(t)$. In the following subsections, we will construct G^1 or G^2 interpolating PH curves by using $U(t)$, $\omega(t)$ and appropriately chosen $\rho(t)$.

3.2. G^1 Interpolation by convex PH curves

Since the rational curve given by Eq. (11) interpolates normals $\mathbf{n}_0$, $\mathbf{n}_1$ and $\mathbf{n}_2$, we further compute a polynomial $\rho_I(t) = \rho_0(1 - t) + \rho_1 t$ such that the curve computed by Eq. (1) interpolates points P_A and P_B at the boundaries.

Substituting $P_0 = P_A$, $\rho(t) = \rho_I(t)$ and $U(t) = \sum_{i=0}^{2} U_i B_{i,2}(t)$ into Eq. (1), we have

$$\begin{aligned} P(t) &= \int_0^t \rho(\tau)U(\tau)d\tau + P_A \\ &= \rho_0\int_0^t (1-\tau)U(\tau)d\tau + \rho_1\int_0^t \tau U(\tau)d\tau + P_A \\ &= \rho_0\int_0^t \sum_{i=0}^{2} \frac{3-i}{3} U_i B_{i,3}(\tau)d\tau \\ &+ \rho_1\int_0^t \sum_{i=0}^{2} \frac{i+1}{3} U_i B_{i+1,3}(\tau)d\tau + P_A. \end{aligned} \tag{12}$$

From Eq. (12), we have $P(1) = \rho_0 A_0 + \rho_1 A_1 + P_A$, where $A_0 = \frac{1}{4}U_0 + \frac{1}{6}U_1 + \frac{1}{12}U_2$ and $A_1 = \frac{1}{12}U_0 + \frac{1}{6}U_1 + \frac{1}{4}U_2$. The solution to the equation $P(1) = P_B$ is

$$(\rho_0\ \ \rho_1) = (P_B - P_A)\begin{pmatrix} A_0 \\ A_1 \end{pmatrix}^{-1}.$$

When ρ_0 and ρ_1 have been obtained, a quartic PH curve interpolating points P_A, P_B and unit normals $\mathbf{n}_0$, $\mathbf{n}_1$, $\mathbf{n}_2$ is obtained by Eqs. (4) and (10). Based on Proposition 1 we know that the interpolating PH curve is convex with no cusps when ρ_0 and ρ_1 have the same sign.

3.3. G^1 Interpolation by convex PH curves with prescribed arc lengths

To construct a convex PH curve that interpolates points P_A and P_B at the boundaries, unit normals $\mathbf{n}_0$, $\mathbf{n}_1$ and $\mathbf{n}_2$ at selected parameter coordinates, as well as with a prescribed arc length L_{arc}, we choose $\rho(t) = \rho_{II}(t) = \sum_{j=0}^{2} \rho_j B_{j,2}(t)$ together with

$U(t)=\sum_{i=0}^{2}U_iB_{i,2}(t)$ for the construction of the curve. Let $P_0 = P_A$. From Eq. (1), a PH curve is obtained as

$$\begin{aligned} P(t) &= \int_0^t \rho(\tau)U(\tau)d\tau + P_A \\ &= \sum_{j=0}^{2}\rho_j\int_0^t\sum_{i=0}^{2}U_iB_{i,2}(\tau)B_{j,2}(\tau)d\tau + P_A \\ &= \sum_{j=0}^{2}\rho_j\int_0^t\sum_{i=0}^{2}\frac{C_2^iC_2^j}{C_4^{i+j}}U_iB_{i+j,4}(\tau)d\tau + P_A, \end{aligned} \tag{13}$$

where $C_n^i = \frac{n!}{i!(n-i)!}$. Therefore, we have

$$P(1) = \rho_0W_0 + \rho_1W_1 + \rho_2W_2 + P_A,$$

where

$$\begin{cases} W_0 = \frac{1}{5}(U_0 + \frac{1}{2}U_1 + \frac{1}{6}U_2) \\ W_1 = \frac{1}{5}(\frac{1}{2}U_0 + \frac{2}{3}U_1 + \frac{1}{2}U_2) \\ W_2 = \frac{1}{5}(\frac{1}{6}U_0 + \frac{1}{2}U_1 + U_2). \end{cases}$$

Let $\omega_0 = 1$, $\omega_1 = \gamma\mu$ and $\omega_2 = \gamma^2$. In the same way as the computation of point $P(1)$, the arc length of the PH curve can be computed by substituting $\omega(t)=\sum_{i=0}^{2}\omega_iB_{i,2}(t)$ and $\rho(t) = \sum_{j=0}^{2}\rho_jB_{j,2}(t)$ into Eq. (2). We have

$$L(1) = \rho_0\bar{\omega}_0 + \rho_1\bar{\omega}_1 + \rho_2\bar{\omega}_2,$$

where

$$\begin{cases} \bar{\omega}_0 = \frac{1}{5}(\omega_0 + \frac{1}{2}\omega_1 + \frac{1}{6}\omega_2) \\ \bar{\omega}_1 = \frac{1}{5}(\frac{1}{2}\omega_0 + \frac{2}{3}\omega_1 + \frac{1}{2}\omega_2) \\ \bar{\omega}_2 = \frac{1}{5}(\frac{1}{6}\omega_0 + \frac{1}{2}\omega_1 + \omega_2). \end{cases}$$

Based on the condition of boundary point interpolation and arc length constraint, a linear system in terms of unknown coefficients ρ_0, ρ_1 and ρ_2 is established as follows

$$\begin{cases} \rho_0W_0 + \rho_1W_1 + \rho_2W_2 = P_B - P_A \\ \rho_0\bar{\omega}_0 + \rho_1\bar{\omega}_1 + \rho_2\bar{\omega}_2 = L_{arc}. \end{cases}$$

By solving the linear system, we have the function $\rho(t) = \sum_{j=0}^{2}\rho_jB_{j,2}(t)$. If the sign of $\rho(t)$ does not change for $0 \le t \le 1$, a quintic interpolating PH curve with a prescribed arc length is obtained by Eqs. (5) and (10).

3.4. G^2 Interpolation by convex PH curves

Besides unit normals $\mathbf{n}_0$, $\mathbf{n}_1$, $\mathbf{n}_2$, boundary points P_A, P_B, a G^2 interpolating PH curve can be constructed to have prescribed curvatures k_A and k_B at the ends. We choose $\rho(t) = \rho_{III}(t) = \sum_{j=0}^{3}\rho_jB_{j,3}(t)$ together with $U(t) = \sum_{i=0}^{2}U_iB_{i,2}(t)$ for the construction of the PH curve. Let $P_0 = P_A$, a PH curve computed by Eq. (1) is obtained as

$$\begin{aligned} P(t) &= \int_0^t \rho(\tau)U(\tau)d\tau + P_A \\ &= \sum_{j=0}^{3}\rho_j\int_0^t\sum_{i=0}^{2}U_iB_{i,2}(\tau)B_{j,3}(\tau)d\tau + P_A \\ &= \sum_{j=0}^{3}\rho_j\int_0^t\sum_{i=0}^{2}\frac{C_2^iC_3^j}{C_5^{i+j}}U_iB_{i+j,5}(\tau)d\tau + P_A. \end{aligned} \tag{14}$$

From Eq. (14), we have

$$P(1) = \rho_0W_0 + \rho_1W_1 + \rho_2W_2 + \rho_3W_3 + P_A,$$

where

$$\begin{cases} W_0 = \frac{1}{6}(U_0 + \frac{2}{5}U_1 + \frac{1}{10}U_2) \\ W_1 = \frac{1}{6}(\frac{3}{5}U_0 + \frac{3}{5}U_1 + \frac{3}{10}U_2) \\ W_2 = \frac{1}{6}(\frac{3}{10}U_0 + \frac{3}{5}U_1 + \frac{3}{5}U_2) \\ W_3 = \frac{1}{6}(\frac{1}{10}U_0 + \frac{2}{5}U_1 + U_2). \end{cases}$$

We solve the parameters ρ_j, $j = 0, 1, 2, 3$, based on the interpolation conditions $P(1) = P_B$, $k(0) = k_A$ and $k(1) = k_B$. Based on expression of $\omega(t)$ we have $\omega(0) = 1$ and $\omega(1) = \gamma^2$. Substituting $k(0) = k_A$, $k(1) = k_B$, $\omega(0) = 1$ and $\omega(1) = \gamma^2$ into Eq. (3), we have

$$\rho_0 = \rho(0) = \frac{U(0)\wedge U'(0)}{\omega^3(0)k(0)} = \frac{2U_0 \wedge U_1}{k_A}$$

and

$$\rho_3 = \rho(1) = \frac{U(1)\wedge U'(1)}{\omega^3(1)k(1)} = \frac{2U_1 \wedge U_2}{k_B\gamma^6}.$$

After computing ρ_0 and ρ_3, ρ_1 and ρ_2 are then computed from the equation $P(1) = P_B$. We have

$$(\rho_1 \;\; \rho_2) = (P_B - P_A - \rho_0W_0 - \rho_3W_3)\begin{pmatrix} W_1 \\ W_2 \end{pmatrix}^{-1}.$$

When the polynomial $\rho_{III}(t)$ has been obtained, a sextic PH curve that interpolates a pair of G^2 Hermite data and an intermediate normal vector is given by Eqs. (6) and (10). Particularly, if all ρ_j have the same sign, the obtained PH curve is convex.

4. Geometric interpolation by PH curves with quartic rational unit normals

This section presents techniques of geometric interpolation by PH curves with single inflection points. Assume P_A and P_B are two distinct points sampled from a planar curve with one inflection point and $\mathbf{n}_0$ and $\mathbf{n}_2$ are unit normals at the two points. Assume $\mathbf{n}_1$ is the unit normal vector at the inflection point. A G^1 interpolating PH curve without or with constraint of arc length will interpolate the two sampled points and three unit normals. If curvatures k_A and k_B at the two boundary points are also given, a G^2 interpolating PH curve with a single inflection point will be constructed to interpolate all the sampled points, normals and curvatures.

4.1. Quartic rational unit normal interpolation

Similar to quadratic rational normal interpolation, we compute quartic rational curve $\mathbf{n}(t)$ satisfying $\|\mathbf{n}(t)\| \equiv 1$ for $0 \le t \le 1$ as well as $\mathbf{n}(0) = \mathbf{n}_0$, $\mathbf{n}(1) = \mathbf{n}_2$ and $\mathbf{n}(t_0) = \mathbf{n}_1$ for some $0 < t_0 < 1$.

Unlike convex PH curve interpolation, the unit normal $\mathbf{n}_1$ at an inflection point does not lie between $\mathbf{n}_0$ and $\mathbf{n}_2$. The boundaries of a circular arc interpolating the three normals can be $\mathbf{n}_1$, $\mathbf{n}_2$ or $\mathbf{n}_1$, $\mathbf{n}_0$. Instead of constructing a rational curve $\mathbf{n}(t)$ satisfying the above interpolation conditions directly, we first construct a quadratic rational curve $\mathbf{v}(s)$ that has a constant norm and satisfies $\mathbf{v}(0) = \mathbf{n}_1$, $\mathbf{v}(1) = \mathbf{n}_2$ or $\mathbf{v}(0) = \mathbf{n}_1$, $\mathbf{v}(1) = \mathbf{n}_0$. After then, we construct a quartic rational curve $\mathbf{n}(t)$ that interpolates $\mathbf{n}_0$ and $\mathbf{n}_2$ at the boundaries by a reparametrization of $\mathbf{v}(s)$.

Assume the quadratic rational Bézier curve is

$$\mathbf{v}(s) = \frac{\mathbf{v}_0B_{0,2}(s) + \mu\mathbf{v}_1B_{1,2}(s) + \mathbf{v}_2B_{2,2}(s)}{B_{0,2}(s) + \mu B_{1,2}(s) + B_{2,2}(s)}, \quad s \in [0, 1].$$

We choose the boundary control points of curve $\mathbf{v}(s)$ based on the relationship among the three unit normals.

1. $\mathbf{n}_0$ lies between $\mathbf{n}_1$ and $\mathbf{n}_2$. As illustrated in Fig. 3, we choose $\mathbf{v}_0 = \mathbf{n}_1$ and $\mathbf{v}_2 = \mathbf{n}_2$ for the construction of $\mathbf{v}(s)$. The vector in between the boundaries is $\mathbf{n}_s = \mathbf{n}_0$.
2. $\mathbf{n}_2$ lies between $\mathbf{n}_1$ and $\mathbf{n}_0$. See Fig. 4 for this case. The boundary control points of $\mathbf{v}(s)$ are chosen as $\mathbf{v}_0 = \mathbf{n}_1$ and $\mathbf{v}_2 = \mathbf{n}_0$. The vector in between the boundaries is $\mathbf{n}_s = \mathbf{n}_2$.

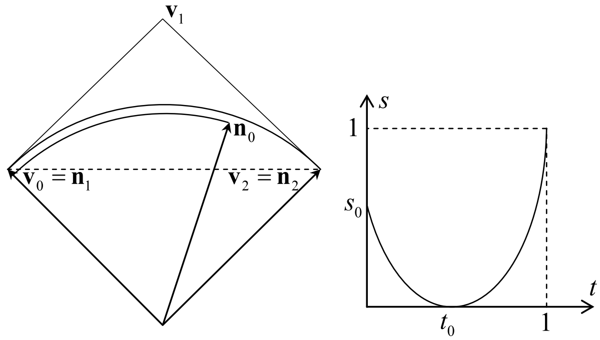

Fig. 3. Quartic rational unit normal interpolation: case 1.

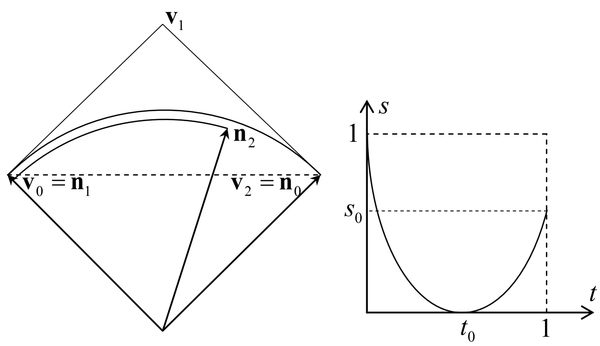

Fig. 4. Quartic rational unit normal interpolation: case 2.

When the boundary control points are determined, the weight μ and control point $\mathbf{v}_1$ are computed by $\mu = \frac{\|\mathbf{v}_0+\mathbf{v}_2\|}{2}$ and $\mathbf{v}_1 = \frac{1}{\mu}\frac{\mathbf{v}_0+\mathbf{v}_2}{\|\mathbf{v}_0+\mathbf{v}_2\|}$. Since $\mathbf{v}(s)$ represents a segment of circular arc, it yields $\|\mathbf{v}(s)\| \equiv 1$ for $0 \le s \le 1$.

As an essential step for reparametrization, we need to find a parameter s_0 which corresponds to the point $\mathbf{n}_s$ on the curve $\mathbf{v}(s)$. We find s_0 by solving equation $\mathbf{v}(s_0) = \mathbf{n}_s$ or $\mathbf{v}(s_0) \wedge \mathbf{n}_s = 0$. From this last expression, we have

$$aB_{0,2}(s_0) + bB_{1,2}(s_0) + cB_{2,2}(s_0) = 0, \tag{15}$$

where $a = \mathbf{v}_0 \wedge \mathbf{n}_s$, $b = \mu\mathbf{v}_1 \wedge \mathbf{n}_s$ and $c = \mathbf{v}_2 \wedge \mathbf{n}_s$. Because $\mathbf{n}_s$ lies between $\mathbf{v}_0$ and $\mathbf{v}_2$, it yields that $ac < 0$. We solve the equation in two cases. (1) $b = 0$. In this case, vectors $\mathbf{v}_1$ and $\mathbf{n}_s$ are parallel with each other. We have $a = -c$. Then, the equation reduces to $(1 - s_0)^2 = s_0^2$ and the solution is $s_0 = 0.5$.

(2) $b \neq 0$. In this case, vectors $\mathbf{v}_1$ and $\mathbf{n}_s$ are not parallel. The two solutions to Eq. (15) are

$$s_{1,2} = \frac{a - b \pm \sqrt{b^2 - ac}}{a + c - 2b}.$$

Since $\mathbf{v}(s)$ just represents a segment of conic section and $\mathbf{n}_s$ lies between $\mathbf{v}_0$ and $\mathbf{v}_2$, there exists a unique solution to Eq. (15) satisfying $0 \le s_0 \le 1$. Therefore, we choose $s_0 = s_1$ when $0 \le s_1 \le 1$. Otherwise, we choose $s_0 = s_2$.

We reparameterize the quadratic rational curve $\mathbf{v}(s)$ by assuming $s = s(t) = k(t - t_0)^2$. Particularly, k and t_0 are chosen based on the relationship of three normals $\mathbf{n}_0$, $\mathbf{n}_1$ and $\mathbf{n}_2$.

1. $\mathbf{n}_0$ lies between $\mathbf{n}_1$ and $\mathbf{n}_2$. In this case, we compute $s(t)$ by assuming $s(0) = s_0$ and $s(1) = 1$. It yields that $k = (1 + \sqrt{s_0})^2$ and $t_0 = \frac{\sqrt{s_0}}{1+\sqrt{s_0}}$. The function $s(t)$ is obtained as $s(t) = (gt + h)^2$, where $g = 1 + \sqrt{s_0}$ and $h = -\sqrt{s_0}$. See Fig. 3 for the reparametrization function.
2. $\mathbf{n}_2$ lies between $\mathbf{n}_1$ and $\mathbf{n}_0$. In this case, we compute $s(t)$ by assuming $s(0) = 1$ and $s(1) = s_0$. The solutions are $k = (1 + \sqrt{s_0})^2$ and $t_0 = \frac{1}{1+\sqrt{s_0}}$. The function $s(t)$ becomes $s(t) = (gt + h)^2$, where $g = 1 + \sqrt{s_0}$ and $h = -1$. The reparametrization function is plotted in Fig. 4.

When we substitute $s = s(t)$ into $\mathbf{v}(s)$, we have a quartic rational curve $\mathbf{n}(t) = \mathbf{v}(s(t))$. For both cases of relationships listed above, we have $\mathbf{n}(0) = \mathbf{n}_0$, $\mathbf{n}(t_0) = \mathbf{n}_1$ and $\mathbf{n}(1) = \mathbf{n}_2$. It also yields that $\|\mathbf{n}(t)\| = \|\mathbf{v}(s(t))\| \equiv 1$. PH curves with normal vector $\mathbf{n}(t)$ each have an inflection point at $t = t_0$.

Concretely, substituting $s(t) = (gt + h)^2$ into $\mathbf{v}(s)$, we have

$$\mathbf{n}(t) = \frac{V(t)}{\omega(t)},$$

where

$$\begin{aligned} V(t) &= (\mathbf{v}_0 - 2\mu\mathbf{v}_1 + \mathbf{v}_2)(g^4t^4 + 4g^3ht^3 \\ &\quad + 6g^2h^2t^2 + 4gh^3t + h^4) \\ &\quad + (-2\mathbf{v}_0 + 2\mu\mathbf{v}_1)(g^2t^2 + 2ght + h^2) + \mathbf{v}_0, \\ \omega(t) &= (2 - 2\mu)(g^4t^4 + 4g^3ht^3 + 6g^2h^2t^2 + 4gh^3t + h^4) \\ &\quad + (-2 + 2\mu)(g^2t^2 + 2ght + h^2) + 1. \end{aligned}$$

Let $U_a = \mathrm{M}_{-\frac{\pi}{2}}(\mathbf{v}_0 - 2\mu\mathbf{v}_1 + \mathbf{v}_2)$, $U_b = \mathrm{M}_{-\frac{\pi}{2}}(-2\mathbf{v}_0 + 2\mu\mathbf{v}_1)$ and $U_c = \mathrm{M}_{-\frac{\pi}{2}}(\mathbf{v}_0)$. A tangent vector field perpendicular to $\mathbf{n}(t)$ is obtained as

$$\begin{aligned} U(t) &= \mathrm{M}_{-\frac{\pi}{2}}(V(t)) \\ &= (1 \;\; t \;\; t^2 \;\; t^3 \;\; t^4)\begin{pmatrix} U_ah^4 + U_bh^2 + U_c \\ 4U_agh^3 + 2U_bgh \\ 6U_ag^2h^2 + U_bg^2 \\ 4U_ag^3h \\ U_ag^4 \end{pmatrix} \\ &\doteq (1 \;\; t \;\; t^2 \;\; t^3 \;\; t^4)\begin{pmatrix} \bar{U}_0 \\ \bar{U}_1 \\ \bar{U}_2 \\ \bar{U}_3 \\ \bar{U}_4 \end{pmatrix}. \end{aligned}$$

By applying basis transformation, the vector field $U(t)$ can be represented by $U(t) = \sum_{i=0}^{4} U_iB_{i,4}(t)$, where the coefficient vectors are obtained as

$$\begin{pmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{4} & 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 1 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \bar{U}_0 \\ \bar{U}_1 \\ \bar{U}_2 \\ \bar{U}_3 \\ \bar{U}_4 \end{pmatrix}.$$

Similar to the reformulation of $U(t)$, the denominator of $\mathbf{n}(t)$ can also be represented using Bernstein basis. Let $\bar{\mu} = 2(1 - \mu)$. We have $\omega(t) = \sum_{i=0}^{4} \omega_iB_{i,n}(t)$, where the coefficients are computed by

$$\begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{4} & 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 1 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \bar{\mu}h^4 - \bar{\mu}h^2 + 1 \\ 4\bar{\mu}gh^3 - 2\bar{\mu}gh \\ 6\bar{\mu}g^2h^2 - \bar{\mu}g^2 \\ 4\bar{\mu}g^3h \\ \bar{\mu}g^4 \end{pmatrix}.$$

Based on the expression of $\omega(t)$, we have $\omega(0) = \omega_0$ and $\omega(1) = \omega_4$. We will construct geometric interpolating PH curves in the following subsections using the obtained functions $U(t)$ and $\omega(t)$.

4.2. G^1 Interpolation by PH curves with single inflection points

When the unit normals $\mathbf{n}_0$, $\mathbf{n}_1$ and $\mathbf{n}_2$ are interpolated by a quartic rational Bézier curve $\mathbf{n}(t)$, we construct a PH curve by Eq. (1) using $U(t) = \sum_{i=0}^{4} U_i B_{i,4}(t)$ and $\rho(t) = \rho_I(t) = \rho_0(1-t) + \rho_1 t$ such that the obtained PH curve can interpolate two given points P_A and P_B at the boundaries.

Substituting $P_0 = P_A$, $\rho(t) = \rho_I(t)$ and $U(t) = \sum_{i=0}^{4} U_i B_{i,4}(t)$ into Eq. (1), we have

$$\begin{aligned} P(t) &= \int_0^t \rho(\tau)U(\tau)d\tau + P_A \\ &= \rho_0 \int_0^t (1-\tau)U(\tau)d\tau + \rho_1 \int_0^t \tau U(\tau)d\tau + P_A \\ &= \rho_0 \int_0^t \sum_{i=0}^{4} \tfrac{5-i}{5} U_i B_{i,5}(\tau)d\tau \\ &+ \rho_1 \int_0^t \sum_{i=0}^{4} \tfrac{i+1}{5} U_i B_{i+1,5}(\tau)d\tau + P_A. \end{aligned} \quad (16)$$

From Eq. (16), we have $P(1) = \rho_0 A_0 + \rho_1 A_1 + P_A$, where $A_0 = \frac{1}{6}\sum_{i=0}^{4} \frac{5-i}{5} U_i$ and $A_1 = \frac{1}{6}\sum_{i=0}^{4} \frac{i+1}{5} U_i$. Based on the interpolation condition $P(1) = P_B$, we have

$$(\rho_0 \ \ \rho_1) = (P_B - P_A)\begin{pmatrix} A_0 \\ A_1 \end{pmatrix}^{-1}.$$

We note that if $\mathbf{n}_0$ and $\mathbf{n}_2$ are parallel, it yields that $A_0 = A_1$ due to the symmetry property of vectors U_is. There is no solution to equation $P(1) = P_B$ if the rank of matrix $(A_0^t \ A_1^t)$ is not equal to the rank of matrix $(A_0^t \ A_1^t \ (P_B - P_A)^t)$, where the upper letter 't' means the transpose of a vector or matrix. If the two mentioned matrices both have rank 1, there are infinitely many solutions to the equation.

When ρ_0 and ρ_1 have been obtained, a sextic PH curve interpolating two pairs of G^1 Hermite data and a prescribed normal vector at the inflection point is obtained by Eqs. (7) and (10). Particularly, the interpolating PH curve is regular when ρ_0 and ρ_1 have the same sign.

4.3. G^1 Interpolation by inflectional PH curves with prescribed arc lengths

To construct an inflectional PH curve that interpolates points P_A and P_B at the boundaries, unit normals $\mathbf{n}_0$, $\mathbf{n}_1$, $\mathbf{n}_2$ at the ends or at the inflection point, as well as with a prescribed arc length L_{arc}, we choose $\rho(t) = \rho_{II}(t) = \sum_{j=0}^{2} \rho_j B_{j,2}(t)$ together with $U(t) = \sum_{i=0}^{4} U_i B_{i,4}(t)$ for the construction of the curve. Let $P_0 = P_A$. From Eq. (1), a PH curve is obtained as

$$\begin{aligned} P(t) &= \int_0^t \rho(\tau)U(\tau)d\tau + P_A \\ &= \sum_{j=0}^{2} \rho_j \int_0^t \sum_{i=0}^{4} U_i B_{i,4}(\tau)B_{j,2}(\tau)d\tau + P_A \\ &= \sum_{j=0}^{2} \rho_j \int_0^t \sum_{i=0}^{4} \frac{C_4^i C_2^j}{C_6^{i+j}} U_i B_{i+j,6}(\tau)d\tau + P_A. \end{aligned} \quad (17)$$

Therefore, we have

$$P(1) = \rho_0 W_0 + \rho_1 W_1 + \rho_2 W_2 + P_A,$$

where

$$\begin{cases} W_0 = \frac{1}{7}(U_0 + \frac{2}{3}U_1 + \frac{2}{5}U_2 + \frac{1}{5}U_3 + \frac{1}{15}U_4) \\ W_1 = \frac{1}{7}(\frac{1}{3}U_0 + \frac{8}{15}U_1 + \frac{3}{5}U_2 + \frac{8}{15}U_3 + \frac{1}{3}U_4) \\ W_2 = \frac{1}{7}(\frac{1}{15}U_0 + \frac{1}{5}U_1 + \frac{2}{5}U_2 + \frac{2}{3}U_3 + U_4). \end{cases}$$

Let $\omega(t) = \sum_{i=0}^{4} \omega_i B_{i,4}(t)$ be the function obtained in Section 4.1. The arc length of the inflectional PH curve can be computed by substituting $\omega(t)$ and $\rho(t)$ into Eq. (2). We have

$$L(1) = \rho_0\bar{\omega}_0 + \rho_1\bar{\omega}_1 + \rho_2\bar{\omega}_2,$$

where

$$\begin{cases} \bar{\omega}_0 = \frac{1}{7}(\omega_0 + \frac{2}{3}\omega_1 + \frac{2}{5}\omega_2 + \frac{1}{5}\omega_3 + \frac{1}{15}\omega_4) \\ \bar{\omega}_1 = \frac{1}{7}(\frac{1}{3}\omega_0 + \frac{8}{15}\omega_1 + \frac{3}{5}\omega_2 + \frac{8}{15}\omega_3 + \frac{1}{3}\omega_4) \\ \bar{\omega}_2 = \frac{1}{7}(\frac{1}{15}\omega_0 + \frac{1}{5}\omega_1 + \frac{2}{5}\omega_2 + \frac{2}{3}\omega_3 + \omega_4). \end{cases}$$

Based on equalities $P(1) = P_B$ and $L(1) = L_{arc}$, we have a linear system in terms of the unknown coefficients ρ_0, ρ_1 and ρ_2 as follows:

$$\begin{cases} \rho_0 W_0 + \rho_1 W_1 + \rho_2 W_2 = P_B - P_A \\ \rho_0\bar{\omega}_0 + \rho_1\bar{\omega}_1 + \rho_2\bar{\omega}_2 = L_{arc}. \end{cases}$$

Solving the linear system, we have function $\rho(t) = \sum_{j=0}^{2} \rho_j B_{j,2}(t)$. A G^1 interpolating inflectional PH curve is obtained by Eqs. (8) and (10). If the function $\rho(t) \neq 0$ for $0 \le t \le 1$, the obtained curve is regular and has a prescribed arc length.

4.4. G^2 Interpolation by PH curves with single inflection points

When the unit normals $\mathbf{n}_0$, $\mathbf{n}_1$ and $\mathbf{n}_2$ have been interpolated by $\mathbf{n}(t)$, we choose $\rho(t) = \rho_{III}(t) = \sum_{j=0}^{3} \rho_j B_{j,3}(t)$ together with $U(t) = \sum_{i=0}^{4} U_i B_{i,4}(t)$ for the construction of a PH curve with one inflection point to interpolate points P_A and P_B as well as prescribed curvatures k_A and k_B at the boundaries.

Let $P_0 = P_A$, a PH curve constructed by Eq. (1) is obtained as

$$\begin{aligned} P(t) &= \int_0^t \rho(\tau)U(\tau)d\tau + P_A \\ &= \sum_{j=0}^{3} \rho_j \int_0^t \sum_{i=0}^{4} U_i B_{i,4}(\tau)B_{j,3}(\tau)d\tau + P_A \\ &= \sum_{j=0}^{3} \rho_j \int_0^t \sum_{i=0}^{4} \frac{C_4^i C_3^j}{C_7^{i+j}} U_i B_{i+j,7}(\tau)d\tau + P_A. \end{aligned} \quad (18)$$

From Eq. (18), we have

$$P(1) = \rho_0 W_0 + \rho_1 W_1 + \rho_2 W_2 + \rho_3 W_3 + P_A,$$

where

$$\begin{cases} W_0 = \frac{1}{8}(U_0 + \frac{4}{7}U_1 + \frac{2}{7}U_2 + \frac{4}{35}U_3 + \frac{1}{35}U_4) \\ W_1 = \frac{1}{8}(\frac{3}{7}U_0 + \frac{4}{7}U_1 + \frac{18}{35}U_2 + \frac{12}{35}U_3 + \frac{1}{7}U_4) \\ W_2 = \frac{1}{8}(\frac{1}{7}U_0 + \frac{12}{35}U_1 + \frac{18}{35}U_2 + \frac{4}{7}U_3 + \frac{3}{7}U_4) \\ W_3 = \frac{1}{8}(\frac{1}{35}U_0 + \frac{4}{35}U_1 + \frac{2}{7}U_2 + \frac{4}{7}U_3 + U_4). \end{cases}$$

The parameters ρ_j, $j = 0, 1, 2, 3$, are solved based on the interpolation conditions $P(1) = P_B$, $k(0) = k_A$ and $k(1) = k_B$. Particularly, ρ_0 and ρ_3 are first computed using the last two equations. Substituting $k(0) = k_A$ and $k(1) = k_B$ into Eq. (3), we have

$$\rho_0 = \rho(0) = \frac{U(0) \wedge U'(0)}{\omega^3(0)k(0)} = \frac{4U_0 \wedge U_1}{\omega^3(0)k_A}$$

and

$$\rho_3 = \rho(1) = \frac{U(1) \wedge U'(1)}{\omega^3(1)k(1)} = \frac{4U_3 \wedge U_4}{\omega^3(1)k_B}.$$

After computing ρ_0 and ρ_3, ρ_1 and ρ_2 are then computed from the equation $P(1) = P_B$. We have

$$(\rho_1 \ \ \rho_2) = (P_B - P_A - \rho_0 W_0 - \rho_3 W_3)\begin{pmatrix} W_1 \\ W_2 \end{pmatrix}^{-1}.$$

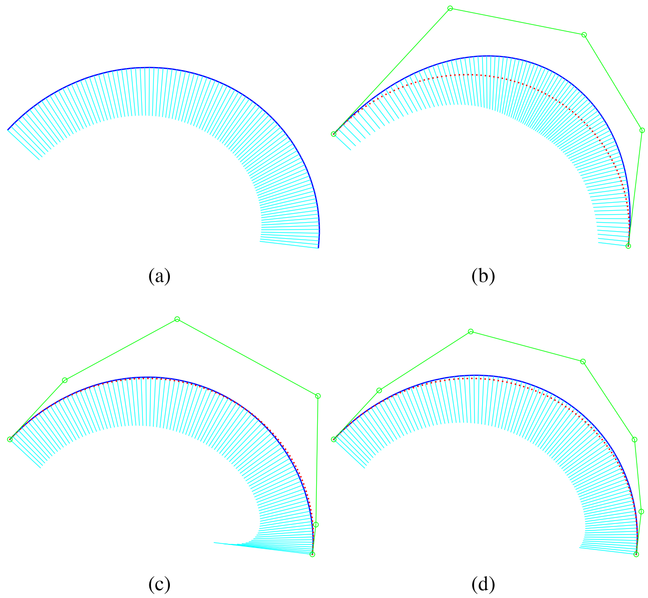

Fig. 5. (a) Original logarithmic spiral; (b) G^1 interpolation by a quartic PH curve with quadratic rational unit normals; (c) G^1 interpolation by a quintic PH curve that has quadratic rational unit normals and a prescribed arc length; (d) G^2 interpolation by a sextic PH curve with quadratic rational unit normals.

Similar to G^1 interpolation by PH curves with single inflection points, there is none or infinite many solutions $(\rho_1 \;\; \rho_2)$ when the normal vectors $\mathbf{n}_0$ and $\mathbf{n}_2$ are parallel.

When the polynomial $\rho_{III}(t)$ has been obtained, a PH curve of degree 8 that interpolates a pair of G^2 Hermite data and a prescribed normal vector at the inflection point is given by Eqs. (9) and (10). Particularly, if all ρ_j have the same sign, the obtained PH curve is regular. Similar to G^1 interpolation of PH curves with arc length constraint, G^2 interpolating PH curves can also have prescribed arc lengths. By choosing $\rho(t)$ as a quartic polynomial, the unknown coefficients of $\rho(t)$ will be determined by solving a linear system that is established by the G^2 interpolation conditions as well as the arc length constraint. We leave the concrete steps of G^2 interpolation of PH curves with arc length constraint to interested readers.

5. Examples

In this section we present a few examples to demonstrate how the proposed PH curve interpolation technique can be used in curve approximation and geometric modeling.

First, we construct convex PH curves interpolating points, normals, curvatures or even the arc length sampled or computed from a logarithmic spiral. Let $r_0 = 0.5$ and $\lambda = 0.12$. A segment of logarithmic spiral is given by

$$\begin{cases} x(t) &= r_0 e^{\lambda t}\cos(t) \\ y(t) &= r_0 e^{\lambda t}\sin(t) \end{cases}$$

with $t \in [0, 0.8\pi]$. Logarithmic spiral has no inflection point and the curvature of the curve is monotone. See Fig. 5(a) for the curvature plot of the original logarithmic spiral.

We sample points P_A and P_B at $t = 0$ or $t = 0.8\pi$, respectively. The unit normal vectors of the original logarithmic spiral at $t = 0$, $t = 0.5\pi$ and $t = 0.8\pi$ are also sampled. The arc length of the curve segment is computed by

$$\begin{aligned} L_{arc} &= \int_0^{0.8\pi} \sqrt{x'^2(t) + y'^2(t)}dt \\ &= \tfrac{r_0}{\lambda}\sqrt{\lambda^2+1}(e^{0.8\lambda} - 1) \end{aligned}$$

Table 1
The maximum approximation errors and the maximum absolute curvature differences for the interpolating curves to the logarithmic spiral.

Interpolation	Max approx. error	Max curvat. difference
G^1	0.082633	0.911133
G^1+Length	0.004541	1.448335
G^2	0.015263	0.137746

Table 2
The maximum approximation errors and the maximum absolute curvature differences for the interpolating curves to the Euler spiral.

Interpolation	Max approx. error	Max curvat. difference
G^1	0.020545	0.216066
G^1+Length	0.081761	0.439625
G^2	0.021921	0.189648

and the curvature of the curve is given by

$$k(t) = \frac{1}{\sqrt{\lambda^2+1}r_0 e^{\lambda t}}.$$

When all the sampled data have been obtained, a quadratic rational Bézier arc $\mathbf{n}(t)$ is constructed that satisfies $\mathbf{n}(0) = \mathbf{n}_0$, $\mathbf{n}(0.5) = \mathbf{n}_1$ and $\mathbf{n}(1) = \mathbf{n}_2$. After then, two G^1 interpolating PH curves without or with constraint of arc length and a G^2 interpolating PH curve are computed by the technique proposed in Section 3. See Figs. 5(b)–(d) for the obtained curves. The maximum approximating errors to the original curve and the maximum curvature differences between the interpolating curves and the original logarithmic spiral are given in Table 1.

Second, we interpolate points and normals sampled from a segment of Euler spiral. Assume the original spiral is defined by $k = s$, where k means the curvature and s is the arc length of the curve. Particularly, the Cartesian coordinates of the Euler spiral is computed by

$$\begin{cases} x(s) &= \int_0^s \cos(\frac{\xi^2}{2})d\xi \\ y(s) &= \int_0^s \sin(\frac{\xi^2}{2})d\xi, \end{cases}$$

where $s \in [-0.4\pi, 0.5\pi]$. From above equation, the points at $s = -0.4\pi$ or $s = 0.5\pi$ are computed by Taylor expansion of the integral while the unit normals at the ends or the inflection point are computed explicitly by the derivatives of the curve. Since the Euler spiral is parameterized by its arc length, the arc length of the curve segment can be computed by the parameters directly. To improve the fairness of the interpolating curve we choose $L_{arc} = (0.5\pi + 0.4\pi) * 0.99 = 0.891\pi$ for PH curve interpolation under arc length constraint.

Fig. 6 illustrates the original Euler spiral and the G^1 or G^2 interpolating PH curves. Differently from the last example, the sampled normals at the ends or the inflection point are interpolated by a quartic rational curve and the obtained PH curves have single inflection points. From the figure we can see that the normal vectors of the interpolating PH curves match the original Euler spiral very well and the shapes of the obtained curves are please-looking too. The approximation errors given in Table 2 show that both positions and curvatures of original Euler spiral have been approximated by the interpolating PH curves with high accuracies.

Third, we design an 8-like shape by geometric interpolation of PH curves. Let $O_a = (0, 0.6)$ and $O_b = (0, -0.6)$ be the centers of two circles both with radius $r = 0.5$ on the plane. Five unit normals $N_i = N(\theta_i) = (\cos\theta_i, \sin\theta_i)$, $i = 0, 1, \ldots, 4$, are given by choosing θ_i as -0.1π, 0.1π, -0.5π, 0.9π and 1.1π, respectively. Five points are then sampled on the two circles as $P_0 = O_a - rN_0$, $P_4 = O_a - rN_4$, and $P_i = O_b + rN_i$ for $i = 1, 2, 3$. See Fig. 7 for the chosen points and normals. Particularly, two sextic

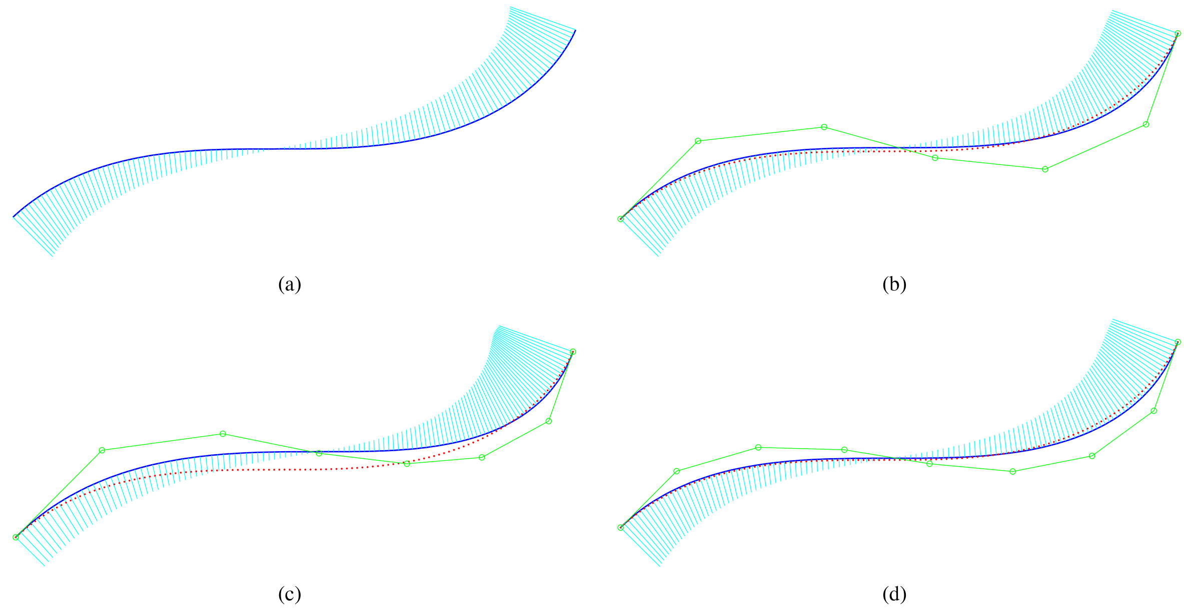

Fig. 6. (a) Original Euler spiral; (b) G^1 interpolation by a sextic PH curve with quartic rational unit normals; (c) G^1 interpolation by a septic PH curve that has quartic rational unit normals and a prescribed arc length; (d) G^2 interpolation by a PH curve of degree 8 with quartic rational unit normals.

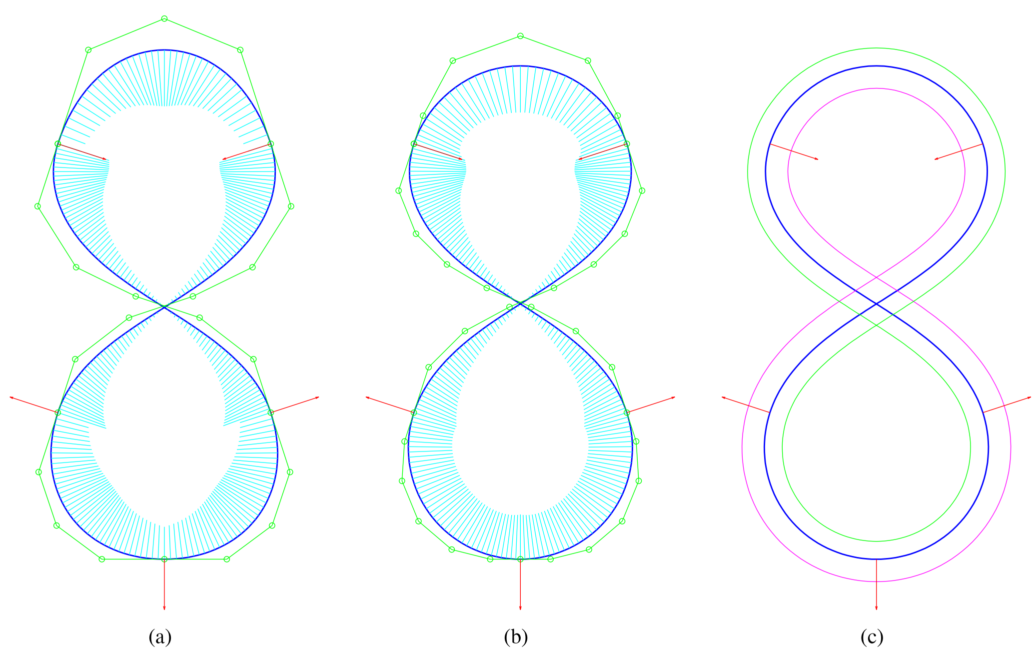

Fig. 7. (a) Piecewise G^1 interpolation to the sampled points and sampled normals; (b) piecewise G^2 interpolation to the sampled points, normals and curvatures; (c) the offsets to the G^2 interpolating PH curves.

PH curves with single inflection points have been constructed to interpolate point-normal pairs $(P_0, N_0; P_1, N_1)$ or $(P_3, N_3; P_4, N_4)$ with prescribed unit normals $N(0.32\pi)$ or $N(0.68\pi)$ at the inflection points, respectively. The point-normal pairs $(P_1, N_1; P_2, N_2)$, $(P_2, N_2; P_3, N_3)$ and $(P_4, N_4; P_0, N_0)$ together with normalized average vectors of each pair of normals are interpolated by convex quartic PH curves. Fig. 7(a) illustrates the piecewise G^1 interpolating PH curves.

Though the G^1 interpolating PH curves are convex or have single inflection points, the curvatures of the curves are not continuous at the joint points. When we set curvatures at respective sampled points as $k_0 = k_4 = 2.4$ and $k_1 = k_2 = k_3 = -2.0$, a sequence of interpolating PH curves with prescribed curvatures as well as with G^2 continuity at the joints are obtained; see Fig. 7(b) for the result. The offsets to the G^2 interpolating PH curves at distances $d = 0.1$ or $d = -0.08$ along normal directions are plotted in Fig. 7(c).

6. Conclusions and discussions

This paper has proposed techniques to interpolate sampled points, normals or curvatures by PH curves with quadratic or quartic rational unit normals. The interpolating PH curves can also have prescribed arc lengths. The parameters for defining each interpolating PH curve are uniquely determined by solving a simple linear system. The singularities of interpolating PH curves can be checked easily based on the signs of a polynomial function and the regular interpolating PH curves are either convex or have expected single inflection points with prescribed normals at the points.

At present we assume that the angles between every two input normals are less than π. If the maximum angle is greater than or equal to π, one can insert one or more points and normals between the original data and construct interpolating PH curves to every pair of neighboring points and normals by the proposed technique. Alternatively, one can construct rational unit normals with even higher degrees from the input normals and compute the interpolating PH curves using the same technique proposed in this paper.

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