Section 4. Variation of arc length, energy and area

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In Riemannian geometry, geodesics and minimal submanifolds play important roles in studying the geometry and topology of manifolds. In this section we study the length, energy and area functions for paths and submanifolds.

1 First variation of arc length.

Given a path $\gamma : [a,b] \to M^n$, its **length** is defined by

$$L(\gamma) := \int_a^b |\gamma'(t)| \, dt.$$

The **distance function** is defined by

$$d_p := d(x,p) := \inf_\gamma L(\gamma),$$

where the infimum is taken over all paths $\gamma : [0,1] \to M^n$ with $\gamma(0) = p$ and $\gamma(1) = x$. A geodesic segment is minimal if its length is equal to the distance between the two endpoints.

We recall the first and second variation of arc length formulas. Let $\gamma_s : [a,b] \to M^n$, $s \in (-\varepsilon, \varepsilon) \subset \mathbb{R}$, be a 1-parameter family of paths. From this we may define the map $\alpha : [a,b] \times (-\varepsilon, \varepsilon) \to M^n$ by

$$\alpha(t,s) := \gamma_s(t).$$

We define the vector fields T and V along γ_s by

$$T = \alpha_* \left(\frac{\partial}{\partial t} \right) \quad \text{and} \quad V = \alpha_* \left(\frac{\partial}{\partial s} \right).$$

We call V the variation vector field and T the tangent vector field. The length of γ_s is given by

$$L(\gamma_s) := \int_a^b |T(\gamma_s(t))| \, dt.$$

The **first variation of arc length formula** is given by

Lemma 4.1 (First variation of arc length). *If γ_0 is parametrized by arc length, that is $|T(\gamma_0(t))| \equiv 1$, then*

$$\boxed{\tfrac{d}{ds}|_{s=0} L(\gamma_s) = -\textstyle\int_a^b \langle V, \nabla_T T \rangle \, dt + \langle V, T \rangle \, |_a^b.} \tag{4.1}$$

Proof. Since

$$\begin{aligned} \frac{d}{ds} L(\gamma_s) &= \frac{1}{2} \int_a^b |T|^{-1} V \langle T, T \rangle \, dt = \int_a^b |T|^{-1} \langle T, \nabla_V T \rangle \, dt \\ &= \int_a^b |T|^{-1} \langle T, \nabla_T V \rangle \, dt \end{aligned} \tag{4.2}$$

where we have used the fact $\nabla_T V - \nabla_V T = [T, V] = \alpha_* \left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right] = 0$. Using $|T| \equiv 1$ and integrating by parts yields the desired formula. QED

Corollary 4.1. *If $\gamma_s : [0, b] \to M^n$, $s \in (-\varepsilon, \varepsilon) \subset R$, is a 1-parameter family of paths emanating from a fixed point $p \in M$ (i.e. $\gamma_s(0) = p$) γ_0 is a geodesic parametrized by arc length, then*

$$\frac{d}{ds}|_{s=0} L(\gamma_s) = \left\langle \frac{d}{ds}|_{s=0} \gamma_s(b), \frac{\partial \gamma_0}{\partial t}(b) \right\rangle.$$

2 Second variation of arc length.

Now suppose we have a 2-parameter family of paths $\gamma_{v,w} : [a, b] \to M^n, v \in (-\varepsilon, \varepsilon) \subset \mathbb{R}$ and $w \in (-\delta, \delta) \subset \mathbb{R}$. Define $\alpha : [a, b] \times (-\varepsilon, \varepsilon) \times (-\delta, \delta) \to M^n$ by

$$\alpha(t, v, w) = \gamma_{v,w}(t)$$

and $T = \alpha_* \left(\frac{\partial}{\partial t} \right), V = \alpha_* \left(\frac{\partial}{\partial v} \right)$ and $W = \alpha_* \left(\frac{\partial}{\partial w} \right)$. The second variation of arc length formula is given by

Lemma 4.2 (Second variation of arc length). *If $\gamma_{0,0}$ is parametrized by arc length, then*

$$\begin{aligned} &\frac{\partial^2}{\partial v \partial w}|_{(v,w)=(0,0)} L\left(\gamma_{v,w}\right) \\ &= \int_a^b \left(\left\langle \nabla_T V, \nabla_T W \right\rangle - \left\langle \nabla_T V, T \right\rangle \left\langle \nabla_T W, T \right\rangle - \left\langle R\left(T, W\right) V, T \right\rangle \right) dt \\ &- \int_a^b \left\langle \nabla_V W, \nabla_T T \right\rangle dt + \left\langle \nabla_V W, T \right\rangle |_a^b. \end{aligned} \tag{4.3}$$

Proof. Differentiating (4.2), we compute that

$$\begin{aligned} &\frac{\partial^2}{\partial v \partial w}|_{(v,w)=(0,0)} L\left(\gamma_{v,w}\right) \\ &= \frac{\partial}{\partial w}|_{(v,w)=(0,0)} \int_a^b \left\langle T/\left|T\right|, \nabla_T V \right\rangle dt \\ &= \int_a^b W \left\langle T/\left|T\right|, \nabla_T V \right\rangle dt \\ &= \int_a^b \left(\left\langle T/\left|T\right|, \nabla_W \nabla_T V \right\rangle + \left\langle \nabla_W \left(T/\left|T\right|\right), \nabla_T V \right\rangle \right) dt \\ &= \int_a^b \left(\left\langle R\left(W, T\right) V, T/\left|T\right| \right\rangle + \left\langle T/\left|T\right|, \nabla_T \nabla_W V \right\rangle \right) dt \\ &+ \int_a^b \left|T\right|^{-1} \left\langle \nabla_W T, \nabla_T V \right\rangle dt - \int_a^b \left|T\right|^{-3} \left\langle \nabla_W T, T \right\rangle \left\langle T, \nabla_T V \right\rangle dt \end{aligned}$$

and the result follows from an integration by parts. QED

Corollary 4.2. *If γ_s is a 1-parameter family of piecewise smooth paths with fixed endpoints and such that γ_0 is a geodesic parametrized by arc length, then*

$$\frac{\partial^2}{\partial s^2}|_{s=0} L\left(\gamma_s\right) = \int_a^b \left(\left| \left(\nabla_T W\right)^\perp \right|^2 - \left\langle R\left(W, T\right) T, W \right\rangle \right) dt, \tag{4.4}$$

where $\left(\nabla_T W\right)^\perp$ is the projection of $\nabla_T W$ onto $T^\perp$, i.e., $\left(\nabla_T W\right)^\perp := \nabla_T W - \left\langle \nabla_T W, T \right\rangle T$.

A geodesic is stable if the second variation of arc length, with respect to variation vector fields which vanish at the endpoints, is nonnegative. In the case of negative curvature we have the following

Corollary 4.3. *If in addition, (M^n, g) has nonpositive sectional curvature and the paths γ_s are smooth and closed, then $\frac{\partial^2}{\partial s^2}|_{s=0} L(\gamma_s) \geq 0$. That is, any smooth closed geodesic γ_0 is stable.*

A **closed geodesic** is a smooth map $\gamma : S^1 \to M$ which is a geodesic. Next we consider the case of positive curvature. With the idea of minimizing in a homotopy class of an element of $\pi_1(M^n)$ and some simple linear algebra, one also obtains

Theorem 4.1 (Synge). *If (M^n, g) is an even-dimensional, orientable, closed Riemannian manifold with positive sectional curvature, then M^n is simply connected.*

Proof. Suppose Θ is a nontrivial free homotopy class of loops. Then since M^n is compact, there exists a smooth closed geodesic γ representing Θ with minimal length among all such loops (see Cheeger-Ebin for a proof). Fix a point $p \in \gamma$ and consider parallel translation around γ to obtain a linear isometry of an even-dimensional vector space: $\iota : T_pM \to T_pM$. Now $\iota\left(\dot{\gamma}\right) = \dot{\gamma}$ because γ is a geodesic. Since M^n is orientable, $\iota : \dot{\gamma}^{\perp} \to \dot{\gamma}^{\perp}$ is an orthogonal transformation with determinant 1 of an odd-dimensional vector space with an inner product. Now the eigenvalues of an orthogonal matrix are complex numbers with length 1 coming in conjugate pairs. Since the dimension of $\dot{\gamma}^{\perp}$ is odd and the determinant of ι is 1, we conclude that 1 is an eigenvalue. Hence there exists a smooth parallel unit vector field W along γ normal to $\dot{\gamma}$. The second variation formula (4.4) implies that

$$\frac{\partial^2}{\partial s^2}|_{s=0} L(\gamma_s) = -\int_a^b \langle R(W, T)T, W \rangle \, dt < 0$$

since the sectional curvatures are positive. This contradicts the fact that γ has minimal length in its free homotopy class. Hence M^n is simply connected. QED

3 Long stable geodesics.

Let $\gamma : \left[0, \bar{t}\right] \to M^n$ be a stable unit speed geodesic in a Riemannian n-manifold with $Rc \leq (n-1) K$ in $B(\gamma(0), r)$ and $B\left(\gamma\left(\bar{t}\right), r\right)$ where $K > 0$ and $2r < \bar{t}$. Let $\{E_i\}_{i=1}^{n-1}$ be a parallel orthonormal frame along γ perpendicular to $\dot{\gamma}$. By the second variation of arc length formula (4.4) we have

$$\begin{aligned}
0 &\leq \sum_{i=1}^{n-1} \int_0^{\bar{t}} \left(\left| \left(\nabla_{\dot{\gamma}} \left(\varphi E_i \right) \right)^{\perp} \right|^2 - \left\langle R \left(\dot{\gamma}, \varphi E_i \right) \varphi E_i, \dot{\gamma} \right\rangle \right) dt \\
&= \int_0^{\bar{t}} \left((n-1) \left(\frac{d\varphi}{dt} \right)^2 - \varphi^2 Rc \left(\dot{\gamma}, \dot{\gamma} \right) \right) dt
\end{aligned}$$

for any function $\varphi : \left[0, \bar{t}\right] \to \mathbb{R}$. Now consider the piecewise smooth function

$$\varphi(t) := \begin{cases} \frac{t}{r} & \text{if } \ 0 \leq t \leq r, \\ 1 & \text{if } r < t < \bar{t} - r, \\ \frac{\bar{t}-t}{r} & \text{if } \bar{t} - r \leq t \leq \bar{t}. \end{cases}$$

We then have $\left| \frac{d\varphi}{dt}(t) \right| = \frac{1}{r}$ for $t \in [0, r]$ and $t \in \left[\bar{t} - r, \bar{t}\right]$, and

$$\begin{aligned}
\int_{\gamma} Rc \left(\dot{\gamma}, \dot{\gamma} \right) dt &\leq \frac{2(n-1)}{r} + \int_0^r \left(1 - \varphi^2\right) Rc \left(\dot{\gamma}, \dot{\gamma} \right)_+ dt \\
&\quad + \int_{\bar{t}-r}^{\bar{t}} \left(1 - \varphi^2\right) Rc \left(\dot{\gamma}, \dot{\gamma} \right)_+ dt \\
&\leq 2(n-1) \left(\frac{1}{r} + Kr \right),
\end{aligned}$$

where $Rc \left(\dot{\gamma}, \dot{\gamma} \right)_+ := \max \left\{ Rc \left(\dot{\gamma}, \dot{\gamma} \right), 0 \right\}$. Here we used

$$\int_0^r \left(1 - \varphi^2\right) dt = \int_{\bar{t}-r}^{\bar{t}} \left(1 - \varphi^2\right) dt = \frac{2}{3} r \leq r.$$

Proposition 4.1 *If $\gamma : [0, L] \to M^n$ is a stable unit speed geodesic in a Riemannian n-manifold with*

$$Rc \leq (n-1) K \qquad \text{in } B \left(\gamma(0), 1/\sqrt{K} \right) \cup B \left(\gamma(L), 1/\sqrt{K} \right),$$

where $K > 0$, then

$$\int_{\gamma} Rc \left(\dot{\gamma}, \dot{\gamma} \right) dt \leq 4(n-1)\sqrt{K}.$$

Let $(M^n; g)$ be a complete Riemannian manifold with nonnegative Ricci curvature. Let $p \in M$ be a given point and assume that $\gamma : [-2R, 2R] \to M$ is a unit speed stable geodesic with $\gamma(0) = p$. Define $\varphi : \mathbb{R} \to [0, 1]$ so that $\varphi(t) = 1$ for $t \in [-R, R]$ and φ changes linearly to 0 on both intervals $[-2R, -R]$ and $[R, 2R]$. Then the second variation formula (4.4) implies that

$$\int_{-R}^{R} Rc\left(\dot{\gamma}, \dot{\gamma}\right) dt \leq \frac{2(n-1)}{R}.$$

From this one can see that if there is a stable geodesic with infinite length (for example a *line*), the Ricci curvature must vanish on such a geodesic.

4 Jacobi fields in relation to the index form and the Laplacian and hessian of the distance function.

Recall that a **Jacobi field** J is a variation of geodesics and satisfies the Jacobi equation:

$$\nabla_T \nabla_T J = R(T, J) T, \tag{4.5}$$

which is derived immediately from the geodesic equation $\nabla_T T = 0$ since

$$0 = \nabla_J \nabla_T T = \nabla_T \nabla_J T - R(T, J) T$$

and $[T, J] = 0$. Given $p \in M^n$ and $V, W \in T_p M$, define a 1-parameter family of geodesics

$$\gamma_s : [0, \infty) \to M^n$$

by

$$\gamma_s(t) := \exp_p t(V + sW) = \gamma_{V+sW}(t).$$

We may define a Jacobi field J_W along $\gamma_0 = \gamma_V$ by

$$J_W(t) := \frac{\partial}{\partial s}|_{s=0} \gamma_{(V+sW)}(t).$$

Definition 4.1. *A point $x \in M^n$ is a* ***conjugate point*** *of $p \in M^n$ if x is a singular value of $\exp_p : T_p M \to M$. That is $x = \exp_p(V)$, for some $V \in T_p M$, where $\left(\exp_p\right)_* : T_V(T_p M) \to T_{\exp_p(V)} M$ is singular (i.e. has nontrivial kernel).*

Equivalently, $\gamma(r)$ is a conjugate point to p along γ if there is a nontrivial Jacobi field along γ vanishing at the endpoints. Given a geodesic $\gamma : [0, L] \to M^n$ without conjugate points and vectors $A \in T_{\gamma(0)}M^n$ and $B \in T_{\gamma(L)}M^n$ with $\langle A, T\rangle = 0$ and $\langle B, T\rangle = 0$, there exists a unique Jacobi field J with $J(0) = A$ and $J(L) = B$.

If $\gamma : [a, b] \to M^n$ is a geodesic and V and W are vector fields along γ, we define the index form by

$$I(V, W) = \int_a^b \left(\langle \nabla_T V, \nabla_T W\rangle - \langle \nabla_T V, T\rangle \langle \nabla_T W, T\rangle - \langle R(V, T) T, W\rangle\right) dt.$$

Note that if either $\langle V, T\rangle \equiv 0$ on γ or $\langle W, T\rangle \equiv 0$ on γ, then

$$I(V, W) = \int_a^b \left(\langle \nabla_T V, \nabla_T W\rangle - \langle R(V, T) T, W\rangle\right) dt.$$

It $\gamma_{v,w}$ is a 1-parameter family of paths with fixed endpoints and if $\gamma_{0,0}$ is a unit speed geodesic, then

$$\frac{\partial^2}{\partial v \partial w}|_{(v,w)=(0,0)} L(\gamma_{v,w}) = I(V, W).$$

Lemma 4.3 (Index Lemma). *Suppose $\gamma : [0, L] \to M^n$ is a geodesic without conjugate points. In the space $\Theta_{A,B}$ of vector fields X along γ with $\langle X, T\rangle \equiv 0, X(0) = A, (L) = B$, the Jacobi field minimizes the (modified) index form:*

$$\mathcal{I}(X) := \widetilde{I}(X, X) := \int_0^L \left(|\nabla_T X|^2 - \langle R(X, T) T, X\rangle\right) dt.$$

Proof. Note that if X and Y are vector fields along γ, then

$$\frac{1}{2}\frac{d}{ds}|_{s=0}\mathcal{I}(X + sY) = -\int_0^L \langle(\nabla_T \nabla_T X + R(X, T) T), Y\rangle dt.$$

Here we used the fact $X(0) = A$ and $X(L) = B$, and $(X + sY)(0) = A, (X + sY)(L) = B$. Then $Y(0) = Y(L) = 0$. Hence the critical points of $\mathcal{I}$ on $\Theta_{A,B}$ are the Jacobi fields. We will see below that if γ has no conjugate points, then

$$\frac{1}{2}\frac{d^2}{ds^2}|_{s=0}\mathcal{I}(X + sY) = \int_0^L \left(|\nabla_T Y|^2 - \langle R(Y, T) T, Y\rangle\right) dt > 0 \tag{4.6}$$

when $Y(0) = Y(L) = 0$ and $Y \neq 0$, so that the index form $\mathcal{I}$ is convex. (Note that the tangent space $T_X \Theta_{A,B}$ is the space of all vector fields along γ which vanish at its endpoints.) Hence the Jacobi fields minimize $\mathcal{I}$ in $\Theta_{A,B}$. we now give a variational proof of inequality (4.6).

Given a unit speed geodesic $\gamma : [0, L] \to M^n$ without conjugate points and a vector field Y along γ, define

$$\mathcal{I}_t(Y) := \int_0^t \left(|\nabla_T Y|^2 - \langle R(Y,T)T, Y \rangle \right) dt$$

for $t \in [0, L]$. Normalize the index by defining

$$\iota(t) := \inf_{\substack{Z(0)=0, Z(t)=0 \\ Z \neq 0}} \frac{\mathcal{I}_t(Z)}{\int_0^t |Z(s)|^2 \, ds}.$$

We claim that $\lim_{t \to 0+} \iota(t) = \infty$. First note

$$\frac{d}{ds} |Z| \leq |\nabla_T Z|, \langle R(Z,T)T, Z \rangle \leq C|Z|^2$$

and

$$\lambda_1([0,t]) = \frac{\pi^2}{t^2},$$

where λ_1 is the first eigenvalue of d^2/ds^2 with Dirichlet boundary conditions (we omit the discussion of eigenvalues and boundary conditions.) Thus the claim follows from

$$\begin{aligned} \iota(t) &\geq \inf_{\substack{Z(0)=0, Z(t)=0 \\ Z \neq 0}} \frac{\int_0^t \left(\left(\frac{d}{dt} |Z| \right)^2 - C|Z|^2 \right) ds}{\int_0^t |Z(s)|^2 \, ds} \\ &\geq \frac{\pi^2}{t^2} - C. \end{aligned}$$

For $t \in (0, L]$ where $\gamma|_{[0,t]}$ is minimizing (e.g. for $t > 0$ small enough), we have $\iota(t) \geq 0$ since $\mathcal{I}_t(Z)$ is a second variation of $\gamma|_{[0,t]}$ vanishing at the endpoints 0 and t. Since $\iota(t)$ continuous, if the lemma is not true, then there exists $t_0 \in (0, L]$ such that $\iota(t_0) = 0$. Then $\mathcal{I}_{t_0}(Z_0) = 0$ for some vector field Z_0 with $Z_0(0) = Z_0(t) = 0$, and $Z_0 \neq 0$. By considering the Euler-Lagrange equation for $E(Z) := \mathcal{I}_{t_0}(Z) / \int_0^{t_0} |Z(s)|^2 \, ds$ at Z_0, we have for all

W vanishing at 0 and t_0

$$\begin{aligned} 0 &= \frac{1}{2}\frac{d}{du}|_{u=0}E\left(Z_0+uW\right) \\ &= -\frac{1}{\int_0^{t_0}\left|Z\left(s\right)\right|^2 ds}\int_0^t \left\langle \left(\nabla_T\nabla_T Z_0 + R\left(Z_0,T\right)T\right), W\right\rangle ds, \end{aligned}$$

since $\mathcal{I}_{t_0}\left(Z_0\right) = 0$. Thus Z_0 is a nontrivial Jacobi field along $\gamma|_{[0,t_0]}$ with $Z_0\left(0\right) = 0$ and $Z_0\left(t_0\right) = 0$. This contradicts the assumption that there are no conjugate points along γ. Hence $\iota\left(t\right) > 0$ for all $t \in (0, L]$. QED

Remark. Since in the definition of $\iota\left(t\right)$ we may take Z to e piecewise smooth, we see that $\iota\left(t\right)$ is nonincreasing in t.

5 Energy.

Analogous to the length functional is the energy functional of a path:

$$Energy\left(\gamma\right) := \int_a^b \left|\frac{d\gamma}{dt}\right|^2 dt.$$

Given a 1-parameter family of paths $\gamma_s : [a,b] \to M^n, s \in (-\varepsilon,\varepsilon) \subset \mathbb{R}$, the first variation formula is given by (V and T defined the same as above)

$$\frac{1}{2}\frac{d}{ds}|_{s=0}Energy\left(\gamma_s\right) = \int_a^b \left\langle \nabla_T V, T\right\rangle dt = -\int_a^b \left\langle V, \nabla_T T\right\rangle dt + \left\langle V,T\right\rangle|_a^b.$$

Note we have not assumed that γ_0 is parameterized by arc length. Hence the critical points of the energy functional, among all paths fixing two endpoints, are the constant speed geodesics γ, which satisfy

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0.$$

The speed of γ is constant since $\dot{\gamma}\left|\dot{\gamma}\right|^2 = 0$.

Now let $\gamma_{v,w} : [a,b] \to M^n, v \in (-\varepsilon,\varepsilon) \subset \mathbb{R}$ and $w \in (-\delta,\delta) \subset \mathbb{R}$, be a 2-parameter family of paths. A similar computation as in the proof of Lemma

4.2 shows that the second variation is given by

$$\begin{aligned}\frac{\partial^2}{\partial v \partial w}|_{(v,w)=(0,0)} Energy\left(\gamma_{v,w}\right) &= \int_a^b W \left\langle T, \nabla_T V \right\rangle dt \\ &= \int_a^b \left(\left\langle T, \nabla_W \nabla_T V \right\rangle + \left\langle \nabla_W T, \nabla_T V \right\rangle \right) dt \\ &= \int_a^b \left(\left\langle \nabla_T W, \nabla_T V \right\rangle + \left\langle R\left(T, W\right) V, T \right\rangle \right) dt \\ &- \int_a^b \left\langle \nabla_V W, \nabla_T T \right\rangle dt + \left\langle \nabla_W V, T \right\rangle |_a^b .\end{aligned}$$

6 First and second variation of area.

Similar to computing the first and second variation of the length of a path, we may compute the first and second variation of the area of a hypersurface. Let $x_t : S \to M^n$ be a parametrized hypersurface in M^n evolving by $\frac{\partial x}{\partial t} = \beta \nu$, where β is some function on $S_t := x_t(S)$. In terms of local coordinates $\{x^i\}$ on S, the area element of S_t is

$$d\sigma = \sqrt{\det g_{ij}} dx^1 \wedge \ldots \wedge dx^{n-1}.$$

Then

$$\frac{\partial}{\partial t} g_{ij} = 2\beta h_{ij}.$$

Hence

$$\frac{\partial}{\partial t} d\sigma = \beta H d\sigma.$$

Thus the first variation of $Area\left(S_t\right) := \int_{S_t} d\sigma$ is

$$\frac{d}{dt} Area\left(S_t\right) = \int_{S_t} \beta H d\sigma.$$

Under the hypersurface flow $\frac{\partial x}{\partial t} = \beta \nu$, we have the equation

$$\frac{\partial}{\partial t} H = -\Delta \beta - |h|^2 \beta - Rc\left(\nu, \nu\right) \beta, \tag{4.7}$$

where H is the mean curvature of the hypersurface, and $H = g^{ij}h_{ij}$. By (4.7) we can compute the second variation of area:

$$\begin{aligned}\frac{d^2}{dt^2}Area\left(S_t\right) &= \int_{S_t} \beta\left(-\Delta\beta - |h|^2\beta - \left(Rc_M\right)(\nu,\nu)\beta + H^2\beta\right)d\sigma \\ &= \int_{S_t}\left(|\nabla\beta|^2 + \left(H^2 - |h|^2 - \left(Rc_M(\nu,\nu)\right)\right)\beta^2\right)d\sigma.\end{aligned}$$

In particular, if $\frac{\partial x}{\partial t} = \nu$ (i.e. $\beta = 1$),then

$$\frac{d^2}{dt^2}Area\left(S_t\right) = \int_{S_t}\left(H^2 - |h|^2 - \left(Rc_M\right)(\nu,\nu)\right)d\sigma.$$

Since

$$R_S = R_M - 2\left(Rc_M\right)(\nu,\nu) + H^2 - |h|^2. \quad \text{(Gauss Equation)}$$

Therefore if $\frac{\partial x}{\partial t} = \nu$ then the second variation of area is given by

$$\frac{d^2}{dt^2}Area\left(S_t\right) = \frac{1}{2}\int_{S_t}\left(R_S - R_M + H^2 - |h|^2\right)d\sigma. \tag{4.8}$$

As an application, we have the following result of Schoen and Yau.

Theorem 4.2. If S is an orientable closed stable minimal surface in a 3-dimension manifold (M^3, g) with positive scalar curvature, then S is diffeomorphic to a 2-sphere.

Proof. Let S_t be a variation of S with $S_0 = S$ and $\beta = 1$. By (4.8), $H = 0$ and the Gauss-Bonnet formula, we have

$$\begin{aligned}0 \le \frac{d^2}{dt^2}|_{t=0}Area\left(S_t\right) &= \frac{1}{2}\int_S\left(R_S - R_M + H^2 - |h|^2\right)d\sigma \\ &= 2\pi\chi(S) - \frac{1}{2}\int_S\left(R_M + |h|^2\right)d\sigma.\end{aligned}$$

Since $R_M > 0$ and $|h|^2 \ge 0$, we conclude $\chi(S) > 0$. Since S is orientable, $S \approxeq S^2$. QED